Complete Solutions to Exercises 3.4

1. For parts (a), (b), (c) and (d) we use definition (3-7) which says that $\dim\left(R^n\right)=n$ and the standard basis is given by $\{\mathbf{e}_1,\ \mathbf{e}_2,\ \cdots,\ \mathbf{e}_n\}$.

(a) $\dim\left(R^5\right)=5$ (b) $\dim\left(R^7\right)=7$ (c) $\dim\left(R^{11}\right)=11$

(d) $\dim\left(R^{13}\right)=13$

(e) The dimension of M_{33} is the number of vectors in the basis for M_{33}. *What does the notation M_{33} mean?*

It is the set of matrices of size 3 by 3. *How many matrices are required in the basis?*

9 because the standard basis is $\left\{\begin{pmatrix}1&0&0\\0&0&0\\0&0&0\end{pmatrix},\ \begin{pmatrix}0&1&0\\0&0&0\\0&0&0\end{pmatrix},\ \cdots,\ \begin{pmatrix}0&0&0\\0&0&0\\0&0&1\end{pmatrix}\right\}$ which has 9 elements. Thus $\dim\left(M_{33}\right)=9$.

(f) Similarly we have $\dim\left(M_{44}\right)=16$.

(g) The standard basis for M_{23} is $\left\{\begin{pmatrix}1&0&0\\0&0&0\end{pmatrix},\ \begin{pmatrix}0&1&0\\0&0&0\end{pmatrix},\ \cdots,\ \begin{pmatrix}0&0&0\\0&0&1\end{pmatrix}\right\}$. This set has 6 matrices so $\dim\left(M_{23}\right)=6$.

(h) *What does P_3 represent?*

It is the set of all cubic polynomials. *What is a basis for P_3?*

The standard basis for P_3 is $\{1,\ t,\ t^2,\ t^3\}$ which has 4 elements therefore $\dim\left(P_3\right)=4$.

(i) Similarly the standard basis for P_5 is $\{1,\ t,\ t^2,\ t^3,\ t^4,\ t^5\}$ which is 6 elements therefore $\dim\left(P_5\right)=6$.

(j) *What is the dimension of the zero vector space* **O**?

As discussed in the text the zero space has a dimension of 0 because there are **no** vectors in the basis.

2. Every vector of *S* is of the form $\begin{pmatrix}a\\0\end{pmatrix}$ which we can express in terms of $\mathbf{e}_1=\begin{pmatrix}1\\0\end{pmatrix}$ as

$$\begin{pmatrix}a\\0\end{pmatrix}=a\begin{pmatrix}1\\0\end{pmatrix}=a\mathbf{e}_1$$

This single vector $\mathbf{e}_1$ forms a basis for *S* and we have $\dim(S)=1$.

3. *What do you notice about the given vectors* $\mathbf{u}=\begin{pmatrix}0\\0\\3\end{pmatrix}$, $\mathbf{v}=\begin{pmatrix}-5\\0\\0\end{pmatrix}$ *and* $\mathbf{w}=\begin{pmatrix}0\\0\\9\end{pmatrix}$*?*

The first and last vectors, $\mathbf{u}$ and $\mathbf{w}$, are linearly dependent because $\mathbf{w}=3\mathbf{u}$. Note that the middle vector **v** is linearly independent of **u** and **w**. *How many vectors are in the basis of the subspace S?*

2 vectors **u** and **v** (or **v** and **w**). *What is* $\dim(S)$ *equal to?*

2 because we only have two vectors in the basis of *S*, that is $\dim(S)=2$.

4. We can write vectors in *S* as

$$at^2+b=a\left(t^2\right)+b(1)$$

Hence the 2 vectors $\{t^2,\ 1\}$ span *S*. *What else do we need to show for these vectors to be a basis for S?*
Need to prove they are linearly independent. Let *k* and *c* be scalars

$$kt^2+c(1)=0$$

Substituting $x=0$ into this gives $c=0$ and equating coefficients of t^2 gives $k=0$.
Since both our scalars are zero, $k=0$ and $c=0$, therefore the vectors t^2 and 1 are **linearly independent**.
Hence the 2 vectors $\{t^2,\ 1\}$ forms a basis for *S* therefore $\dim(S)=2$.

