Complete Solutions to Exercises 1.2

1. (a) The augmented matrix is given by

$$\begin{array}{c} \mathrm{R}_1 \\ \mathrm{R}_2 \end{array} \left(\begin{array}{cc|c} 1 & 1 & 7 \\ 1 & -2 & 4 \end{array}\right)$$

We need to convert the 1 in the bottom row to zero. *How?*

We execute the row operation $\mathrm{R}_2 - \mathrm{R}_1$:

$$\begin{array}{c} \mathrm{R}_1 \\ \mathrm{R}_2{}' = \mathrm{R}_2 - \mathrm{R}_1 \end{array} \left(\begin{array}{cc|c} 1 & 1 & 7 \\ 1-1 & -2-1 & 4-7 \end{array}\right)$$

Simplifying the entries in the bottom row we have

$$\begin{array}{c} \\ \mathrm{R}_1 \\ \mathrm{R}_2{}' \end{array} \begin{array}{c} \begin{array}{cc} x & y \end{array} \\ \left(\begin{array}{cc|c} 1 & 1 & 7 \\ 0 & -3 & -3 \end{array}\right) \end{array}$$

Expanding the bottom row we have

$$-3y = -3 \text{ gives } y = 1$$

Substituting $y = 1$ into the expansion of the top row:

$$x + y = 7 \quad \text{implies } x = 7 - y = 7 - 1 = 6 \quad [\text{Substituting } y = 1]$$

Our solution is $x = 6$ and $y = 1$.

(b) The augmented matrix is

$$\begin{array}{c} R_1 \\ R_2 \\ R_3 \end{array} \left(\begin{array}{ccc|c} 1 & 2 & -3 & 3 \\ 2 & -1 & -1 & 11 \\ 3 & 2 & 1 & -5 \end{array}\right)$$

Need zeros in these entries.

We need 0's in the bottom left hand corner of the matrix, that is 0 in place of 3, 2 and 2.

We execute the following row operations to achieve this:

$$\begin{array}{c} \mathrm{R}_1 \\ \mathrm{R}_2{}^* = \mathrm{R}_2 - 2\mathrm{R}_1 \\ \mathrm{R}_3{}^* = \mathrm{R}_3 - 3\mathrm{R}_1 \end{array} \left(\begin{array}{ccc|c} 1 & 2 & -3 & 3 \\ 2-2(1) & -1-2(2) & -1-2(-3) & 11-2(3) \\ 3-3(1) & 2-3(2) & 1-3(-3) & -5-3(3) \end{array}\right)$$

Simplifying these entries gives

$$\begin{array}{c} \mathrm{R}_1 \\ \mathrm{R}_2{}^* \\ \mathrm{R}_3{}^* \end{array} \left(\begin{array}{ccc|c} 1 & 2 & -3 & 3 \\ 0 & -5 & 5 & 5 \\ 0 & -4 & 10 & -14 \end{array}\right)$$

How do we get 0 in place of -4 *in the bottom row?*

$R_3{}^* - \frac{4}{5}R_2{}^*$ because $-4 - \frac{4}{5}(-5) = 0$:

$$\begin{array}{c} R_1 \\ R_2{}^* \\ R_3{}^{**} = R_3{}^* - \frac{4}{5}R_2{}^* \end{array} \left(\begin{array}{ccc|c} 1 & 2 & -3 & 3 \\ 0 & -5 & 5 & 5 \\ 0 - \frac{4}{5}(0) & -4 - \frac{4}{5}(-5) & 10 - \frac{4}{5}(5) & -14 - \frac{4}{5}(5) \end{array}\right)$$

Simplifying the bottom row entries gives:

$$\begin{array}{c|c} & \begin{matrix} x & y & z & \end{matrix} \\ \begin{matrix} R_1 \\ R_2\,* \\ R_3\,** \end{matrix} & \left(\begin{array}{ccc|c} 1 & 2 & -3 & 3 \\ 0 & -5 & 5 & 5 \\ 0 & 0 & 6 & -18 \end{array}\right) \end{array}$$

From the last row $R_3\,**$ we have $6z=-18$ which gives $z=-3$. Substituting this $z=-3$ into the second row $R_2\,*$ gives

$$-5y+5z=5$$
$$-5y+5(-3)=5$$
$$-5y-15=5 \text{ which gives } y=-4$$

How do we find the last unknown x?

