Complete solutions to Exercise 1.6

1. (a) Swapping rows and columns gives $\mathbf{A}^T = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}^T = \begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix}$.

(b) Again interchanging rows and columns gives

$$\mathbf{A}^T = \begin{pmatrix} 1 & 2 & 3 \\ -1 & -2 & -3 \end{pmatrix}^T = \begin{pmatrix} 1 & -1 \\ 2 & -2 \\ 3 & -3 \end{pmatrix}$$

(c) We have a row matrix so transposing this gives a column matrix (vector):

$$\mathbf{A}^T = \begin{pmatrix} -1 & 5 & 9 & 100 \end{pmatrix}^T = \begin{pmatrix} -1 \\ 5 \\ 9 \\ 100 \end{pmatrix}$$

(d) Transposing $\mathbf{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ gives $\mathbf{A}^T = \begin{pmatrix} a & c \\ b & d \end{pmatrix}$.

(e) In general transposing a $m \times n$ zero matrix gives a $n \times m$ zero matrix, that is the number of rows becomes the number of columns and the number of columns becomes the number of rows in the transposed matrix.

$$\mathbf{A}^T = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}^T = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

(f) Swapping over rows and columns gives

$$\mathbf{A}^T = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}^T = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

2. (a) Substituting for the matrices **A** and **B** we have

$$(\mathbf{AB})^T = \left[\begin{pmatrix} -1 & 4 & 8 \\ -9 & 1 & 2 \end{pmatrix} \begin{pmatrix} 5 & 8 \\ 0 & -6 \\ 5 & 6 \end{pmatrix} \right]^T$$
$$= \begin{pmatrix} 35 & 16 \\ -35 & -66 \end{pmatrix}^T = \begin{pmatrix} 35 & -35 \\ 16 & -66 \end{pmatrix}$$

(b) Similarly we have

$$(\mathbf{BC})^T = \left[\begin{pmatrix} 5 & 8 \\ 0 & -6 \\ 5 & 6 \end{pmatrix} \begin{pmatrix} -4 & 1 \\ 6 & 5 \end{pmatrix} \right]^T$$
$$= \begin{pmatrix} 28 & 45 \\ -36 & -30 \\ 16 & 35 \end{pmatrix}^T = \begin{pmatrix} 28 & -36 & 16 \\ 45 & -30 & 35 \end{pmatrix}$$

(c) We have

$$\mathbf{C}-\mathbf{C}^T=\begin{pmatrix}-4 & 1\\ 6 & 5\end{pmatrix}-\begin{pmatrix}-4 & 6\\ 1 & 5\end{pmatrix}=\begin{pmatrix}0 & -5\\ 5 & 0\end{pmatrix}$$

(d) Similarly we have

$$\mathbf{D}-\mathbf{D}^T=\begin{pmatrix}-6 & 3 & 1\\ 8 & 9 & -2\\ 6 & -1 & 5\end{pmatrix}-\begin{pmatrix}-6 & 8 & 6\\ 3 & 9 & -1\\ 1 & -2 & 5\end{pmatrix}=\begin{pmatrix}0 & -5 & -5\\ 5 & 0 & -1\\ 5 & 1 & 0\end{pmatrix}$$

(e) By Theorem (1-4) property (a) we have $\left(\mathbf{D}^T\right)^T=\mathbf{D}=\begin{pmatrix}-6 & 3 & 1\\ 8 & 9 & -2\\ 6 & -1 & 5\end{pmatrix}$.

(f) To find the matrix $\left(2\mathbf{C}\right)^T$ we multiply the matrix **C** by 2 and then take the transpose or we could change the order by taking the transpose first and then multiply by 2:

$$\left(2\mathbf{C}\right)^T=\left[2\begin{pmatrix}-4 & 1\\ 6 & 5\end{pmatrix}\right]^T=\begin{pmatrix}-8 & 2\\ 12 & 10\end{pmatrix}^T=\begin{pmatrix}-8 & 12\\ 2 & 10\end{pmatrix}$$

(g) To evaluate $\mathbf{A}^T+\mathbf{B}$ we take the transpose of **A** and add **B** to the result:

$$\mathbf{A}^T+\mathbf{B}=\begin{pmatrix}-1 & 4 & 8\\ -9 & 1 & 2\end{pmatrix}^T+\begin{pmatrix}5 & 8\\ 0 & -6\\ 5 & 6\end{pmatrix}$$

$$=\begin{pmatrix}-1 & -9\\ 4 & 1\\ 8 & 2\end{pmatrix}+\begin{pmatrix}5 & 8\\ 0 & -6\\ 5 & 6\end{pmatrix}=\begin{pmatrix}4 & -1\\ 4 & -5\\ 13 & 8\end{pmatrix}$$

Note that $\mathbf{A}+\mathbf{B}$ is **not** defined because of the **different** size matrices but $\mathbf{A}^T+\mathbf{B}$ is valid.

(h) Similarly we have

$$\mathbf{A}+\mathbf{B}^T=\begin{pmatrix}-1 & 4 & 8\\ -9 & 1 & 2\end{pmatrix}+\begin{pmatrix}5 & 8\\ 0 & -6\\ 5 & 6\end{pmatrix}^T$$

$$=\begin{pmatrix}-1 & 4 & 8\\ -9 & 1 & 2\end{pmatrix}+\begin{pmatrix}5 & 0 & 5\\ 8 & -6 & 6\end{pmatrix}=\begin{pmatrix}4 & 4 & 13\\ -1 & -5 & 8\end{pmatrix}$$

Again $\mathbf{A}+\mathbf{B}$ is **not** defined but $\mathbf{A}+\mathbf{B}^T$ is possible.