5. To evaluate the dimension of the given subspace of symmetric matrices $\begin{pmatrix} a & b \\ b & c \end{pmatrix}$ we need to find a basis for *S*. *What matrices could be contenders for a basis?*
We could try $\mathbf{X}=\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, $\mathbf{Y}=\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and $\mathbf{Z}=\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$. *Do these matrices span S?*
We have

$$a\mathbf{X}+b\mathbf{Y}+c\mathbf{Z}=a\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}+b\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}+c\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$
$$=\begin{pmatrix} a & b \\ b & c \end{pmatrix}$$

Thus the matrices **X**, **Y** and **Z** span the given subspace *S*. *What else do we need to check?*
X, **Y** and **Z** are linearly independent. *How?*
Check that:

$$k_1\mathbf{X}+k_2\mathbf{Y}+k_3\mathbf{Z}=\mathbf{O} \quad \text{gives} \quad k_1=k_2=k_3=0$$

We have

$$k_1\mathbf{X}+k_2\mathbf{Y}+k_3\mathbf{Z}=k_1\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}+k_2\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}+k_3\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}=\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

By equating the corresponding entries of the matrices we have that all the scalars are zero, $k_1=k_2=k_3=0$, which means that **X**, **Y** and **Z** are linearly independent. Hence **X**, **Y** and **Z** form a basis for the subspace *S* and because there are 3 matrices therefore $\dim(S)=3$.

6. (a) *What does the notation* M_{mn} *mean?*
Matrices of size *m* by *n*. *What is the dimension of this space?*
The standard basis for this space is

$$M_{ij}=\begin{cases} 1 & \text{in the position } ij \text{ of the matrix} \\ 0 & \text{in the rest of the matrix} \end{cases}$$

This means that M_{11} will have 1 in the first row and column and zeros in the remaining entries, M_{12} will have 1 in the first row and second column and zeros in the remaining positions etc. *How many matrices are there in this basis?*

mn. Thus $\dim\left(M_{mn}\right)=mn$. This is why the table in the main text gives the dimension of M_{mn} as mn.

(b) *What is the dimension of the subspace $S=\left\{at^3+bt^2+c\right\}$ of P_3?*

The standard basis for $S=\left\{at^3+bt^2+c\right\}$ is $\left\{1,\ t^2,\ t^3\right\}$. [We do not need a t in the basis because there is no linear term *t* in the set *S*.]

What is the dimension of S equal to?

3 because we have 3 elements in the basis of *S*, that is $\dim\left(S\right)=3$.

(c) *What do you notice about the given subspace $S=\left\{at^3+bt^2+ct+d\right\}$?*

It is the whole vector space of cubic polynomials P_3 and the dimension of this is 4 because the standard basis for this space is $\left\{1,\ t,\ t^2,\ t^3\right\}$. We have $\dim\left(P_3\right)=\dim\left(S\right)=4$.

7. (a) Need to show that vectors $\mathbf{u}=\begin{pmatrix}1\\5\end{pmatrix}$ and $\mathbf{v}=\begin{pmatrix}2\\1\end{pmatrix}$ are a basis for ^{2}. ^{2} is of dimension 2 therefore we only need to show that the given vectors are linearly independent. *How?*

Vector **u** is not a scalar multiple of vector **v**; that is

$$\begin{pmatrix}1\\5\end{pmatrix}\neq m\begin{pmatrix}2\\1\end{pmatrix}\text{ where } m \text{ is a scalar}$$

Hence the given vectors **u** and **v** are linearly independent. The given vectors **u** and **v** form a basis for ^{2}.

(b) Need to prove that the vectors $\mathbf{u}=\begin{pmatrix}0\\3\\4\end{pmatrix}$, $\mathbf{v}=\begin{pmatrix}1\\1\\1\end{pmatrix}$ and $\mathbf{w}=\begin{pmatrix}1\\0\\1\end{pmatrix}$ form a basis for ^{3}.

The dimension of ^{3} is 3 therefore we only need to show that the given vectors are linearly independent (or span ^{3}). Let $k_1,\ k_2$ and k_3 be scalars such that

$$k_1\mathbf{u}+k_2\mathbf{v}+k_3\mathbf{w}=k_1\begin{pmatrix}0\\3\\4\end{pmatrix}+k_2\begin{pmatrix}1\\1\\1\end{pmatrix}+k_3\begin{pmatrix}1\\0\\1\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix}$$

Expanding and solving the three simultaneous equations

$$\left.\begin{array}{rcrcrcl} & & k_2 & + & k_3 & = & 0\\ 3k_1 & + & k_2 & & & = & 0\\ 4k_1 & + & k_2 & + & k_3 & = & 0\end{array}\right\}\text{ gives }\ k_1=0,\quad k_2=0\ \text{ and }\ k_3=0$$

Hence the given vectors **u**, **v** and **w** are linearly independent so they form a basis for ^{3}.

