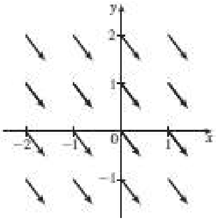
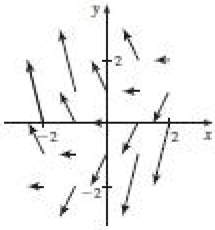
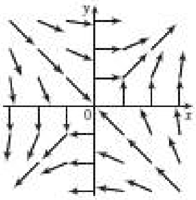
부록 E 해답

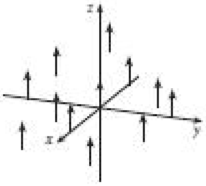
13장

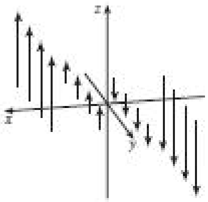
연습문제 13.1

01. 

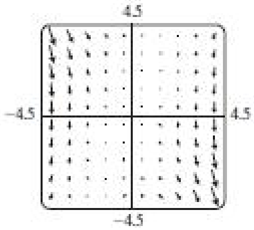
02. 

03. 

04. 

05. 

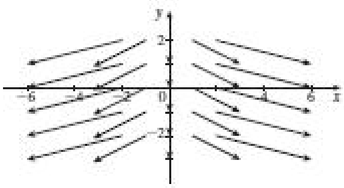
06. IV 07. III 08. I 09. II 10. IV 11. I 12. III 13. II

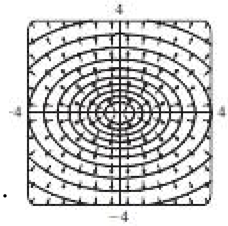
14. 

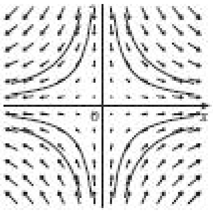
; 직선 $y = 2x$

15. $\nabla f(x, y) = (xy+1)e^{xy}\mathrm{i} + x^2 e^{xy}\mathrm{j}$

16. $\nabla f(x, y, z) = \dfrac{x}{\sqrt{x^2+y^2+z^2}}\mathrm{i} + \dfrac{y}{\sqrt{x^2+y^2+z^2}}\mathrm{j} + \dfrac{z}{\sqrt{x^2+y^2+z^2}}\mathrm{k}$

17. $\nabla f(x, y) = 2x\mathrm{i} - \mathrm{j}$; 

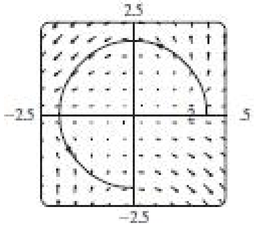
18. 

19. $(2.04, 1.03)$ 20. (a)  ; $y = C/x$ (b) $y = 1/x,\ x > 0$

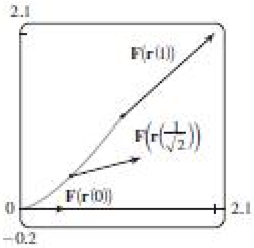
연습문제 13.2

01. $\dfrac{1}{54}(145^{3/2} - 1)$ 02. 1638.4 03. $\dfrac{243}{8}$ 04. $\dfrac{5}{2}$ 05. $\sqrt{5}\,\pi$

06. $\dfrac{1}{12}\sqrt{14}\,(e^6 - 1)$ 07. $\dfrac{2}{5}(e - 1)$ 08. $\dfrac{35}{3}$ 09. (a) 양수 (b) 음수

10. 45 11. $\dfrac{6}{5} - \cos 1 - \sin 1$ 12. 1.9633 13. 

; $3\pi + \dfrac{2}{3}$

14. (a) $\dfrac{11}{8} - 1/e$ (b) 

15. $\dfrac{945}{16777216}\pi$

16. $2\pi k,\ (4/\pi, 0)$

17. (a) $\overline{x} = (1/m)\displaystyle\int_C x\rho(x, y, z)\,ds,\ \overline{y} = (1/m)\int_C y\rho(x, y, z)\,ds,$

$\overline{z} = (1/m)\displaystyle\int_C z\rho(x, y, z)\,ds$, 여기서 $m = \displaystyle\int_C \rho(x, y, z)\,ds$

(b) $(0, 0, 3\pi)$

18. $I_x = k\left(\dfrac{1}{2}\pi - \dfrac{4}{3}\right),\ I_y = k\left(\dfrac{1}{2}\pi - \dfrac{2}{3}\right)$

19. $2\pi^2$ 20. $\dfrac{7}{3}$ 21. (a) $2ma\,\mathbf{i} + 6mbt\,\mathbf{j}$ (b) $2ma^2 + \dfrac{9}{2}mb^2$

22. $\approx 1.67 \times 10^4$ ft·lb

23. $\mathbf{r}(t) = \langle x(t), y(t), z(t)\rangle$, $\mathbf{v} = \langle v_1, v_2, v_3\rangle$

$\int_C \mathbf{v} \cdot d\mathbf{r} = \int_a^b \langle v_1, v_2, v_3\rangle \cdot \langle x'(t), y'(t), z'(t)\rangle\, dt$

$$= \int_a^b [v_1\, x'(t) + v_2\, y'(t) + v_3\, z'(t)]\, dt$$

$$= \Big[v_1\, x(t) + v_2\, y(t) + v_3\, z(t)\Big]_a^b$$

$$= [v_1\, x(b) + v_2\, y(b) + v_3\, z(b)] - [v_1\, x(a) + v_2\, y(a) + v_3\, z(a)]$$

$$= v_1\, [x(b) - x(a)] + v_2\, [y(b) - y(a)] + v_3\, [z(b) - z(a)]$$

$$= \langle v_1, v_2, v_3 \rangle \cdot \langle x(b) - x(a), y(b) - y(a), z(b) - z(a) \rangle$$

$$= \langle v_1, v_2, v_3 \rangle \cdot [\langle x(b), y(b), z(b) \rangle - \langle x(a), y(a), z(a) \rangle]$$

$$= \mathbf{v} \cdot [\mathbf{r}(b) - \mathbf{r}(a)]$$

24. (a) 생략 (b) 참이다.