(i) Applying Theorem (1-4) property (c), $\left(\mathbf{X}+\mathbf{Y}\right)^T=\mathbf{X}^T+\mathbf{Y}^T$, we have

$$\left(\mathbf{A}^T+\mathbf{B}\right)^T=\underbrace{\left(\mathbf{A}^T\right)^T}_{=\mathbf{A}}+\mathbf{B}^T$$

$$=\mathbf{A}+\mathbf{B}^T$$

By part (h) above we have $\mathbf{A}+\mathbf{B}^T=\begin{pmatrix}4 & 4 & 13\\ -1 & -5 & 8\end{pmatrix}$. Hence $\left(\mathbf{A}^T+\mathbf{B}\right)^T=\begin{pmatrix}4 & 4 & 13\\ -1 & -5 & 8\end{pmatrix}$.

(j) Similarly we have

$$\left(2\mathbf{A}^T-5\mathbf{B}\right)^T=\left(2\mathbf{A}^T\right)^T-\left(5\mathbf{B}\right)^T \qquad \left[\text{By } \left(\mathbf{X}+\mathbf{Y}\right)^T=\mathbf{X}^T+\mathbf{Y}^T\right]$$

$$=2\left(\mathbf{A}^T\right)^T-5\mathbf{B}^T=2\mathbf{A}-5\mathbf{B}^T \qquad \left[\text{By Theorem (1-4) } \left(\mathbf{A}^T\right)^T=\mathbf{A}\right]$$

$$=2\begin{pmatrix}-1&4&8\\-9&1&2\end{pmatrix}-5\begin{pmatrix}5&8\\0&-6\\5&6\end{pmatrix}^T$$

$$=\begin{pmatrix}-2&8&16\\-18&2&4\end{pmatrix}-5\begin{pmatrix}5&0&5\\8&-6&6\end{pmatrix}$$

$$=\begin{pmatrix}-2&8&16\\-18&2&4\end{pmatrix}-\begin{pmatrix}25&0&25\\40&-30&30\end{pmatrix}=\begin{pmatrix}-27&8&-9\\-58&32&-26\end{pmatrix}$$

(k) $$\left(-\mathbf{D}\right)^T=\left(-\begin{pmatrix}-6&3&1\\8&9&-2\\6&-1&5\end{pmatrix}\right)^T \underbrace{=}_{\substack{\text{Taking the}\\\text{negative sign into}\\\text{matrix}}} \begin{pmatrix}6&-3&-1\\-8&-9&2\\-6&1&-5\end{pmatrix}^T \underbrace{=}_{\text{Transposing}} \begin{pmatrix}6&-8&-6\\-3&-9&1\\-1&2&-5\end{pmatrix}.$$

(l) We have $-\left(\mathbf{D}^T\right)=\left(-\mathbf{D}^T\right)=\left(-\mathbf{D}\right)^T \underbrace{=}_{\text{By Part (k)}} \begin{pmatrix}6&-8&-6\\-3&-9&1\\-1&2&-5\end{pmatrix}.$

(m) We have

$$\left(\mathbf{C}^2\right)^T=\left[\begin{pmatrix}-4&1\\6&5\end{pmatrix}^2\right]^T$$

$$=\left[\begin{pmatrix}-4&1\\6&5\end{pmatrix}\begin{pmatrix}-4&1\\6&5\end{pmatrix}\right]^T=\begin{pmatrix}22&1\\6&31\end{pmatrix}^T=\begin{pmatrix}22&6\\1&31\end{pmatrix}$$

(n) We have

$$\left(\mathbf{C}^T\right)^2=\left[\begin{pmatrix}-4&1\\6&5\end{pmatrix}^T\right]^2$$

$$=\left[\begin{pmatrix}-4&6\\1&5\end{pmatrix}\right]^2=\begin{pmatrix}-4&6\\1&5\end{pmatrix}\times\begin{pmatrix}-4&6\\1&5\end{pmatrix}=\begin{pmatrix}22&6\\1&31\end{pmatrix}$$

In question 7 you are asked to prove that for any square matrix **A** we have $\left(\mathbf{A}^T\right)^2=\left(\mathbf{A}^2\right)^T$.

3. We are given that $\mathbf{u}=\begin{pmatrix}1\\2\\3\end{pmatrix}$ and $\mathbf{v}=\begin{pmatrix}4\\5\\6\end{pmatrix}$.

(a) $$\mathbf{u}^T\mathbf{v}=\begin{pmatrix}1\\2\\3\end{pmatrix}^T\begin{pmatrix}4\\5\\6\end{pmatrix}=\begin{pmatrix}1&2&3\end{pmatrix}\begin{pmatrix}4\\5\\6\end{pmatrix}=\left(1\times4\right)+\left(2\times5\right)+\left(3\times6\right)=32$$

(b) $\mathbf{v}^T\mathbf{u} = \begin{pmatrix}4\\5\\6\end{pmatrix}^T \begin{pmatrix}1\\2\\3\end{pmatrix} = \begin{pmatrix}4 & 5 & 6\end{pmatrix}\begin{pmatrix}1\\2\\3\end{pmatrix} = (4\times 1)+(5\times 2)+(6\times 3) = 32$

(c) The notation $\mathbf{u}\cdot\mathbf{v}$ is the dot product of the vectors **u** and **v** which was defined in section C. We have

$$\mathbf{u}\cdot\mathbf{v} = \begin{pmatrix}1\\2\\3\end{pmatrix}\cdot\begin{pmatrix}4\\5\\6\end{pmatrix} = (1\times 4)+(2\times 5)+(3\times 6) = 32$$

Note that $\mathbf{u}^T\mathbf{v} = \mathbf{v}^T\mathbf{u} = \mathbf{u}\cdot\mathbf{v}$.