(c) We need to show that the set of vectors $\left\{t^n,\ t^{n-1},\ \cdots,\ t,\ 1\right\}$ are a basis for P_n. *What is the dimension of the vector space of polynomials of degree n?*

$n+1$, so any basis for P_n must have $n+1$ vectors. *How many vectors are in the set* $\{t^n,\ t^{n-1},\ \cdots,\ t,\ 1\}$ *?*

$n+1$. We only need to show that the given set of vectors are linearly independent or span P_n. Each vector in $\{t^n,\ t^{n-1},\ \cdots,\ t,\ 1\}$ is **not** a linear combination of the preceding vector therefore the set is linearly independent and we conclude that the given set forms a basis for P_n.

(d) Need to show that $\left\{\begin{pmatrix}2&0\\0&2\end{pmatrix},\ \begin{pmatrix}0&3\\0&2\end{pmatrix},\ \begin{pmatrix}0&0\\1&1\end{pmatrix},\ \begin{pmatrix}0&0\\0&5\end{pmatrix}\right\}$ is a basis for M_{22}. We know the dimension of M_{22} is 4 so that means we need 4 matrices in the basis for M_{22}. We are given 4 matrices therefore they might form a basis for M_{22}. *How do we show this?*

Show that the given matrices are linearly independent. Let $k_1,\ k_2,\ k_3$ and k_4 be scalars. We have

$$k_1\begin{pmatrix}2&0\\0&2\end{pmatrix}+k_2\begin{pmatrix}0&3\\0&2\end{pmatrix}+k_3\begin{pmatrix}0&0\\1&1\end{pmatrix}+k_4\begin{pmatrix}0&0\\0&5\end{pmatrix}=\begin{pmatrix}0&0\\0&0\end{pmatrix}$$

$$\begin{pmatrix}2k_1&0\\0&2k_1\end{pmatrix}+\begin{pmatrix}0&3k_2\\0&2k_2\end{pmatrix}+\begin{pmatrix}0&0\\k_3&k_3\end{pmatrix}+\begin{pmatrix}0&0\\0&5k_4\end{pmatrix}=\begin{pmatrix}0&0\\0&0\end{pmatrix}$$

Adding the corresponding entries of the matrices and equating them to zero yields:

$$2k_1=0 \quad \text{gives} \quad k_1=0$$
$$3k_2=0 \quad \text{gives} \quad k_2=0$$
$$k_3=0$$
$$2k_1+2k_2+k_3+5k_4=0 \quad \text{gives} \quad k_4=0 \quad \text{because } k_1=k_2=k_3=0$$

Thus **all** the scalars are zero, $k_1=0,\ k_2=0,\ k_3=0$ and $k_4=0$, therefore the given matrices are linearly independent which means that they form a basis for M_{22}.

8. (a) The vectors in the set $\{1+t,\ 1+t^2,\ 1+2t+t^2,\ 1+2t\}$ do **not** form a basis for P_2. *Why not?*

Because we have 4 vectors in the set whilst the dimension of P_2 is 3 therefore they **cannot** form a basis.

(b) *Why doesn't* $\{1+t,\ 1+2t+t^2\}$ *form a basis for* P_2*?*

Because the dimension of P_2 is 3 but we only have 2 vectors in the given set therefore the set **cannot** form a basis for P_2.

(c) The given set $\{(1+t)^2,\ 1+t^2,\ 2+4t+2t^2\}$ does **not** form a basis for P_2 because the last element $2+4t+2t^2$ is a multiple of the first element $(1+t)^2$:

$$(1+t)^2=1+2t+t^2$$
$$2(1+t)^2=2(1+2t+t^2)=2+4t+2t^2$$

Thus we have

$$2+4t+2t^2=2(1+t)^2 \quad \text{or} \quad 2+4t+2t^2-2(1+t)^2=0$$

Hence the given set is linearly dependent because polynomial $(1+t)^2$ is a multiple of another polynomial $2+4t+2t^2$ so **cannot** form a basis for P_2.

(d) *Why doesn't* $\left\{\mathbf{A}=\begin{pmatrix}1&0\\0&1\end{pmatrix},\ \mathbf{B}=\begin{pmatrix}2&0\\0&2\end{pmatrix},\ \mathbf{C}=\begin{pmatrix}0&0\\1&1\end{pmatrix},\ \mathbf{D}=\begin{pmatrix}0&0\\0&5\end{pmatrix}\right\}$ *form a basis for* M_{22}*?*

Examining the matrices **A** and **B** we observe that $\mathbf{B}=2\mathbf{A}$ or $\mathbf{B}-2\mathbf{A}=\mathbf{O}$. *What does this mean?*

Means that the given set is linearly dependent because matrix **A** is a multiple of another matrix **B**. Thus the given set **cannot** form a basis for M_{22}.