Use the first row R_1. By substituting $y=-4$ and $z=-3$ into this row gives

$$x+2y-3z=3$$
$$x+2(-4)-3(-3)=3$$
$$x+1=3 \text{ which gives } x=2$$

Hence our solution is $x=2$, $y=-4$ and $z=-3$.

(c) The augmented matrix is given by

$$\begin{matrix} \mathrm{R}_1 \\ \mathrm{R}_2 \\ \mathrm{R}_3 \end{matrix} \left(\begin{array}{ccc|c} 2 & 2 & 1 & 10 \\ 1 & -3 & 4 & 0 \\ 3 & -1 & 6 & 12 \end{array}\right)$$

Interchange the first two rows:

$$\begin{matrix} \mathrm{R}_1'=\mathrm{R}_2 \\ \mathrm{R}_2'=\mathrm{R}_1 \\ \mathrm{R}_3 \end{matrix} \left(\begin{array}{ccc|c} 1 & -3 & 4 & 0 \\ 2 & 2 & 1 & 10 \\ 3 & -1 & 6 & 12 \end{array}\right)$$

We carry out the following row operations to get 0's in the bottom left hand corner of this augmented matrix:

$$\begin{matrix} \mathrm{R}_1' \\ \mathrm{R}_2''=\mathrm{R}_2'-2\mathrm{R}_1' \\ \mathrm{R}_3'=\mathrm{R}_3-3\mathrm{R}_1' \end{matrix} \left(\begin{array}{ccc|c} 1 & -3 & 4 & 0 \\ 2-2(1) & 2-2(-3) & 1-2(4) & 10-2(0) \\ 3-3(1) & -1-3(-3) & 6-3(4) & 12-3(0) \end{array}\right)$$

Simplifying these entries gives

$$\begin{matrix} \mathrm{R}_1' \\ \mathrm{R}_2'' \\ \mathrm{R}_3' \end{matrix} \left(\begin{array}{ccc|c} 1 & -3 & 4 & 0 \\ 0 & 8 & -7 & 10 \\ 0 & 8 & -6 & 12 \end{array}\right)$$

Carrying out the row operation $\mathrm{R}_3'-\mathrm{R}_2''$:

$$\begin{matrix} \mathrm{R}_1' \\ \mathrm{R}_2'' \\ \mathrm{R}_3''=\mathrm{R}_3'-\mathrm{R}_2'' \end{matrix} \left(\begin{array}{ccc|c} 1 & -3 & 4 & 0 \\ 0 & 8 & -7 & 10 \\ 0-0 & 8-8 & -6-(-7) & 12-10 \end{array}\right)$$

Simplifying the entries in the last row gives

$$\begin{array}{c|c} & \begin{array}{ccc|c} x & y & z & \end{array} \\ \begin{array}{c} R_1' \\ R_2'' \\ R_3'' \end{array} & \left(\begin{array}{ccc|c} 1 & -3 & 4 & 0 \\ 0 & 8 & -7 & 10 \\ 0 & 0 & 1 & 2 \end{array}\right) \end{array}$$

From the last row we have $z=2$. *How can we find the other two unknowns x and y?* Using the middle row and substituting $z=2$ we have

$$8y-7z=10$$

$$8y-7(2)=10 \quad \text{which gives } 8y-14=10$$

$$8y=24 \text{ therefore } y=3$$

To find *x* we use the first row and substitute $y=3$ and $z=2$

$$x-3y+4z=0$$

$$x-3(3)+4(2)=0$$

$$x-9+8=0 \quad \text{which gives } x=1$$

Our solution is $x=1$, $y=3$ and $z=2$.