4. *Proof*. By definition (1-1) we have $\mathbf{I} = (i_{kj})$ then

$$\mathbf{I}^T = (i_{kj})^T = (i_{jk}) \qquad [\text{Interchanging Rows and Columns}]$$
$$= \left(\begin{cases}1 & \text{if } k=j\\ 0 & \text{if } k\neq j\end{cases}\right) = \mathbf{I}$$

■

5. *Proof*. Remember that $k^{-1} = \dfrac{1}{k}$ where k is a scalar. Using this we have

$$(k\mathbf{A})^{-1} = \mathbf{A}^{-1}k^{-1} = \mathbf{A}^{-1}\frac{1}{k} = \frac{1}{k}\mathbf{A}^{-1}$$

■

6. *Proof*. Since k is a scalar (a 1×1 matrix) we have $k^T = k$ therefore

$$(k\mathbf{A})^T = \mathbf{A}^T k^T = \mathbf{A}^T k = k\mathbf{A}^T$$

■

7. *Proof*. Since **A** is a square matrix we can find $\mathbf{A}^2$:

$$(\mathbf{A}^2)^T = (\mathbf{A}\mathbf{A})^T$$
$$= \mathbf{A}^T\mathbf{A}^T \qquad \left[\text{Because by (1-4) (d) we have } (\mathbf{X}\mathbf{Y})^T = \mathbf{Y}^T\mathbf{X}^T\right]$$
$$= (\mathbf{A}^T)^2$$

■

8. *Proof*. We are given that **A** has a left inverse **B** which means that

$$\mathbf{BA} = \mathbf{I} \qquad (*)$$

We are also given that **A** has a right inverse **C**:

$$\mathbf{AC} = \mathbf{I} \qquad (**)$$

Required to prove $\mathbf{B} = \mathbf{C}$. Left multiplying this (**) by matrix **B** yields

$$\mathbf{B}(\mathbf{AC}) = \mathbf{BI} = \mathbf{B}$$

Remember matrix multiplication is associative which means we can move the brackets:

$$\mathbf{B}(\mathbf{AC}) = \underbrace{(\mathbf{BA})}_{=\mathbf{I}\ \text{by}\ (*)}\mathbf{C} = \mathbf{IC} = \mathbf{C} = \mathbf{B}$$

Hence $\mathbf{B} = \mathbf{C}$ which means the right and left inverse of an invertible matrix are identical.

■

9. Choose matrices **A** and **B** such that $\mathbf{A}+\mathbf{B}=\mathbf{O}$ [zero matrix]. The zero matrix **O** is singular or non-invertible. Select $\mathbf{A}=-\mathbf{B}$.

10. *Proof*. Matrices **A** and **B** are of size $m\times r$ and $r\times n$ respectively therefore the matrix multiplications $(\mathbf{AB})^T$ and $\mathbf{B}^T\mathbf{A}^T$ are valid and are of size $n\times m$. We consider an arbitrary ij entry. First examine the ij entry of $\mathbf{B}^T\mathbf{A}^T$:

$$\begin{aligned}\left(\mathbf{B}^T\mathbf{A}^T\right)_{ij} &= \left(b_{i1}\right)^T\left(a_{1j}\right)^T+\left(b_{i2}\right)^T\left(a_{2j}\right)^T+\left(b_{i3}\right)^T\left(a_{3j}\right)^T+\cdots+\left(b_{ir}\right)^T\left(a_{rj}\right)^T && \left[\text{By (1.13)}\right]\\ &= b_{1i}a_{j1}+b_{2i}a_{j2}+b_{3i}a_{j3}+\cdots+b_{ri}a_{jr} && \left[\text{By (1.15)}\right]\\ &= a_{j1}b_{1i}+a_{j2}b_{2i}+a_{j3}b_{3i}+\cdots+a_{jr}b_{ri} && (\dagger)\end{aligned}$$

Since we need to prove $(\mathbf{AB})^T=\mathbf{B}^T\mathbf{A}^T$ we now consider ij entry of $(\mathbf{AB})^T$:

$$\begin{aligned}(\mathbf{AB})^T_{ij} &= (\mathbf{AB})_{ji}\\ &= a_{j1}b_{1i}+a_{j2}b_{2i}+a_{j3}b_{3i}+\cdots+a_{jr}b_{ri} && \left[\text{By (1.13)}\right]\end{aligned}$$

Hence this last line is equal to equation (†) which means that ij entry of $(\mathbf{AB})^T$ and $\mathbf{B}^T\mathbf{A}^T$ are equal. Therefore we conclude that $(\mathbf{AB})^T=\mathbf{B}^T\mathbf{A}^T$ which is what we needed to show.