(e) *What do notice about the matrices in the given set ?*

$$\left\{\mathbf{A}=\begin{pmatrix}1&0\\0&1\end{pmatrix},\ \mathbf{B}=\begin{pmatrix}2&0\\2&2\end{pmatrix},\ \mathbf{C}=\begin{pmatrix}1&0\\3&6\end{pmatrix},\ \mathbf{D}=\begin{pmatrix}3&0\\4&5\end{pmatrix}\right\}$$

Note that we cannot get a non-zero entry in first row second column because

$$\begin{aligned}a\mathbf{A}+b\mathbf{B}+c\mathbf{C}+d\mathbf{D}&=a\begin{pmatrix}1&0\\0&1\end{pmatrix}+b\begin{pmatrix}2&0\\2&2\end{pmatrix}+c\begin{pmatrix}1&0\\3&6\end{pmatrix}+d\begin{pmatrix}3&0\\4&5\end{pmatrix}\\&=\begin{pmatrix}a&0\\0&a\end{pmatrix}+\begin{pmatrix}2b&0\\2b&2b\end{pmatrix}+\begin{pmatrix}c&0\\3c&6c\end{pmatrix}+\begin{pmatrix}3d&0\\4d&5d\end{pmatrix}\\&=\begin{pmatrix}a+2b+c+3d&0\\2b+3c+4d&a+2b+6c+5d\end{pmatrix}\end{aligned}$$

This entry will always be zero.

We **cannot generate** the matrix $\begin{pmatrix}e&f\\g&h\end{pmatrix}$ where $f\neq 0$ by the given matrices so they do **not** span M_{22} which means they fail to be a basis for M_{22}.

9. *How do we show that* $S=\left\{\begin{pmatrix}1\\2\\1\\0\\0\end{pmatrix},\begin{pmatrix}-1\\-5\\2\\1\\0\end{pmatrix},\begin{pmatrix}-3\\5\\2\\0\\0\end{pmatrix},\begin{pmatrix}2\\3\\3\\1\\1\end{pmatrix},\begin{pmatrix}1\\0\\0\\0\\0\end{pmatrix}\right\}$ *is a basis for* ^{5}*?*

Dimension of ^{5} is 5 and we have 5 vectors in *S* therefore we only need to show that they are linearly independent or span ^{5}. In this case we show that they are linearly independent. Let $k_1,\ k_2,\ k_3,\ k_4$ and k_5 be scalars such that

$$k_1\begin{pmatrix}1\\2\\1\\0\\0\end{pmatrix}+k_2\begin{pmatrix}-1\\-5\\2\\1\\0\end{pmatrix}+k_3\begin{pmatrix}-3\\5\\2\\0\\0\end{pmatrix}+k_4\begin{pmatrix}2\\3\\3\\1\\1\end{pmatrix}+k_5\begin{pmatrix}1\\0\\0\\0\\0\end{pmatrix}=\begin{pmatrix}0\\0\\0\\0\\0\end{pmatrix}$$

By expanding the bottom row we have $k_4=0$. Expanding the penultimate row we have

$$k_2+k_4=0 \text{ gives } k_2=0 \text{ because } k_4=0$$

Expanding the second and third rows and substituting $k_2 = 0$ and $k_4 = 0$ we have the simultaneous equations:

$$\left.\begin{aligned} 2k_1 + 5k_3 &= 0 \\ k_1 + 2k_3 &= 0 \end{aligned}\right\} \text{ solving these gives } k_1 = 0 \text{ and } k_3 = 0$$

By expanding the first row and substituting $k_1 = 0,\ k_2 = 0,\ k_3 = 0$ and $k_4 = 0$ yields $k_5 = 0$.
What conclusion can you draw about the given vectors?
They are linearly independent because all the scalars are zero. Hence the given set of 5 vectors is a basis for $\mathbb{R}^5$.

10. We need to prove that if *S* be a subspace of a *n* – dimensional vector space *V* then $\dim(S) \le n$.

Proof.
Let $\{\mathbf{v}_1,\ \mathbf{v}_2,\ \mathbf{v}_3,\ \cdots,\ \mathbf{v}_r\}$ be a basis for *S* so that $\dim(S) = r$ because we have *r* vectors in the basis for *S*. *What do we need to prove?*
$r \le n$. Suppose $r > n$ then by Lemma (3-12) part (a):

Lemma (3-12) (a) Let $\{\mathbf{v}_1,\ \mathbf{v}_2,\ \mathbf{v}_3,\ \cdots,\ \mathbf{v}_n\}$ be a set of linearly independent vectors. Then $\{\mathbf{v}_1,\ \mathbf{v}_2,\ \cdots,\ \mathbf{v}_m\}$ where $m > n$ (*m* is greater than n) is linearly dependent.