(d) The augmented matrix we have is

$$\begin{array}{c} R_1 \\ R_2 \\ R_3 \end{array} \left(\begin{array}{ccc|c} 1 & 2 & 1 & 1 \\ 2 & 2 & 3 & 2 \\ 5 & 8 & 2 & 4 \end{array}\right)$$

Executing the row operations R_2-2R_1 and R_3-5R_1 we have

$$\begin{array}{c} R_1 \\ R_2'=R_2-2R_1 \\ R_3'=R_3-5R_1 \end{array} \left(\begin{array}{ccc|c} 1 & 2 & 1 & 1 \\ 2-2(1) & 2-2(2) & 3-2(1) & 2-2(1) \\ 5-5(1) & 8-5(2) & 2-5(1) & 4-5(1) \end{array}\right)$$

Simplifying the entries gives

$$\begin{array}{c} R_1 \\ R_2' \\ R_3' \end{array} \left(\begin{array}{ccc|c} 1 & 2 & 1 & 1 \\ 0 & -2 & 1 & 0 \\ 0 & -2 & -3 & -1 \end{array}\right)$$

Subtracting the last two rows we have

$$\begin{array}{c} R_1 \\ R_2' \\ R_3''=R_3'-R_2' \end{array} \left(\begin{array}{ccc|c} 1 & 1 & 2 & 1 \\ 0 & -2 & 1 & 0 \\ 0-0 & -2-(-2) & -3-1 & -1-0 \end{array}\right)$$

Simplifying gives

$$\begin{array}{c|c} & \begin{array}{ccc|c} x & y & z & \end{array} \\ \begin{array}{c} R_1 \\ R_2' \\ R_3'' \end{array} & \left(\begin{array}{ccc|c} 1 & 2 & 1 & 1 \\ 0 & -2 & 1 & 0 \\ 0 & 0 & -4 & -1 \end{array}\right) \end{array}$$

From the bottom row we have $-4z=-1$ which gives $z=\dfrac{1}{4}$. Substituting this into second row gives

$$-2y+z=0$$

$$-2y+\frac{1}{4}=0 \text{ implies that } y=\frac{1}{8}$$

Using the first row and substituting the values already obtained we have

$$x+2y+z=1$$

$$x+2\left(\frac{1}{8}\right)+\frac{1}{4}=1 \quad \text{gives} \quad x=\frac{1}{2}$$

Our solution is $x=\frac{1}{2}$, $y=\frac{1}{8}$ and $z=\frac{1}{4}$.

(e) The augmented matrix in this case is

$$\begin{array}{c} R_1 \\ R_2 \\ R_3 \end{array} \left(\begin{array}{ccc|c} 10 & 1 & -5 & 18 \\ -20 & 3 & 20 & 14 \\ 5 & 3 & 5 & 9 \end{array}\right)$$

Interchanging the first and last rows we have

$$\begin{array}{c} R_1^{\dagger}=R_3 \\ R_2 \\ R_3^{\dagger}=R_1 \end{array} \left(\begin{array}{ccc|c} 5 & 3 & 5 & 9 \\ -20 & 3 & 20 & 14 \\ 10 & 1 & -5 & 18 \end{array}\right)$$

Carrying out the row operations stated gives

$$\begin{array}{c} R_1^{\dagger} \\ R_2^{\dagger}=R_2+4R_1^{\dagger} \\ R_3^{\dagger\dagger}=R_3^{\dagger}-2R_1^{\dagger} \end{array} \left(\begin{array}{ccc|c} 5 & 3 & 5 & 9 \\ -20+4(5) & 3+4(3) & 20+4(5) & 14+4(9) \\ 10-2(5) & 1-2(3) & -5-2(5) & 18-2(9) \end{array}\right)$$

Simplifying our entries gives

$$\begin{array}{c} R_1^{\dagger} \\ R_2^{\dagger} \\ R_3^{\dagger\dagger} \end{array} \left(\begin{array}{ccc|c} 5 & 3 & 5 & 9 \\ 0 & 15 & 40 & 50 \\ 0 & -5 & -15 & 0 \end{array}\right)$$

We need a 0 in place of the -5 in the bottom row. Executing the combined row operation $3R_3^{\dagger\dagger}+R_2^{\dagger}$ gives:

$$\begin{array}{c} R_1^{\dagger} \\ R_2^{\dagger} \\ R_3^{\dagger\dagger\dagger}=3R_3^{\dagger\dagger}+R_2^{\dagger} \end{array} \left(\begin{array}{ccc|c} 5 & 3 & 5 & 9 \\ 0 & 15 & 40 & 50 \\ 0 & 3(-5)+15 & 3(-15)+40 & 3(0)+50 \end{array}\right)$$