연습문제 13.3

01. 40 02. $f(x,\, y) = x^2 - 3xy + 2y^2 - 8y + K$ 03. 보존적이 아니다.
04. $f(x,\, y) = y\,e^x + x\,\sin y + K$ 05. $f(x,\, y) = x\,\ln y + x^2 y^3 + K$
06. (a) $f(x,\, y) = \dfrac{1}{2}x^2 y^2$ (b) 2 07. (a) $f(x,\, y,\, z) = xyz + z^2$ (b) 77
08. (a) $f(x,\, y,\, z) = y\,e^{xz}$ (b) 4 09. $4/e$ 10. 30 11. 아니다.
12. 보존적이다.
13. 클레로의 정리에 의하면 $\dfrac{\partial P}{\partial y} = f_{xy} = f_{yx} = \dfrac{\partial Q}{\partial x}$, $\dfrac{\partial P}{\partial z} = f_{xz} = f_{zx} = \dfrac{\partial R}{\partial x}$,

$\dfrac{\partial Q}{\partial z} = f_{yz} = f_{zy} = \dfrac{\partial R}{\partial y}$ 이다.
14. (a) 열린 집합 (b) 연결 집합 (c) 단순연결 집합
15. (a) 아니다. (b) 연결 집합 (c) 단순연결 집합
16. (a) $P = -\dfrac{y}{x^2 + y^2}$, $\dfrac{\partial P}{\partial y} = \dfrac{y^2 - x^2}{(x^2 + y^2)^2}$, $Q = \dfrac{x}{x^2 + y^2}$, $\dfrac{\partial Q}{\partial x} = \dfrac{y^2 - x^2}{(x^2 + y^2)^2}$

따라서 $\dfrac{\partial P}{\partial y} = \dfrac{\partial Q}{\partial x}$ 이다.

(b) 생략

연습문제 13.4

01. 8π 02. $\dfrac{2}{3}$ 03. 12 04. $\dfrac{1}{3}$ 05. -24π 06. $-\dfrac{16}{3}$

07. 4π 08. $-8e + 48e^{-1}$ 09. $-\dfrac{1}{12}$ 10. 3π

11. (a) $\int_C x\,dy - y\,dx$

$= \int_0^1 [(1-t)x_1 + tx_2](y_2 - y_1)\,dt + [(1-t)y_1 + ty_2](x_2 - x_1)\,dt$

$= \int_0^1 (x_1(y_2 - y_1) - y_1(x_2 - x_1) + t[(y_2 - y_1)(x_2 - x_1) - (x_2 - x_1)(y_2 - y_1)])\,dt$

$= \int_0^1 (x_1 y_2 - x_2 y_1)\,dt = x_1 y_2 - x_2 y_1$

(b) 생략 (c) $\frac{9}{2}$

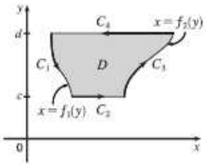
12. 그린의 정리에 의해 $\frac{1}{2A}\oint_C x^2\,dy = \frac{1}{2A}\iint_D 2x\,dA = \frac{1}{A}\iint_D x\,dA = \overline{x}$,

$-\frac{1}{2A}\oint_C y^2\,dx = -\frac{1}{2A}\iint_D (-2y)\,dA = \frac{1}{A}\iint_D y\,dA = \overline{y}$이다.

13. 영역이 제1사분면에서 원판 $x^2 + y^2 = a^2$의 부분이면 $(4a/3\pi,\, 4a/3\pi)$이다.

14. 그린의 정리에 의해 $-\frac{\rho}{3}\oint_C y^3\,dx = -\frac{1}{3}\rho\iint_D (-3y^2)\,dA = \iint_D y^2 \rho\,dA = I_x$,

$\frac{\rho}{3}\oint_C x^3\,dy = \frac{1}{3}\rho\iint_D (3x^2)\,dA = \iint_D x^2\rho\,dA = I_y$이다.

15. 0 16. 생략 17. 

18. 생략

연습문제 13.5

01. (a) 0 (b) 3

02. (a) $ze^x\mathbf{i} + (xye^z - yze^x)\mathbf{j} - xe^z\mathbf{k}$ (b) $y(e^z + e^x)$

03. (a) 0 (b) $2/\sqrt{x^2 + y^2 + z^2}$

04. (a) $\langle -e^y \cos z,\ -e^z \cos x,\ -e^x \cos y \rangle$ (b) $e^x \sin y + e^y \sin z + e^z \sin x$

05. (a) 0 (b) curl F 는 음의 z방향을 가리킨다.

06. $f(x, y, z) = xy^2z^3 + K$ 07. 보존적이 아니다.

08. $f(x, y, z) = xe^{yz} + K$ 09. 존재하지 않는다.

10. $$\operatorname{curl}\mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ f(x) & g(y) & h(z) \end{vmatrix} = (0-0)\,\mathbf{i} + (0-0)\,\mathbf{j} + (0-0)\,\mathbf{k} = \mathbf{0}$$

그러므로 $\mathbf{F}(x, y, z) = f(x)\mathbf{i} + g(y)\mathbf{j} + h(z)\mathbf{k}$는 비회전적이다.