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11. *Proof*. Applying Proposition (1-4) property (d) $(\mathbf{XY})^T=\mathbf{Y}^T\mathbf{X}^T$ we have

$$\begin{aligned}(\mathbf{ABC})^T &= \left((\mathbf{AB})\mathbf{C}\right)^T && \left[\text{Because } \mathbf{ABC}=(\mathbf{AB})\mathbf{C}\right]\\ &= \left(\mathbf{C}^T(\mathbf{AB})^T\right) && \left[\text{By (1-4) property (d)} \quad (\mathbf{XY})^T=\mathbf{Y}^T\mathbf{X}^T\right]\\ &= \mathbf{C}^T\mathbf{B}^T\mathbf{A}^T && \left[\text{By (1-4) property (d)} \quad (\mathbf{XY})^T=\mathbf{Y}^T\mathbf{X}^T\right]\end{aligned}$$

■

12. The procedure for mathematical induction is:
(i) Check the result for some base 1 or n_0.
(ii) Assume the result is true for $n=k$.
(iii) Prove the result for $n=k+1$.

Proof. We need to show $\left(\mathbf{A}_1\mathbf{A}_2\cdots\mathbf{A}_{n-1}\mathbf{A}_n\right)^T=\mathbf{A}_n^T\mathbf{A}_{n-1}^T\cdots\mathbf{A}_2^T\mathbf{A}_1^T$.

Check the result for $n=2$:

$$\left(\mathbf{A}_1\mathbf{A}_2\right)^T=\mathbf{A}_2^T\mathbf{A}_1^T \qquad \left[\text{Because } (\mathbf{AB})^T=\mathbf{B}^T\mathbf{A}^T\right]$$

Assume the result is true for $n=k$:

$$\left(\mathbf{A}_1\mathbf{A}_2\cdots\mathbf{A}_{k-1}\mathbf{A}_k\right)^T=\mathbf{A}_k^T\mathbf{A}_{k-1}^T\cdots\mathbf{A}_2^T\mathbf{A}_1^T \qquad (*)$$

Required to prove the result for $n=k+1$, that is we need to prove

$$\left(\mathbf{A}_1\mathbf{A}_2\cdots\mathbf{A}_{k-1}\mathbf{A}_k\mathbf{A}_{k+1}\right)^T=\mathbf{A}_{k+1}^T\mathbf{A}_k^T\mathbf{A}_{k-1}^T\cdots\mathbf{A}_2^T\mathbf{A}_1^T$$

Expanding the Left Hand Side, $\left(\mathbf{A}_1\mathbf{A}_2\cdots\mathbf{A}_{k-1}\mathbf{A}_k\mathbf{A}_{k+1}\right)^T$, gives

$$\begin{aligned}\left(\mathbf{A}_1\mathbf{A}_2\cdots\mathbf{A}_{k-1}\mathbf{A}_k\mathbf{A}_{k+1}\right)^T &= \left(\left(\mathbf{A}_1\mathbf{A}_2\cdots\mathbf{A}_{k-1}\mathbf{A}_k\right)\mathbf{A}_{k+1}\right)^T \\ &= \mathbf{A}_{k+1}^T\left(\mathbf{A}_1\mathbf{A}_2\cdots\mathbf{A}_{k-1}\mathbf{A}_k\right)^T \qquad \left[\text{By } \left(\mathbf{AB}\right)^T=\mathbf{B}^T\mathbf{A}^T\right] \\ &= \mathbf{A}_{k+1}^T\mathbf{A}_k^T\mathbf{A}_{k-1}^T\cdots\mathbf{A}_2^T\mathbf{A}_1^T \qquad \left[\text{By } (*)\right]\end{aligned}$$

By mathematical induction we have our result $\left(\mathbf{A}_1\mathbf{A}_2\cdots\mathbf{A}_{n-1}\mathbf{A}_n\right)^T=\mathbf{A}_n^T\mathbf{A}_{n-1}^T\cdots\mathbf{A}_2^T\mathbf{A}_1^T$.

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13. *Proof*. Since **A** is a square matrix we can find $\mathbf{A}^n$:

$$\begin{aligned}\left(\mathbf{A}^n\right)^T &= \Big(\underbrace{\mathbf{A}\times\mathbf{A}\times\mathbf{A}\times\cdots\times\mathbf{A}}_{n\text{ copies}}\Big)^T \\ &= \underbrace{\mathbf{A}^T\times\mathbf{A}^T\times\mathbf{A}^T\times\cdots\times\mathbf{A}^T}_{n\text{ copies}} \qquad \left[\begin{array}{l}\text{Using } \left(\mathbf{A}_1\mathbf{A}_2\cdots\mathbf{A}_{n-1}\mathbf{A}_n\right)^T=\mathbf{A}_n^T\mathbf{A}_{n-1}^T\cdots\mathbf{A}_2^T\mathbf{A}_1^T \\ \text{from question 9.}\end{array}\right] \\ &= \left(\mathbf{A}^T\right)^n\end{aligned}$$

■

14. First we show that matrix $\mathbf{A}=\begin{pmatrix}0&0\\0&0\end{pmatrix}$ is a non-invertible matrix.

Proof. Multiplying matrix **A** by a general 2×2 matrix we have

$$\begin{pmatrix}0&0\\0&0\end{pmatrix}\begin{pmatrix}a&b\\c&d\end{pmatrix}=\begin{pmatrix}0&0\\0&0\end{pmatrix}\neq\begin{pmatrix}1&0\\0&1\end{pmatrix}=\mathbf{I}$$

For any real numbers $a,\ b,\ c$ and d the zero matrix can **never** equal the identity matrix **I**. Hence the zero matrix **A** does not have an inverse which means it is a singular matrix.