Simplifying these gives

$$\begin{array}{cc} & \begin{array}{ccc} x & y & z \end{array} \\ \begin{array}{c} R_1^{\dagger} \\ R_2^{\dagger} \\ R_3^{\dagger\dagger\dagger}=3R_3^{\dagger\dagger}+R_2^{\dagger} \end{array} & \left(\begin{array}{ccc|c} 5 & 3 & 5 & 9 \\ 0 & 15 & 40 & 50 \\ 0 & 0 & -5 & 50 \end{array}\right) \end{array}$$

From the last row we have $-5z=50$ which gives $z=-10$. Substituting this into the second row we have

$$15y + 40z = 50$$
$$15y + 40(-10) = 50$$
$$15y - 400 = 50 \text{ gives } y = \frac{450}{15} = 30$$

Substituting $y = 30$ and $z = -10$ into the first row we have

$$5x + 3y + 5z = 9$$
$$5x + 3(30) + 5(-10) = 9$$
$$5x + 90 - 50 = 9 \quad \text{implies that } 5x = 9 - 40 = -31 \text{ so } x = -\frac{31}{5} = -6.2$$

Our solution is $x = -6.2$, $y = 30$ and $z = -10$.

2. (a) The augmented matrix is given by

$$\begin{array}{c} R_1 \\ R_2 \\ R_3 \end{array} \left(\begin{array}{ccc|c} 1 & 2 & 3 & 12 \\ 2 & -1 & 5 & 3 \\ 3 & 3 & 6 & 21 \end{array}\right)$$

Need to get 0's in the bottom left hand corner:

$$\begin{array}{c} R_1 \\ R_2' = R_2 - 2R_1 \\ R_3' = R_3 - 3R_1 \end{array} \left(\begin{array}{ccc|c} 1 & 2 & 3 & 12 \\ 2-2(1) & -1-2(2) & 5-2(3) & 3-2(12) \\ 3-3(1) & 3-3(2) & 6-3(3) & 21-3(12) \end{array}\right)$$

Simplifying the entries in the bottom two rows gives:

$$\begin{array}{c} R_1 \\ R_2' \\ R_3' \end{array} \left(\begin{array}{ccc|c} 1 & 2 & 3 & 12 \\ 0 & -5 & -1 & -21 \\ 0 & -3 & -3 & -15 \end{array}\right)$$

Dividing the second row by -5 and third row by -3 we have

$$\begin{array}{c} R_1 \\ R_2'' = R_2'/(-5) \\ R_3'' = R_3'/(-3) \end{array} \left(\begin{array}{ccc|c} 1 & 2 & 3 & 12 \\ 0 & 1 & 0.2 & 4.2 \\ 0 & 1 & 1 & 5 \end{array}\right)$$

Carrying out the row operation $R_3'' - R_2''$ gives

$$\begin{array}{c} R_1 \\ R_2'' \\ R_3''' = R_3'' - R_2'' \end{array} \left(\begin{array}{ccc|c} 1 & 2 & 3 & 12 \\ 0 & 1 & 0.2 & 4.2 \\ 0-0 & 1-1 & 1-0.2 & 5-4.2 \end{array}\right)$$

Simplifying the entries in the last row

$$\begin{array}{c} R_1 \\ R_2'' \\ R_3''' \end{array} \left(\begin{array}{ccc|c} 1 & 2 & 3 & 12 \\ 0 & 1 & 0.2 & 4.2 \\ 0 & 0 & 0.8 & 0.8 \end{array}\right)$$

Multiply the last row by 1.25 to give 1 in place of 0.8:

$$\begin{array}{c} \\ R_1 \\ R_2'' \\ R_3^* = 1.25R_3''' \end{array} \begin{array}{c} \begin{array}{cccc} x & y & z & \end{array} \\ \left(\begin{array}{ccc|c} 1 & 2 & 3 & 12 \\ 0 & 1 & 0.2 & 4.2 \\ 0 & 0 & 1 & 1 \end{array}\right) \end{array}$$

From the last row we have $z=1$ and by the second row we have

$$y+0.2z=4.2$$

$$y+0.2=4.2 \quad \text{which gives } y=4$$

By the first row we have

$$x+2y+3z=12$$

$$x+2(4)+3(1)=12 \qquad [\text{Substituting } y=4 \text{ and } z=1]$$

$$x+8+3=12 \text{ which gives } x=1$$

Our solution is $x=1$, $y=4$ and $z=1$.