11. $\operatorname{div}(\mathbf{F}+\mathbf{G}) = \operatorname{div}\langle P_1+P_2, Q_1+Q_2, R_1+R_2\rangle = \dfrac{\partial(P_1+P_2)}{\partial x} + \dfrac{\partial(Q_1+Q_2)}{\partial y} + \dfrac{\partial(R_1+R_2)}{\partial z}$

$$= \frac{\partial P_1}{\partial x} + \frac{\partial P_2}{\partial x} + \frac{\partial Q_1}{\partial y} + \frac{\partial Q_2}{\partial y} + \frac{\partial R_1}{\partial z} + \frac{\partial R_2}{\partial z}$$

$$= \left(\frac{\partial P_1}{\partial x} + \frac{\partial Q_1}{\partial y} + \frac{\partial R_1}{\partial z}\right) + \left(\frac{\partial P_2}{\partial x} + \frac{\partial Q_2}{\partial y} + \frac{\partial R_2}{\partial z}\right)$$

$$= \operatorname{div}\langle P_1, Q_1, R_1\rangle + \operatorname{div}\langle P_2, Q_2, R_2\rangle = \operatorname{div}\mathbf{F} + \operatorname{div}\mathbf{G}$$

12. $\operatorname{curl}\mathrm{F} + \operatorname{curl}\mathrm{G} = \left[\left(\dfrac{\partial R_1}{\partial y} - \dfrac{\partial Q_1}{\partial z}\right)\mathbf{i} + \left(\dfrac{\partial P_1}{\partial z} - \dfrac{\partial R_1}{\partial x}\right)\mathbf{j} + \left(\dfrac{\partial Q_1}{\partial x} - \dfrac{\partial P_1}{\partial y}\right)\mathbf{k}\right]$

$$+ \left[\left(\frac{\partial R_2}{\partial y} - \frac{\partial Q_2}{\partial z}\right)\mathbf{i} + \left(\frac{\partial P_2}{\partial z} - \frac{\partial R_2}{\partial x}\right)\mathbf{j} + \left(\frac{\partial Q_2}{\partial x} - \frac{\partial P_2}{\partial y}\right)\mathbf{k}\right]$$

$$= \left[\frac{\partial(R_1+R_2)}{\partial y} - \frac{\partial(Q_1+Q_2)}{\partial z}\right]\mathbf{i} + \left[\frac{\partial(P_1+P_2)}{\partial z} - \frac{\partial(R_1+R_2)}{\partial x}\right]\mathbf{j}$$

$$+ \left[\frac{\partial(Q_1+Q_2)}{\partial x} - \frac{\partial(P_1+P_2)}{\partial y}\right]\mathbf{k}$$

$$= \operatorname{curl}(\mathrm{F}+\mathrm{G})$$

13. $\operatorname{div}(f\mathbf{F}) = \operatorname{div}(f\,\langle P_1, Q_1, R_1\rangle) = \operatorname{div}\langle fP_1, fQ_1, fR_1\rangle = \dfrac{\partial(fP_1)}{\partial x} + \dfrac{\partial(fQ_1)}{\partial y} + \dfrac{\partial(fR_1)}{\partial z}$

$$= \left(f\,\frac{\partial P_1}{\partial x} + P_1\,\frac{\partial f}{\partial x}\right) + \left(f\,\frac{\partial Q_1}{\partial y} + Q_1\,\frac{\partial f}{\partial y}\right) + \left(f\,\frac{\partial R_1}{\partial z} + R_1\,\frac{\partial f}{\partial z}\right)$$

$$= f\left(\frac{\partial P_1}{\partial x} + \frac{\partial Q_1}{\partial y} + \frac{\partial R_1}{\partial z}\right) + \langle P_1, Q_1, R_1\rangle \cdot \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right\rangle = f\operatorname{div}\mathbf{F} + \mathbf{F}\cdot\nabla f$$

14. $\operatorname{curl}(f\mathbf{F}) = \left[\dfrac{\partial(fR_1)}{\partial y} - \dfrac{\partial(fQ_1)}{\partial z}\right]\mathbf{i} + \left[\dfrac{\partial(fP_1)}{\partial z} - \dfrac{\partial(fR_1)}{\partial x}\right]\mathbf{j} + \left[\dfrac{\partial(fQ_1)}{\partial x} - \dfrac{\partial(fP_1)}{\partial y}\right]\mathbf{k}$

$$= \left[f\,\frac{\partial R_1}{\partial y} + R_1\,\frac{\partial f}{\partial y} - f\,\frac{\partial Q_1}{\partial z} - Q_1\,\frac{\partial f}{\partial z}\right]\mathbf{i} + \left[f\,\frac{\partial P_1}{\partial z} + P_1\,\frac{\partial f}{\partial z} - f\,\frac{\partial R_1}{\partial x} - R_1\,\frac{\partial f}{\partial x}\right]\mathbf{j}$$

$$+ \left[f\,\frac{\partial Q_1}{\partial x} + Q_1\,\frac{\partial f}{\partial x} - f\,\frac{\partial P_1}{\partial y} - P_1\,\frac{\partial f}{\partial y}\right]\mathbf{k}$$

$$= f\left[\frac{\partial R_1}{\partial y} - \frac{\partial Q_1}{\partial z}\right]\mathbf{i} + f\left[\frac{\partial P_1}{\partial z} - \frac{\partial R_1}{\partial x}\right]\mathbf{j} + f\left[\frac{\partial Q_1}{\partial x} - \frac{\partial P_1}{\partial y}\right]\mathbf{k}$$

$$+ \left[R_1\frac{\partial f}{\partial y} - Q_1\frac{\partial f}{\partial z}\right]\mathbf{i} + \left[P_1\frac{\partial f}{\partial z} - R_1\frac{\partial f}{\partial x}\right]\mathbf{j} + \left[Q_1\frac{\partial f}{\partial x} - P_1\frac{\partial f}{\partial y}\right]\mathbf{k}$$