Next we prove $\mathbf{B}=\begin{pmatrix}1&1\\1&1\end{pmatrix}$ is a singular matrix.

Proof. Similarly to the first proof we have

$$\begin{pmatrix}1&1\\1&1\end{pmatrix}\begin{pmatrix}a&b\\c&d\end{pmatrix}=\begin{pmatrix}a+c&b+d\\a+c&b+d\end{pmatrix}=\begin{pmatrix}1&0\\0&1\end{pmatrix}=\mathbf{I}$$

By equating these entries in the matrix we have

$$\begin{aligned}a+c&=1\\a+c&=0\end{aligned}\quad\text{and}\quad\begin{aligned}b+d&=0\\b+d&=1\end{aligned}$$

There is no solution which satisfies these equations. Hence we **cannot** find real numbers $a,\ b,\ c$ and d such that the matrix $\mathbf{B}=\begin{pmatrix}1&1\\1&1\end{pmatrix}$ multiplied by the matrix $\begin{pmatrix}a&b\\c&d\end{pmatrix}$ gives the identity matrix **I**. Hence **B** does not have an inverse therefore we conclude that the matrix **B** is non-invertible or singular.

Proof. Similarly we have

$$\begin{pmatrix}1&3\\2&6\end{pmatrix}\begin{pmatrix}a&b\\c&d\end{pmatrix}=\begin{pmatrix}a+3c&b+3d\\2a+6c&2b+6d\end{pmatrix}=\begin{pmatrix}1&0\\0&1\end{pmatrix}=\mathbf{I}$$

Equating the first column gives

$$a+3c=1 \qquad (1)$$

$$2a+6c=0 \qquad (2)$$

The bottom equation is double the first on the Left Hand Side but **not** on the right. Hence this system is inconsistent which means there are no values of a, b, c and d which satisfy these equations. Hence no inverse, which means the given matrix in non-invertible or singular matrix.

15. (a) Multiplying $\mathbf{A}=\begin{pmatrix}9 & 2\\ 13 & 3\end{pmatrix}$ and $\mathbf{B}=\begin{pmatrix}3 & -2\\ -13 & 9\end{pmatrix}$ gives

$$\mathbf{AB}=\begin{pmatrix}9 & 2\\ 13 & 3\end{pmatrix}\begin{pmatrix}3 & -2\\ -13 & 9\end{pmatrix}=\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}=\mathbf{I}$$

Similarly we have

$$\mathbf{BA}=\begin{pmatrix}3 & -2\\ -13 & 9\end{pmatrix}\begin{pmatrix}9 & 2\\ 13 & 3\end{pmatrix}=\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}=\mathbf{I}$$

Hence **A** and **B** are inverses of each other.

(b) Multiplying $\mathbf{A}=\begin{pmatrix}1 & 2 & 3\\ 0 & 1 & 4\\ 0 & 0 & 1\end{pmatrix}$ and $\mathbf{B}=\begin{pmatrix}1 & -2 & 5\\ 0 & 1 & -4\\ 0 & 0 & 1\end{pmatrix}$ gives

$$\mathbf{AB}=\begin{pmatrix}1 & 2 & 3\\ 0 & 1 & 4\\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix}1 & -2 & 5\\ 0 & 1 & -4\\ 0 & 0 & 1\end{pmatrix}=\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}=\mathbf{I}$$

Similarly

$$\mathbf{BA}=\begin{pmatrix}1 & -2 & 5\\ 0 & 1 & -4\\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix}1 & 2 & 3\\ 0 & 1 & 4\\ 0 & 0 & 1\end{pmatrix}=\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}=\mathbf{I}$$

Hence **A** and **B** are inverses of each other.

(c) Similarly for $\mathbf{A}=\begin{pmatrix}1 & 0 & 2\\ 2 & -1 & 3\\ 4 & 1 & 8\end{pmatrix}$ and $\mathbf{B}=\begin{pmatrix}-11 & 2 & 2\\ -4 & 0 & 1\\ 6 & -1 & -1\end{pmatrix}$ we have $\mathbf{AB}=\mathbf{I}$ and $\mathbf{BA}=\mathbf{I}$.

The given linear system can be written as $\mathbf{Ax}=\mathbf{b}$ where

$$\mathbf{A}=\begin{pmatrix}1 & 0 & 2\\ 2 & -1 & 3\\ 4 & 1 & 8\end{pmatrix},\ \mathbf{x}=\begin{pmatrix}x\\ y\\ z\end{pmatrix}\text{ and }\mathbf{b}=\begin{pmatrix}1\\ 2\\ 3\end{pmatrix}$$

Hence $\mathbf{x}=\mathbf{A}^{-1}\mathbf{b}$ and $\mathbf{A}^{-1}=\mathbf{B}$ because by part (c) we have $\mathbf{AB}=\mathbf{I}$. Therefore

$$\mathbf{x}=\mathbf{A}^{-1}\mathbf{b}=\begin{pmatrix}-11 & 2 & 2\\ -4 & 0 & 1\\ 6 & -1 & -1\end{pmatrix}\begin{pmatrix}1\\ 2\\ 3\end{pmatrix}=\begin{pmatrix}-1\\ -1\\ 1\end{pmatrix}$$

The solution is $x=-1$, $y=-1$ and $z=1$.