(b) The augmented matrix is

$$\begin{array}{c} R_1 \\ R_2 \\ R_3 \end{array}\left(\begin{array}{ccc|c} 2 & -1 & -4 & 0 \\ 3 & 5 & 2 & 5 \\ 4 & -3 & 6 & -16 \end{array}\right)$$

Divide the first row by 2:

$$\begin{array}{c} R_1'=R_1/2 \\ R_2 \\ R_3 \end{array}\left(\begin{array}{ccc|c} 1 & -1/2 & -2 & 0 \\ 3 & 5 & 2 & 5 \\ 4 & -3 & 6 & -16 \end{array}\right)$$

To get 0's in place of 3 and 4 we execute the row operations:

$$\begin{array}{c} R_1' \\ R_2'=R_2-3R_1' \\ R_3'=R_3-4R_1' \end{array}\left(\begin{array}{ccc|c} 1 & -1/2 & -2 & 0 \\ 3-3(1) & 5-3(-1/2) & 2-3(-2) & 5-3(0) \\ 4-4(1) & -3-4(-1/2) & 6-4(-2) & -16-4(0) \end{array}\right)$$

Simplifying the entries gives

$$\begin{array}{c} R_1' \\ R_2' \\ R_3' \end{array}\left(\begin{array}{ccc|c} 1 & -1/2 & -2 & 0 \\ 0 & 13/2 & 8 & 5 \\ 0 & -1 & 14 & -16 \end{array}\right)$$

How do we get 0 in place of -1 *in the bottom row?*

$R_3'+\frac{2}{13}R_2'$ because $-1+\frac{2}{13}\left(\frac{13}{2}\right)=0$. We have

$$\begin{array}{c} R_1' \\ R_2' \\ R_3''=R_3'+\frac{2}{13}R_2' \end{array}\left(\begin{array}{ccc|c} 1 & -1/2 & -2 & 0 \\ 0 & 13/2 & 8 & 5 \\ 0 & -1+\frac{2}{13}\left(\frac{13}{2}\right) & 14+\frac{2}{13}(8) & -16+\frac{2}{13}(5) \end{array}\right)$$

Simplifying the entries gives

$$\begin{array}{c} R_1' \\ R_2' \\ R_3'' \end{array}\left(\begin{array}{ccc|c} 1 & -1/2 & -2 & 0 \\ 0 & 13/2 & 8 & 5 \\ 0 & 0 & 198/13 & -198/13 \end{array}\right)$$

Multiplying the second row by 2/13 and the last row by 13/198 we have

$$\begin{array}{c} \\ R_1{}' \\ R_2{}''=2R_2{}'/13 \\ R_3{}^*=13R_3{}''/198 \end{array}\begin{array}{c} \begin{array}{ccc} x & y & z \end{array} \\ \left(\begin{array}{ccc|c} 1 & -1/2 & -2 & 0 \\ 0 & 1 & 16/13 & 10/13 \\ 0 & 0 & 1 & -1 \end{array}\right) \end{array}$$

This is now in row echelon form and from the last row we have $z=-1$. Examining the second row we have

$$y+\frac{16}{13}(z)=\frac{10}{13}$$

$$y+\frac{16}{13}(-1)=\frac{10}{13} \qquad \left[\text{Substituting } z=-1\right]$$

$$y=\frac{26}{13}=2$$

Substituting $y=2$ and $z=-1$ into the first row $R_1{}'$ we have

$$x-\frac{1}{2}y-2z=0$$

$$x-\frac{1}{2}(2)-2(-1)=0 \quad \text{which gives } x=-1$$

Our solution is $x=-1$, $y=2$ and $z=-1$.