$$= \mathrm{f}\operatorname{curl}\mathrm{F} + (\nabla f)\times\mathrm{F}$$

15. $\operatorname{div}(\mathbf{F}\times\mathbf{G}) = \nabla\cdot(\mathbf{F}\times\mathbf{G}) = \begin{vmatrix} \partial/\partial x & \partial/\partial y & \partial/\partial z \\ P_1 & Q_1 & R_1 \\ P_2 & Q_2 & R_2 \end{vmatrix} = \dfrac{\partial}{\partial x}\begin{vmatrix} Q_1 & R_1 \\ Q_2 & R_2 \end{vmatrix} - \dfrac{\partial}{\partial y}\begin{vmatrix} P_1 & R_1 \\ P_2 & R_2 \end{vmatrix} + \dfrac{\partial}{\partial z}\begin{vmatrix} P_1 & Q_1 \\ P_2 & Q_2 \end{vmatrix}$

$$= \left[Q_1 \frac{\partial R_2}{\partial x} + R_2 \frac{\partial Q_1}{\partial x} - Q_2 \frac{\partial R_1}{\partial x} - R_1 \frac{\partial Q_2}{\partial x}\right] - \left[P_1 \frac{\partial R_2}{\partial y} + R_2 \frac{\partial P_1}{\partial y} - P_2 \frac{\partial R_1}{\partial y} - R_1 \frac{\partial P_2}{\partial y}\right]$$

$$+ \left[P_1 \frac{\partial Q_2}{\partial z} + Q_2 \frac{\partial P_1}{\partial z} - P_2 \frac{\partial Q_1}{\partial z} - Q_1 \frac{\partial P_2}{\partial z}\right]$$

$$= \left[P_2 \left(\frac{\partial R_1}{\partial y} - \frac{\partial Q_1}{\partial z}\right) + Q_2 \left(\frac{\partial P_1}{\partial z} - \frac{\partial R_1}{\partial x}\right) + R_2 \left(\frac{\partial Q_1}{\partial x} - \frac{\partial P_1}{\partial y}\right)\right]$$

$$- \left[P_1 \left(\frac{\partial R_2}{\partial y} - \frac{\partial Q_2}{\partial z}\right) + Q_1 \left(\frac{\partial P_2}{\partial z} - \frac{\partial R_2}{\partial x}\right) + R_1 \left(\frac{\partial Q_2}{\partial x} - \frac{\partial P_2}{\partial y}\right)\right]$$

$$= \mathrm{G} \cdot \mathrm{curl}\,\mathrm{F} - \mathrm{F} \cdot \mathrm{curl}\,\mathrm{G}$$

16. $\operatorname{div}(\nabla f \times \nabla g) = \nabla g \cdot \operatorname{curl}(\nabla f) - \nabla f \cdot \operatorname{curl}(\nabla g)$

17. $$\operatorname{curl}(\operatorname{curl} \mathbf{F}) = \nabla \times (\nabla \times \mathbf{F}) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ \partial R_1/\partial y - \partial Q_1/\partial z & \partial P_1/\partial z - \partial R_1/\partial x & \partial Q_1/\partial x - \partial P_1/\partial y \end{vmatrix}$$

$$= \left(\frac{\partial^2 Q_1}{\partial y \partial x} - \frac{\partial^2 P_1}{\partial y^2} - \frac{\partial^2 P_1}{\partial z^2} + \frac{\partial^2 R_1}{\partial z \partial x}\right) \mathbf{i} + \left(\frac{\partial^2 R_1}{\partial z \partial y} - \frac{\partial^2 Q_1}{\partial z^2} - \frac{\partial^2 Q_1}{\partial x^2} + \frac{\partial^2 P_1}{\partial x \partial y}\right) \mathbf{j}$$

$$+ \left(\frac{\partial^2 P_1}{\partial x \partial z} - \frac{\partial^2 R_1}{\partial x^2} - \frac{\partial^2 R_1}{\partial y^2} + \frac{\partial^2 Q_1}{\partial y \partial z}\right) \mathbf{k}$$

$$\operatorname{grad}(\operatorname{div} \mathbf{F}) - \nabla^2 \mathbf{F} = \left[\left(\frac{\partial^2 P_1}{\partial x^2} + \frac{\partial^2 Q_1}{\partial x \partial y} + \frac{\partial^2 R_1}{\partial x \partial z}\right) \mathbf{i} + \left(\frac{\partial^2 P_1}{\partial y \partial x} + \frac{\partial^2 Q_1}{\partial y^2} + \frac{\partial^2 R_1}{\partial y \partial z}\right) \mathbf{j} + \left(\frac{\partial^2 P_1}{\partial z \partial x} + \frac{\partial^2 Q_1}{\partial z \partial y} + \frac{\partial^2 R_1}{\partial z^2}\right) \mathbf{k}\right]$$

$$- \left[\left(\frac{\partial^2 P_1}{\partial x^2} + \frac{\partial^2 P_1}{\partial y^2} + \frac{\partial^2 P_1}{\partial z^2}\right) \mathbf{i} + \left(\frac{\partial^2 Q_1}{\partial x^2} + \frac{\partial^2 Q_1}{\partial y^2} + \frac{\partial^2 Q_1}{\partial z^2}\right) \mathbf{j}\right.$$

$$\left. + \left(\frac{\partial^2 R_1}{\partial x^2} + \frac{\partial^2 R_1}{\partial y^2} + \frac{\partial^2 R_1}{\partial z^2}\right) \mathbf{k}\right]$$

$$= \left(\frac{\partial^2 Q_1}{\partial x \partial y} + \frac{\partial^2 R_1}{\partial x \partial z} - \frac{\partial^2 P_1}{\partial y^2} - \frac{\partial^2 P_1}{\partial z^2}\right) \mathbf{i} + \left(\frac{\partial^2 P_1}{\partial y \partial x} + \frac{\partial^2 R_1}{\partial y \partial z} - \frac{\partial^2 Q_1}{\partial x^2} - \frac{\partial^2 Q_1}{\partial z^2}\right) \mathbf{j}$$

$$+ \left(\frac{\partial^2 P_1}{\partial z \partial x} + \frac{\partial^2 Q_1}{\partial z \partial y} - \frac{\partial^2 R_1}{\partial x^2} - \frac{\partial^2 R_2}{\partial y^2}\right) \mathbf{k}$$