16. We have

$$\mathbf{AB}=\begin{pmatrix}\sin(\theta) & \cos(\theta)\\ -\cos(\theta) & \sin(\theta)\end{pmatrix}\begin{pmatrix}\sin(\theta) & -\cos(\theta)\\ \cos(\theta) & \sin(\theta)\end{pmatrix}$$
$$=\begin{pmatrix}\sin^2(\theta)+\cos^2(\theta) & -\sin(\theta)\cos(\theta)+\cos(\theta)\sin(\theta)\\ -\cos(\theta)\sin(\theta)+\sin(\theta)\cos(\theta) & \cos^2(\theta)+\sin^2(\theta)\end{pmatrix}$$
$$=\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}=\mathbf{I}$$

Similarly we have

$$\mathbf{BA}=\begin{pmatrix}\sin(\theta) & -\cos(\theta)\\ \cos(\theta) & \sin(\theta)\end{pmatrix}\begin{pmatrix}\sin(\theta) & \cos(\theta)\\ -\cos(\theta) & \sin(\theta)\end{pmatrix}=\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}=\mathbf{I}$$

Hence **A** and **B** are inverses of each other.
Writing the given linear system in matrix form $\mathbf{Ax}=\mathbf{b}$ where

$$\mathbf{A}=\begin{pmatrix}\sin(\theta) & \cos(\theta)\\ -\cos(\theta) & \sin(\theta)\end{pmatrix},\ \mathbf{x}=\begin{pmatrix}x\\ y\end{pmatrix}\ \text{and}\ \mathbf{b}=\begin{pmatrix}1\\ -1\end{pmatrix}$$

By the above we have $\mathbf{A}^{-1}=\mathbf{B}=\begin{pmatrix}\sin(\theta) & -\cos(\theta)\\ \cos(\theta) & \sin(\theta)\end{pmatrix}$ therefore

$$\mathbf{x}=\mathbf{A}^{-1}\mathbf{b}=\begin{pmatrix}\sin(\theta) & -\cos(\theta)\\ \cos(\theta) & \sin(\theta)\end{pmatrix}\begin{pmatrix}1\\ -1\end{pmatrix}=\begin{pmatrix}\sin(\theta)+\cos(\theta)\\ \cos(\theta)-\sin(\theta)\end{pmatrix}$$

We have $x=\sin(\theta)+\cos(\theta)$, $y=\cos(\theta)-\sin(\theta)$. Check your answer by substituting these solutions into the given linear system. [Remember $\cos^2(\theta)+\sin^2(\theta)=1$.]

17. An invertible matrix **A** has to be a square matrix because we know it has an inverse, say **B**, which satisfies $\mathbf{AB}=\mathbf{BA}=\mathbf{I}$. The rows and columns of matrix **A** must be equal because both multiplications **AB** and **BA** are possible.

18. Need to prove $\left((\mathbf{AB})^{-1}\right)^T=\left(\mathbf{A}^T\right)^{-1}\left(\mathbf{B}^T\right)^{-1}$.

Proof.

$$\left((\mathbf{AB})^{-1}\right)^T=\left(\mathbf{B}^{-1}\mathbf{A}^{-1}\right)^T \qquad \left[\text{Because by (1-8)}\quad (\mathbf{AB})^{-1}=\mathbf{B}^{-1}\mathbf{A}^{-1}\right]$$
$$=\left(\mathbf{A}^{-1}\right)^T\left(\mathbf{B}^{-1}\right)^T \qquad \left[\text{Using (1-4) (d)}\quad (\mathbf{XY})^T=\mathbf{Y}^T\mathbf{X}^T\right]$$
$$=\left(\mathbf{A}^T\right)^{-1}\left(\mathbf{B}^T\right)^{-1} \qquad \left[\text{Because by (1-10)}\ \left(\mathbf{A}^{-1}\right)^T=\left(\mathbf{A}^T\right)^{-1}\right]$$

■

19. Need to prove that $(\mathbf{ABC})^{-1}=\mathbf{C}^{-1}\mathbf{B}^{-1}\mathbf{A}^{-1}$.

Proof. Consider the Left Hand Side:

$$(\mathbf{ABC})^{-1}=((\mathbf{AB})\mathbf{C})^{-1}$$
$$=\mathbf{C}^{-1}(\mathbf{AB})^{-1} \qquad \left[\text{Using (1-8)}\quad (\mathbf{XY})^{-1}=\mathbf{Y}^{-1}\mathbf{X}^{-1}\right]$$
$$=\mathbf{C}^{-1}\mathbf{B}^{-1}\mathbf{A}^{-1} \qquad \left[\text{Using (1-8)}\quad (\mathbf{XY})^{-1}=\mathbf{Y}^{-1}\mathbf{X}^{-1}\right]$$

Hence we have our result.

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20. As the question says we need to use mathematical induction to prove this result:

$$\left(\mathbf{A}_1\times\mathbf{A}_2\times\cdots\times\mathbf{A}_{n-1}\times\mathbf{A}_n\right)^{-1}=\mathbf{A}_n^{-1}\times\mathbf{A}_{n-1}^{-1}\times\cdots\times\mathbf{A}_3^{-1}\times\mathbf{A}_2^{-1}\times\mathbf{A}_1^{-1}$$

Proof.