(c) The augmented matrix is

$$\begin{array}{c} R_1 \\ R_2 \\ R_3 \end{array}\left(\begin{array}{ccc|c} 3 & -1 & 7 & 9 \\ 5 & 3 & 2 & 10 \\ 9 & 2 & -5 & 6 \end{array}\right)$$

To get 0's in place of 5 and 9 we carry out the following combined row operations:

$$\begin{array}{r} R_1 \\ r_2=3R_2-5R_1 \\ r_3=R_3-3R_1 \end{array}\left(\begin{array}{ccc|c} 3 & -1 & 7 & 9 \\ 3(5)-5(3) & 3(3)-5(-1) & 3(2)-5(7) & 3(10)-5(9) \\ 9-3(3) & 2-3(-1) & -5-3(7) & 6-3(9) \end{array}\right)$$

Simplifying the entries gives

$$\begin{array}{c} R_1 \\ r_2 \\ r_3 \end{array}\left(\begin{array}{ccc|c} 3 & -1 & 7 & 9 \\ 0 & 14 & -29 & -15 \\ 0 & 5 & -26 & -21 \end{array}\right)$$

To get 0 in place of 5 in the bottom row we execute the row operation $r_3-\frac{5}{14}r_2$:

$$\begin{array}{r} R_1 \\ r_2 \\ r_3{}'=r_3-\frac{5}{14}r_2 \end{array}\left(\begin{array}{ccc|c} 3 & -1 & 7 & 9 \\ 0 & 14 & -29 & -15 \\ 0 & 5-\frac{5}{14}(14) & -26-\frac{5}{14}(-29) & -21-\frac{5}{14}(-15) \end{array}\right)$$

Simplifying the arithmetic in the last row we have

$$\begin{array}{c} R_1 \\ r_2 \\ r_3{}' \end{array}\left(\begin{array}{ccc|c} 3 & -1 & 7 & 9 \\ 0 & 14 & -29 & -15 \\ 0 & 0 & -219/14 & -219/14 \end{array}\right)$$

Multiply the last row by $-14/219$, the second row by 1/14 and the first row by 1/3:

$$\begin{array}{r} \\ r_1 = R_1/3 \\ r_2' = r_2/14 \\ r_3'' = -14r_3'/219 \end{array} \begin{array}{c} \begin{array}{ccc} x & y & z \end{array} \\ \left(\begin{array}{ccc|c} 1 & -1/3 & 7/3 & 3 \\ 0 & 1 & -29/14 & -15/14 \\ 0 & 0 & 1 & 1 \end{array}\right) \end{array}$$

The augmented matrix is now in row echelon form. From the last row we have $z=1$ and substituting this into the second row we have

$$y - \frac{29}{14} = -\frac{15}{14} \text{ which gives } y = \frac{14}{14} = 1$$

How can we determine the last unknown x?

By substituting $y=1$ and $z=1$ into the first row r_1:

$$x - \frac{1}{3} + \frac{7}{3} = 3 \text{ gives } x = 1$$

The final solution is $x = y = z = 1$.

3. (a) The augmented matrix is given by

$$\begin{array}{c} R_1 \\ R_2 \\ R_3 \end{array} \left(\begin{array}{ccc|c} 1 & 1 & 2 & 9 \\ 4 & 4 & -3 & 3 \\ 5 & 1 & 2 & 13 \end{array}\right)$$

To obtain 0's in place of 4 and 5 in the first column we execute $R_2 - 4R_1$ and $R_3 - 5R_1$:

$$\begin{array}{r} R_1 \\ R_2' = R_2 - 4R_1 \\ R_3' = R_3 - 5R_1 \end{array} \left(\begin{array}{ccc|c} 1 & 1 & 2 & 9 \\ 0 & 0 & -11 & -33 \\ 0 & -4 & -8 & -32 \end{array}\right)$$

Interchange the new rows R_2' and R_3':

$$\begin{array}{c} R_1 \\ R_3' \\ R_2' \end{array} \left(\begin{array}{ccc|c} 1 & 1 & 2 & 9 \\ 0 & -4 & -8 & -32 \\ 0 & 0 & -11 & -33 \end{array}\right)$$