18. (a) $$\nabla r = \nabla \sqrt{x^2 + y^2 + z^2} = \frac{x}{\sqrt{x^2 + y^2 + z^2}} \mathbf{i} + \frac{y}{\sqrt{x^2 + y^2 + z^2}} \mathbf{j} + \frac{z}{\sqrt{x^2 + y^2 + z^2}} \mathbf{k}$$

$$= \frac{x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}}{\sqrt{x^2 + y^2 + z^2}} = \frac{\mathbf{r}}{r}$$

(b) $$\nabla \times \mathbf{r} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ x & y & z \end{vmatrix} = \left[\frac{\partial}{\partial y}(z) - \frac{\partial}{\partial z}(y)\right] \mathbf{i} + \left[\frac{\partial}{\partial z}(x) - \frac{\partial}{\partial x}(z)\right] \mathbf{j} + \left[\frac{\partial}{\partial x}(y) - \frac{\partial}{\partial y}(x)\right] \mathbf{k} = \mathbf{0}$$

(c) $$\nabla\left(\frac{1}{r}\right) = \nabla\left(\frac{1}{\sqrt{x^2 + y^2 + z^2}}\right)$$

$$= \frac{-\dfrac{1}{2\sqrt{x^2+y^2+z^2}}(2x)}{x^2+y^2+z^2}\,\mathbf{i} - \frac{\dfrac{1}{2\sqrt{x^2+y^2+z^2}}(2y)}{x^2+y^2+z^2}\,\mathbf{j} - \frac{\dfrac{1}{2\sqrt{x^2+y^2+z^2}}(2z)}{x^2+y^2+z^2}\,\mathbf{k}$$

$$= -\frac{x\,\mathbf{i}+y\,\mathbf{j}+z\,\mathbf{k}}{(x^2+y^2+z^2)^{3/2}} = -\frac{\mathbf{r}}{r^3}$$

(d) $\nabla \ln r = \nabla \ln(x^2+y^2+z^2)^{1/2} = \frac{1}{2}\nabla \ln(x^2+y^2+z^2)$

$$= \frac{x}{x^2+y^2+z^2}\,\mathbf{i} + \frac{y}{x^2+y^2+z^2}\,\mathbf{j} + \frac{z}{x^2+y^2+z^2}\,\mathbf{k} = \frac{x\,\mathbf{i}+y\,\mathbf{j}+z\,\mathbf{k}}{x^2+y^2+z^2} = \frac{\mathbf{r}}{r^2}$$

19. 생략 20. 생략

21. (a) 생략

(b) $\mathbf{v} = \mathbf{w}\times\mathbf{r} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & \omega \\ x & y & z \end{vmatrix} = (0\cdot z - \omega y)\,\mathbf{i} + (\omega x - 0\cdot z)\,\mathbf{j} + (0\cdot y - x\cdot 0)\,\mathbf{k} = -\omega y\,\mathbf{i} + \omega x\,\mathbf{j}$

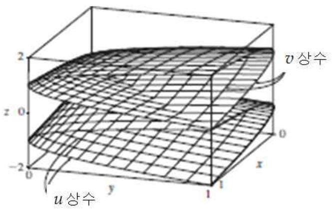
(c) $\operatorname{curl}\mathbf{v} = \nabla\times\mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ -\omega y & \omega x & 0 \end{vmatrix}$

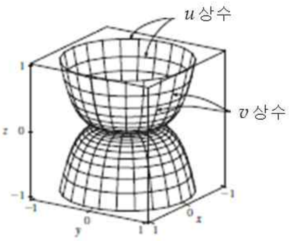
$$= \left[\frac{\partial}{\partial y}(0) - \frac{\partial}{\partial z}(\omega x)\right]\mathbf{i} + \left[\frac{\partial}{\partial z}(-\omega y) - \frac{\partial}{\partial x}(0)\right]\mathbf{j} + \left[\frac{\partial}{\partial x}(\omega x) - \frac{\partial}{\partial y}(-\omega y)\right]\mathbf{k}$$

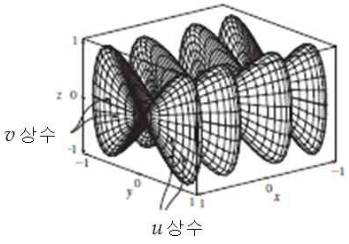
$$= [\omega - (-\omega)]\,\mathbf{k} = 2\omega\,\mathbf{k} = 2\mathbf{w}$$

연습문제 13.6

01. 벡터 $\langle 1, 0, 4\rangle, \langle 1, -1, 5\rangle$를 포함하고 $(0, 3, 1)$을 지나는 평면 02. 쌍곡포물면

03. 

04. 

05. 

06. IV 07. I 08. II 09. V

10. $x = u,\ y = v - u,\ z = -v$

11. $y = y,\ z = z,\ x = \sqrt{1 + y^2 + \frac{1}{4}z^2}$

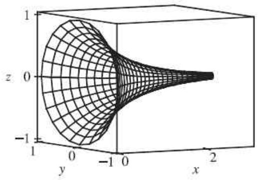
12. $x = 2\sin\phi\cos\theta,\ y = 2\sin\phi\sin\theta,\ z = 2\cos\phi,\ 0 \le \phi \le \frac{\pi}{4},\ 0 \le \theta \le 2\pi$

또는 $x = x,\ y = y,\ z = \sqrt{4 - x^2 - y^2},\ x^2 + y^2 \le 2$

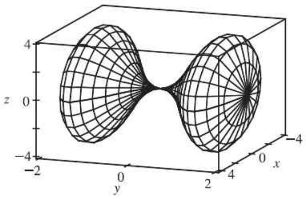
13. $x = x,\ y = 4\cos\theta,\ z = 4\sin\theta,\ 0 \le x \le 5,\ 0 \le \theta \le 2\pi$

14. 생략

15. $x = x,\ y = e^{-x}\cos\theta,\ z = e^{-x}\sin\theta,\ 0 \le x \le 3,\ 0 \le \theta \le 2\pi$;



16. $x = (4y^2 - y^4)\cos\theta$, $y = y$, $z = (4y^2 - y^4)\sin\theta$, $-2 \le y \le 2$, $0 \le \theta \le 2\pi$;



17. (a) 역방향　(b) 나선의 수가 두 배로 된다.