By Proposition (1-8) we have $\left(\mathbf{AB}^{-}\right)^{1}\ =\mathbf{B}^{-1}\mathbf{A}^{-1}$ therefore the result for $n\ =2$ is true:

$$\left(\mathbf{A}_1\mathbf{A}_2\right)^{-1}=\mathbf{A}_2^{-1}\mathbf{A}_1^{-1}\quad\checkmark$$

Assume it is true for $n=k$ that is:

$$\left(\mathbf{A}_1\times\mathbf{A}_2\times\mathbf{A}_3\cdots\times\mathbf{A}_{k-1}\times\mathbf{A}_k\right)^{-1}=\mathbf{A}_k^{-1}\times\mathbf{A}_{k-1}^{-1}\times\cdots\times\mathbf{A}_3^{-1}\times\mathbf{A}_2^{-1}\times\mathbf{A}_1^{-1}\qquad(*)$$

Required to prove the result for $n=k+1$, that is we need to prove:

$$\left(\mathbf{A}_1\times\mathbf{A}_2\times\cdots\times\mathbf{A}_k\times\mathbf{A}_{k+1}\right)^{-1}=\mathbf{A}_{k+1}^{-1}\times\mathbf{A}_k^{-1}\times\mathbf{A}_{k-1}^{-1}\times\cdots\times\mathbf{A}_2^{-1}\times\mathbf{A}_1^{-1}\qquad(**)$$

How do we prove this?

By using (*). We examine the Left Hand Side of (**):

$$\begin{aligned}\left(\mathbf{A}_1\times\mathbf{A}_2\times\cdots\times\mathbf{A}_k\times\mathbf{A}_{k+1}\right)^{-1}&=\left(\left(\mathbf{A}_1\times\mathbf{A}_2\times\cdots\times\mathbf{A}_k\right)\times\mathbf{A}_{k+1}\right)^{-1}\\&=\mathbf{A}_{k+1}^{-1}\times\left(\mathbf{A}_1\times\mathbf{A}_2\times\cdots\times\mathbf{A}_k\right)^{-1}&&\left[\text{Using }\left(\mathbf{XY}\right)^{-1}=\mathbf{Y}^{-1}\mathbf{X}^{-1}\right]\\&=\mathbf{A}_{k+1}^{-1}\times\mathbf{A}_k^{-1}\times\mathbf{A}_{k-1}^{-1}\times\cdots\times\mathbf{A}_3^{-1}\times\mathbf{A}_2^{-1}\times\mathbf{A}_1^{-1}&&\left[\text{By }(*)\right]\end{aligned}$$

Hence by induction we have our result:

$$\left(\mathbf{A}_1\times\mathbf{A}_2\times\cdots\times\mathbf{A}_{n-1}\times\mathbf{A}_n\right)^{-1}=\mathbf{A}_n^{-1}\times\mathbf{A}_{n-1}^{-1}\times\cdots\times\mathbf{A}_3^{-1}\times\mathbf{A}_2^{-1}\times\mathbf{A}_1^{-1}$$

■

21. *Proof.* We have

$$\begin{aligned}\mathbf{A}^{-n}\underbrace{=}_{\text{By Definition}}\left(\mathbf{A}^{-1}\right)^n&=\underbrace{\mathbf{A}^{-1}\times\mathbf{A}^{-1}\times\mathbf{A}^{-1}\times\cdots\times\mathbf{A}^{-1}}_{n\text{ copies}}\\&\underbrace{=}_{\substack{\text{By result of}\\\text{question 20}}}\left(\underbrace{\mathbf{A}\times\mathbf{A}\times\mathbf{A}\times\cdots\times\mathbf{A}}_{n\text{ copies}}\right)^{-1}=\left(\mathbf{A}^n\right)^{-1}\end{aligned}$$

■

22. (a) *Given* $\mathbf{AC}=\mathbf{BC}$ *how do we show* $\mathbf{A}=\mathbf{B}$ *?*

Proof. Since $\mathbf{C}$ is an invertible matrix we can right or post-multiply $\mathbf{AC}=\mathbf{BC}$ by $\mathbf{C}^{-1}$:

$$\left(\mathbf{AC}\right)\mathbf{C}^{-1}=\left(\mathbf{BC}\right)\mathbf{C}^{-1}$$

$$\mathbf{A}\underbrace{\left(\mathbf{CC}^{-1}\right)}_{=\mathbf{I}}=\mathbf{B}\underbrace{\left(\mathbf{CC}^{-1}\right)}_{=\mathbf{I}}$$

$$\mathbf{A}\left(\mathbf{I}\right)=\mathbf{B}\left(\mathbf{I}\right)\quad\text{which gives }\ \mathbf{A}=\mathbf{B}$$

We have our required result.

■

(b) *Proof.* To show $\mathbf{CA}=\mathbf{CB}\ \Rightarrow\ \mathbf{A}=\mathbf{B}$ is very similar to part (a). Since $\mathbf{C}$ is an invertible matrix we can left or pre-multiply $\mathbf{CA}=\mathbf{CB}$ by $\mathbf{C}^{-1}$:

$$\mathbf{C}^{-1}\left(\mathbf{CA}\right)=\mathbf{C}^{-1}\left(\mathbf{CB}\right)$$

$$\underbrace{\left(\mathbf{CC}^{-1}\right)}_{=\mathbf{I}}\mathbf{A}=\underbrace{\left(\mathbf{CC}^{-1}\right)}_{=\mathbf{I}}\mathbf{B}$$

$$\left(\mathbf{I}\right)\mathbf{A}=\left(\mathbf{I}\right)\mathbf{B}\quad\text{which gives }\ \mathbf{A}=\mathbf{B}$$

Hence we have proven our result.