We have the set $\{\mathbf{v}_1,\ \mathbf{v}_2,\ \mathbf{v}_3,\ \cdots,\ \mathbf{v}_r\}$ is linearly dependent which means it cannot be a basis for *S*. Thus $r \le n$ which implies that $\dim(S) = r \le n$.

■

11. We need to prove that if $\dim(S) = n$ then $S = V$ where *S* is subspace of a *n* – dimensional vector space *V*.

Proof.
Let $\{\mathbf{v}_1,\ \mathbf{v}_2,\ \mathbf{v}_3,\ \cdots,\ \mathbf{v}_n\}$ be a basis for *S*. These vectors are linearly independent because they are a basis for *S*. By Theorem (3-13) part (a):

Theorem (3-13) (a) Any linearly independent set of *n* vectors $\{\mathbf{v}_1,\ \mathbf{v}_2,\ \cdots,\ \mathbf{v}_n\}$ forms a basis.

We can say that these vectors are also a basis for *V* because they are *n* linearly independent vectors. By Question 17 of the last Exercise 3.3:

If *S* and *V* have the same basis then $S = V$.

We conclude that $S = V$.

■

12. Required to prove that if *n* is the dimension of a vector space then *n* is a positive integer or zero.

Proof.
By definition (3-7) we have that the dimension of a non-zero vector space is the number of vectors in the basis of the vector space. Thus $n \ge 1$ because we cannot have a negative number of vectors in the basis. Also *n* **must** be a whole number as it counts the number of basis vectors which means that *n* is an integer. The zero vector space $\{\mathbf{O}\}$ has zero dimension because it has **no** vectors in its basis. Hence we have our required result, n is a positive integer or zero.

■

13. We need to prove that a single vector $\mathbf{v} \neq \mathbf{O}$ in a 1 – dimensional vector space *V* is a basis for *V*.
Proof.
By result of Question 9 of the last Exercise 3.4:

Any $\mathbf{v} \neq \mathbf{O}$ is a linearly independent vector.

We have that $\mathbf{v} \neq \mathbf{O}$ is linearly independent. Thus $\mathbf{v} \neq \mathbf{O}$ is a basis for *V* because we have one dimensional vector space which means we only need a single non-zero vector for a basis.
■

14. Required to prove that if $S = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \cdots, \mathbf{v}_n\}$ is a set of vectors in *V* such that **none** of vectors is a linear combination of the preceding vectors then it forms a basis for *V*.
Proof.
The set *S* has *n* vectors so we only need to show that this set of vectors is either linearly independent or spans *V*. *Why?*
Because by Theorem (3-13):

Theorem (3-13) (a) Any linearly independent set of *n* vectors $\{\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_n\}$ forms a basis.

If we have *n* vectors in a *n* – dimensional vector space then we only need to show one of the characteristics of a basis.
The given set *S* is linearly independent because it is **not** a linear combination of the preceding vectors by Proposition (3-9):

Proposition (3-9). The vectors $S = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \cdots, \mathbf{v}_n\}$ are linearly dependent $\Leftrightarrow$ one of these vectors is a linear combination of the preceding vectors.

Thus, the set *S* forms a basis for *V*.
■

15. We need to prove that if *P* is the vector space of all polynomials then it has infinite dimension. *How do we prove this result?*
By contradiction - Suppose *P* has a finite dimension and show that this leads to a contradiction.
Proof.
Suppose the vector space of all polynomials *P* has a finite dimension, *n* say. This means that it has *n* basis vectors. Let the linearly independent set $S = \{1, t, t^2, \cdots, t^{n-1}\}$ be a basis for *P*. Since *P* is vector space of all polynomials therefore t^n is a member of *P*. Because *S* is a basis for *P* therefore we must be able to write t^n as a linear combination of vectors of *S*. However this is impossible because t^n **cannot** be generated by $S = \{1, t, t^2, \cdots, t^{n-1}\}$ since t^n is **not** a linear combination of the vectors in *S*. We have a contradiction which means that our supposition *P* has a finite dimension must be wrong. Thus the vector space of all polynomials *P* is infinite dimensional.
■