18.

19. $3x - y + 3z = 3$

20. $\frac{\sqrt{3}}{2}x - \frac{1}{2}y + z = \frac{\pi}{3}$

21. $3\sqrt{14}$

22. $\sqrt{14}\,\pi$

23. $\frac{\sqrt{2}}{6}$

24. $\frac{4}{15}(3^{5/2} - 2^{7/2} + 1)$

25. $(2\pi/3)(2\sqrt{2} - 1)$

26. $\frac{1}{2}\sqrt{21} + \frac{17}{4}\left[\ln(2 + \sqrt{21}) - \ln\sqrt{17}\right]$

27. $\pi\left(2\sqrt{6} - \frac{8}{3}\right)$

28. $\pi R^2 \le A(S) \le \sqrt{3}\,\pi R^2$

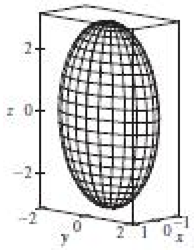
29. 13.9783

30. (a) 24.2055　(b) 24.2476

31. $\frac{45}{8}\sqrt{14} + \frac{15}{16}\ln\left(\frac{11\sqrt{5} + 3\sqrt{70}}{3\sqrt{5} + \sqrt{70}}\right)$

32. (a) $x = a\sin u\cos v,\ y = b\sin u\sin v,\ z = c\cos u$

$$\Rightarrow \begin{aligned} \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} &= (\sin u\cos v)^2 + (\sin u\sin v)^2 + (\cos u)^2 \\ &= \sin^2 u + \cos^2 u = 1 \end{aligned}$$

(b) 

(c) $\int_0^{2\pi}\int_0^{\pi}\sqrt{36\sin^4 u\cos^2 v+9\sin^4 u\sin^2 v+4\cos^2 u\sin^2 u}\,du\,dv$

33. 4π

34. $\mathbf{r}_x\times\mathbf{r}_\theta=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ 1 & f'(x)\cos\theta & f'(x)\sin\theta\\ 0 & -f(x)\sin\theta & f(x)\cos\theta\end{vmatrix}$

$$=\left[f(x)f'(x)\cos^2\theta+f(x)f'(x)\sin^2\theta\right]\mathbf{i}-f(x)\cos\theta\,\mathbf{j}-f(x)\sin\theta\,\mathbf{k}$$

$$=f(x)f'(x)\mathbf{i}-f(x)\cos\theta\,\mathbf{j}-f(x)\sin\theta\,\mathbf{k}$$

$$|\mathbf{r}_x\times\mathbf{r}_\theta|=\sqrt{[f(x)f'(x)]^2+[f(x)]^2\cos^2\theta+[f(x)]^2\sin^2\theta}$$

$$=\sqrt{[f(x)]^2([f'(x)]^2+1)}=f(x)\sqrt{1+[f'(x)]^2}\ [\,f(x)\geq 0]$$

$$A(S)=\iint_D|\mathbf{r}_x\times\mathbf{r}_\theta|\,dA=\int_a^b\int_0^{2\pi}f(x)\sqrt{1+[f'(x)]^2}\,d\theta\,dx$$

$$=\int_a^b f(x)\sqrt{1+[f'(x)]^2}\,[\theta]_0^{2\pi}\,dx=2\pi\int_a^b f(x)\sqrt{1+[f'(x)]^2}\,dx$$

35. $(\pi/6)(37\sqrt{37}-17\sqrt{17})$

연습문제 13.7

01. $8(1+\sqrt{2}+\sqrt{3})\approx 33.17$ 02. 900π 03. $11\sqrt{14}$ 04. $\frac{2}{3}(2\sqrt{2}-1)$

05. $171\sqrt{14}$ 06. $\sqrt{21}/3$ 07. $364\sqrt{2}\pi/3$ 08. $(\pi/60)(391\sqrt{17}+1)$

09. 16π 10. 12 11. 4 12. $\frac{1}{6}\pi^3$ 13. $\frac{713}{180}$

14. $-\frac{4}{3}\pi$ 15. 0 16. 48 17. $2\pi+\frac{8}{3}$ 18. 3.4895

19. $\iint_S \mathbf{F}\cdot d\mathbf{S}=\iint_D[P(\partial h/\partial x)-Q+R(\partial h/\partial z)]\,dA$, 여기서 D는 xz평면 위로의 S의 사영

20. $\iint_D\left(P-Q\frac{\partial k}{\partial y}-R\frac{\partial k}{\partial z}\right)dA$ 21. $(0,\,0,\,a/2)$ 22. $108\sqrt{2}\pi$

23. (a) $I_z = \iint_S (x^2+y^2)\,\rho(x,\,y,\,z)\,dS$ (b) $4329\sqrt{2}\,\pi/5$

24. $0\,\mathrm{kg/s}$ 25. $\dfrac{8}{3}\pi a^3 \varepsilon_0$ 26. 1248π

27. S를 원점을 중심으로 하는 반지름이 a인 구라 하면 $|\mathbf{r}| = a$,

$\mathbf{F}(\mathbf{r}) = c\mathbf{r}/|\mathbf{r}|^3 = (c/a^3)\,(x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k})$이다. S에 대한 매개변수 표현은

$\mathbf{r}(\phi,\theta) = a\sin\phi\,\cos\theta\,\mathbf{i} + a\sin\phi\,\sin\theta\,\mathbf{j} + a\cos\phi\,\mathbf{k},\ 0 \le \phi \le \pi, 0 \le \theta \le 2\pi$ 이다.