■

23. *Proof. How do we show* $\mathbf{AB}=\mathbf{O} \;\Rightarrow\; \mathbf{B}=\mathbf{O}$ *?*

Since **A** is invertible therefore inverse **A** exists and we can pre-multiply $\mathbf{AB}=\mathbf{O}$ by $\mathbf{A}^{-1}$:

$$\mathbf{A}^{-1}(\mathbf{AB})=\mathbf{A}^{-1}\mathbf{O}$$

$$\underbrace{(\mathbf{A}^{-1}\mathbf{A})}_{=\mathbf{I}}\mathbf{B}=\mathbf{O} \qquad \left[\text{Because } \mathbf{A}^{-1}\mathbf{O}=\mathbf{O}\right]$$

$$\mathbf{IB}=\mathbf{O} \quad \text{which gives } \mathbf{B}=\mathbf{O} \qquad \left[\text{Because } \mathbf{IB}=\mathbf{B}\right]$$

Hence we have our result.

■

24. (a) Expanding $\left(\mathbf{P}^{-1}\mathbf{AP}\right)^2$ gives

$$\begin{aligned}\left(\mathbf{P}^{-1}\mathbf{AP}\right)^2 &= \left(\mathbf{P}^{-1}\mathbf{AP}\right)\left(\mathbf{P}^{-1}\mathbf{AP}\right)\\ &= \mathbf{P}^{-1}\mathbf{A}\underbrace{\left(\mathbf{PP}^{-1}\right)}_{=\mathbf{I}}\mathbf{AP}\\ &= \mathbf{P}^{-1}\left(\mathbf{AIA}\right)\mathbf{P}\\ &= \mathbf{P}^{-1}\left(\mathbf{AA}\right)\mathbf{P}=\mathbf{P}^{-1}\left(\mathbf{A}^2\right)\mathbf{P}=\mathbf{P}^{-1}\mathbf{A}^2\mathbf{P}\end{aligned}$$

■

(b) Similarly, expanding $\left(\mathbf{P}^{-1}\mathbf{AP}\right)^3$ gives

$$\begin{aligned}\left(\mathbf{P}^{-1}\mathbf{AP}\right)^3 &= \left(\mathbf{P}^{-1}\mathbf{AP}\right)^2\left(\mathbf{P}^{-1}\mathbf{AP}\right)\\ &= \mathbf{P}^{-1}\mathbf{A}^2\mathbf{P}\left(\mathbf{P}^{-1}\mathbf{AP}\right)\\ &= \mathbf{P}^{-1}\mathbf{A}^2\left(\mathbf{PP}^{-1}\right)\mathbf{AP}\\ &= \mathbf{P}^{-1}\mathbf{A}^2\left(\mathbf{I}\right)\mathbf{AP}\\ &= \mathbf{P}^{-1}\left(\mathbf{A}^2\mathbf{A}\right)\mathbf{P}=\mathbf{P}^{-1}\mathbf{A}^3\mathbf{P}\end{aligned}$$

■

(c) We use proof by mathematical induction to show $\left(\mathbf{P}^{-1}\mathbf{AP}\right)^n=\mathbf{P}^{-1}\mathbf{A}^n\mathbf{P}$. The procedure is to check the result for $n=2$ assume it is true for $n=k$ and then prove it for $n=k+1$.

Proof. The result is true for $n=2$ by part (a):

$$\left(\mathbf{P}^{-1}\mathbf{AP}\right)^2=\mathbf{P}^{-1}\mathbf{A}^2\mathbf{P}$$

Assume it is true for $n=k$:

$$\left(\mathbf{P}^{-1}\mathbf{AP}\right)^k=\mathbf{P}^{-1}\mathbf{A}^k\mathbf{P} \qquad (\dagger)$$

Show the result for $n=k+1$, that is we need to prove

$$\left(\mathbf{P}^{-1}\mathbf{AP}\right)^{k+1}=\mathbf{P}^{-1}\mathbf{A}^{k+1}\mathbf{P}$$

Expanding $\left(\mathbf{P}^{-1}\mathbf{AP}\right)^{k+1}$ we have

$$\begin{aligned}\left(\mathbf{P}^{-1}\mathbf{AP}\right)^{k+1} &= \left(\mathbf{P}^{-1}\mathbf{AP}\right)\left(\mathbf{P}^{-1}\mathbf{AP}\right)^{k} \\ &= \left(\mathbf{P}^{-1}\mathbf{AP}\right)\left(\mathbf{P}^{-1}\mathbf{A}^{k}\mathbf{P}\right) \qquad \left[\text{By } (\dagger)\right] \\ &= \mathbf{P}^{-1}\mathbf{A}\underbrace{\left(\mathbf{PP}^{-1}\right)}_{=\mathbf{I}}\mathbf{A}^{k}\mathbf{P} \\ &= \mathbf{P}^{-1}\left(\mathbf{AIA}^{k}\right)\mathbf{P} = \mathbf{P}^{-1}\left(\mathbf{AA}^{k}\right)\mathbf{P} = \mathbf{P}^{-1}\mathbf{A}^{k+1}\mathbf{P}\end{aligned}$$

■