따라서 $\mathbf{r}_\phi = a\cos\phi\,\cos\theta\,\mathbf{i} + a\cos\phi\,\sin\theta\,\mathbf{j} - a\sin\phi\,\mathbf{k}$,

$\mathbf{r}_\theta = -a\sin\phi\,\sin\theta\,\mathbf{i} + a\sin\phi\,\cos\theta\,\mathbf{j}$이고

$\mathbf{r}_\phi \times \mathbf{r}_\theta = a^2\sin^2\phi\,\cos\theta\,\mathbf{i} + a^2\sin^2\phi\,\sin\theta\,\mathbf{j} + a^2\sin\phi\,\cos\phi\,\mathbf{k}$에 의해 바깥쪽 방향이 주어진다.

S를 가로지르는 F의 유량은

$$\iint_S \mathbf{F}\cdot d\mathbf{S} = \int_0^\pi \int_0^{2\pi} \frac{c}{a^3}\left(a\sin\phi\,\cos\theta\,\mathbf{i} + a\sin\phi\,\sin\theta\,\mathbf{j} + a\cos\phi\,\mathbf{k}\right)$$

$$\cdot\left(a^2\sin^2\phi\,\cos\theta\,\mathbf{i} + a^2\sin^2\phi\,\sin\theta\,\mathbf{j} + a^2\sin\phi\,\cos\phi\,\mathbf{k}\right)d\theta\,d\phi$$

$$= \frac{c}{a^3}\int_0^\pi \int_0^{2\pi} a^3\left(\sin^3\phi + \sin\phi\,\cos^2\phi\right)d\theta\,d\phi$$

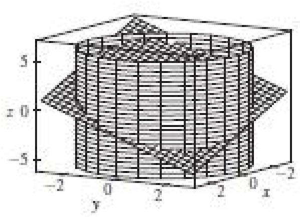
$$= c\int_0^\pi \int_0^{2\pi} \sin\phi\,d\theta\,d\phi = 4\pi c$$ 이므로

유량은 반지름 a에 독립이다.

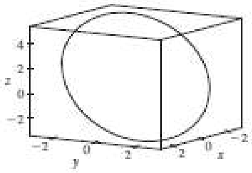
연습문제 13.8

01. 0 02. 0 03. -1 04. 80π

05. (a) $81\pi/2$ (b)



(c) $x = 3\cos t,\ y = 3\sin t,\ z = 1 - 3(\cos t + \sin t),\ 0 \le t \le 2\pi$;



06. 생략

07. $\mathbf{r}_\phi \times \mathbf{r}_\theta = \sin^2\phi\cos\theta\,\mathbf{i} + \sin^2\phi\sin\theta\,\mathbf{j} + \sin\phi\cos\phi\,\mathbf{k}$,

$$\iint_S \operatorname{curl}\mathbf{F}\cdot d\mathbf{S} = \iint_{x^2+z^2\le 1} \operatorname{curl}\mathbf{F}\cdot(\mathbf{r}_\phi\times\mathbf{r}_\theta)\,dA$$

$$= \int_0^\pi\int_0^\pi(-\sin^2\phi\cos\theta - \sin^2\phi\sin\theta - \sin\phi\cos\phi)\,d\theta\,d\phi$$

$$= \int_0^\pi(-2\sin^2\phi - \pi\sin\phi\cos\phi)\,d\phi = \left[\tfrac{1}{2}\sin 2\phi - \phi - \tfrac{\pi}{2}\sin^2\phi\right]_0^\pi = -\pi$$

08. 3　　　　　　　09. 생략

연습문제 13.9

01. $\operatorname{div}\mathbf{F} = 3 + x + 2x = 3 + 3x$,

$$\iiint_E \operatorname{div}\mathbf{F}\,dV = \int_0^1\int_0^1\int_0^1(3x+3)\,dx\,dy\,dz = \tfrac{9}{2}$$

S_1: $\mathbf{n}=\mathbf{i}$, $\mathbf{F} = 3\,\mathbf{i} + y\,\mathbf{j} + 2z\,\mathbf{k}$, and $\iint_{S_1}\mathbf{F}\cdot d\mathbf{S} = \iint_{S_1} 3\,dS = 3$;

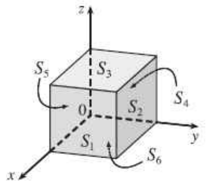
S_2: $\mathbf{F} = 3x\,\mathbf{i} + x\,\mathbf{j} + 2xz\,\mathbf{k}$, $\mathbf{n}=\mathbf{j}$ and $\iint_{S_2}\mathbf{F}\cdot d\mathbf{S} = \iint_{S_2} x\,dS = \tfrac{1}{2}$;

S_3: $\mathbf{F} = 3x\,\mathbf{i} + xy\,\mathbf{j} + 2x\,\mathbf{k}$, $\mathbf{n}=\mathbf{k}$ and $\iint_{S_3}\mathbf{F}\cdot d\mathbf{S} = \iint_{S_3} 2x\,dS = 1$;

S_4: $\mathbf{F} = \mathbf{0}$, $\iint_{S_4}\mathbf{F}\cdot d\mathbf{S} = 0$;

S_5: $\mathbf{F} = 3x\,\mathbf{i} + 2x\,\mathbf{k}$, $\mathbf{n} = -\mathbf{j}$ and $\iint_{S_5}\mathbf{F}\cdot d\mathbf{S} = \iint_{S_5} 0\,dS = 0$;

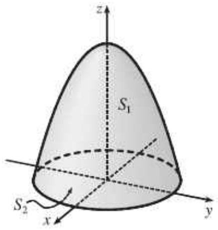
S_6: $\mathbf{F} = 3x\,\mathbf{i} + xy\,\mathbf{j}$, $\mathbf{n} = -\mathbf{k}$ and $\iint_{S_6}\mathbf{F}\cdot d\mathbf{S} = \iint_{S_6} 0\,dS = 0$



따라서 $\iint_S \mathbf{F}\cdot d\mathbf{S} = \frac{9}{2}$이다.

02. $\operatorname{div}\mathbf{F} = 2x + x + 1 = 3x + 1$

$$\begin{aligned}\iiint_E \operatorname{div}\mathbf{F}\,dV &= \iiint_E(3x+1)\,dV = \int_0^{2\pi}\int_0^2\int_0^{4-r^2}(3r\cos\theta+1)\,r\,dz\,dr\,d\theta\\ &= \int_0^2\int_0^{2\pi} r(3r\cos\theta+1)(4-r^2)\,d\theta\,dr\\ &= \int_0^{2\pi} r(4-r^2)\big[3r\sin\theta+\theta\big]_{\theta=0}^{\theta=2\pi}\,dr\\ &= 2\pi\int_0^2(4r-r^3)\,dr = 2\pi\big[2r^2-\tfrac{1}{4}r^4\big]_0^2\\ &= 2\pi(8-4) = 8\pi\end{aligned}$$



03. $\dfrac{9}{2}$　　04. $9\pi/2$　　05. 0　　06. $32\pi/3$　　07. 2π

08. $341\sqrt{2}/60 + \dfrac{81}{20}\arcsin(\sqrt{3}/3)$

09. $13\pi/20$　　10. P_1에서 음, P_2에서 양

11. I, II 사분면에서 $\operatorname{div}\mathbf{F} > 0$; III, IV 사분면에서 $\operatorname{div}\mathbf{F} < 0$　　12. 생략

13. $\iint_S \mathbf{a}\cdot\mathbf{n}\,dS = \iiint_E \operatorname{div}\mathbf{a}\,dV = 0$ since $\operatorname{div}\mathbf{a} = 0$

14. $\iint_S \operatorname{curl} \mathbf{F} \cdot d\mathbf{S} = \iiint_E \operatorname{div}(\operatorname{curl} \mathbf{F})\, dV = 0$

15. $\iint_S (f\nabla g) \cdot \mathbf{n}\, dS = \iiint_E \operatorname{div}(f\nabla g)\, dV = \iiint_E (f\nabla^2 g + \nabla g \cdot \nabla f)\, dV$

13장 복습문제

참-거짓 질문

01. 거짓 02. 참 03. 거짓 04. 거짓 05. 참 06. 참

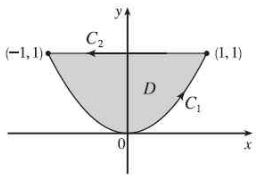
연습문제

01. (a) 음수 (b) 양수 02. $6\sqrt{10}$ 03. $\dfrac{4}{15}$ 04. $\dfrac{110}{3}$

05. $\dfrac{11}{12} - 4/e$ 06. $f(x, y) = e^y + x e^{xy}$ 07. 0

08. C_1: $\mathbf{r}(t) = t\,\mathbf{i} + t^2\,\mathbf{j}, -1 \le t \le 1$, C_2: $\mathbf{r}(t) = -t\,\mathbf{i} + \mathbf{j}, -1 \le t \le 1$

$$\int_C xy^2\, dx - x^2 y\, dy = \int_{-1}^{1} (t^5 - 2t^5)\, dt + \int_{-1}^{1} t\, dt$$
$$= \left[-\tfrac{1}{6}t^6\right]_{-1}^{1} + \left[\tfrac{1}{2}t^2\right]_{-1}^{1} = 0$$



그린 정리를 사용하면

$$\int_C xy^2\, dx - x^2 y\, dy = \iint_D \left[\frac{\partial}{\partial x}(-x^2 y) - \frac{\partial}{\partial y}(xy^2)\right] dA$$
$$= \iint_D (-2xy - 2xy)\, dA = \int_{-1}^{1}\int_{x^2}^{1} -4xy\, dy\, dx$$
$$= \int_{-1}^{1} \left[-2xy^2\right]_{y=x^2}^{y=1} dx = \int_{-1}^{1} (2x^5 - 2x)\, dx$$
$$= \left[\tfrac{1}{3}x^6 - x^2\right]_{-1}^{1} = 0$$

09. -8π 10. 생략 11. 생략

12. 그린 정리를 적용하면

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C f_y\, dx - f_x\, dy = \iint_D \left[\tfrac{\partial}{\partial x}(-f_x) - \tfrac{\partial}{\partial y}(f_y)\right] dA$$
$$= -\iint_D (f_{xx} + f_{yy})\, dA = -\iint_D 0\, dA = 0$$

따라서 선적분은 경로와 독립이다.

13. $\frac{1}{6}(27-5\sqrt{5})$ 14. $(\pi/60)(391\sqrt{17}+1)$ 15. $-64\pi/3$

16. $\oint_C \mathbf{F}\cdot d\mathbf{r} = \int_0^{2\pi}(-\cos^2 t\,\sin t + \sin^2 t\,\cos t)\,dt = \frac{1}{3}\cos^3 t + \frac{1}{3}\sin^3 t\Big]_0^{2\pi} = 0$

17. $-\frac{1}{2}$ 18. 생략 19. 21

20. $\mathbf{F} = \mathbf{a}\times\mathbf{r} = \langle a_1, a_2, a_3\rangle \times \langle x, y, z\rangle = \langle a_2 z - a_3 y, a_3 x - a_1 z, a_1 y - a_2 x\rangle$

$\operatorname{curl}\mathbf{F} = \langle 2a_1, 2a_2, 2a_3\rangle = 2\mathbf{a}$

$\iint_S 2\mathbf{a}\cdot d\mathbf{S} = \iint_S \operatorname{curl}\mathbf{F}\cdot d\mathbf{S} = \int_C \mathbf{F}\cdot d\mathbf{r} = \int_C (\mathbf{a}\times\mathbf{r})\cdot d\mathbf{r}$