Divide the second row by -4 and the third row by -11 so we have

$$\begin{array}{r} R_1 \\ R_2'' = R_3'/(-4) \\ R_3'' = R_2'/(-11) \end{array} \left(\begin{array}{ccc|c} 1 & 1 & 2 & 9 \\ 0 & 1 & 2 & 8 \\ 0 & 0 & 1 & 3 \end{array}\right)$$

This is now in row echelon form but **not** in reduced row echelon form (rref). Subtract the first two rows, that is $R_1 - R_2''$:

$$\begin{array}{r} R_1 - R_2'' \\ R_2'' \\ R_3'' \end{array} \left(\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 2 & 8 \\ 0 & 0 & 1 & 3 \end{array}\right)$$

Execute the row operation $\mathrm{R}_2'' - 2\mathrm{R}_3''$:

$$\begin{array}{c} \\ R_1 - R_2'' \\ R_2{*} = R_2'' - 2R_3'' \\ R_3'' \end{array} \begin{array}{c} \begin{array}{ccc} x & y & z \end{array} \\ \left(\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{array}\right) \end{array}$$

Hence our solution is $x=1$, $y=2$ and $z=3$.

(b) The augmented matrix is

$$\begin{array}{c} R_1 \\ R_2 \\ R_3 \end{array} \left(\begin{array}{ccc|c} 1 & 1 & 1 & -2 \\ 2 & -1 & -1 & -4 \\ 4 & 2 & -3 & -3 \end{array}\right)$$

To get 0's in place of 4 in the bottom row and 2 in the second row we execute the following row operations:

$$\begin{array}{c} R_1 \\ r_2 = R_2 - 2R_1 \\ r_3 = R_3 - 4R_1 \end{array} \left(\begin{array}{ccc|c} 1 & 1 & 1 & -2 \\ 2-2(1) & -1-2(1) & -1-2(1) & -4-2(-2) \\ 4-4(1) & 2-4(1) & -3-4(1) & -3-4(-2) \end{array}\right)$$

Simplifying the entries gives

$$\begin{array}{c} R_1 \\ r_2 \\ r_3 \end{array} \left(\begin{array}{ccc|c} 1 & 1 & 1 & -2 \\ 0 & -3 & -3 & 0 \\ 0 & -2 & -7 & 5 \end{array}\right)$$

Divide the second row by -3 and the bottom row by -2:

$$\begin{array}{c} R_1 \\ r_2' = r_2/(-3) \\ r_3' = r_3/(-2) \end{array} \left(\begin{array}{ccc|c} 1 & 1 & 1 & -2 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 7/2 & -5/2 \end{array}\right)$$

Need to get 0 in place of the 1 in the bottom row. *How?*

Subtract the bottom two rows.

$$\begin{array}{c} R_1 \\ r_2' \\ r_3'' = r_3' - r_2' \end{array} \left(\begin{array}{ccc|c} 1 & 1 & 1 & -2 \\ 0 & 1 & 1 & 0 \\ 0-0 & 1-1 & 7/2-1 & -5/2-0 \end{array}\right)$$

Simplifying the last row we have

$$\begin{array}{c} R_1 \\ r_2' \\ r_3'' \end{array} \left(\begin{array}{ccc|c} 1 & 1 & 1 & -2 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 5/2 & -5/2 \end{array}\right)$$

Multiplying the last row by 2/5 gives:

$$\begin{array}{c} R_1 \\ r_2' \\ r_3^{\$} = (2/5)r_3'' \end{array} \left(\begin{array}{ccc|c} 1 & 1 & 1 & -2 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & -1 \end{array}\right)$$

This is now in row echelon form but **not** in reduced row echelon form. We need to convert the 1's in the upper right hand corner of the matrix to 0's. We execute the following row operations:

$$\begin{array}{r} r_1 = R_1 - r_3^{\$} \\ r_2'' = r_2' - r_3^{\$} \\ r_3^{\$} \end{array} \left(\begin{array}{ccc|c} 1-0 & 1-0 & 1-1 & -2-(-1) \\ 0-0 & 1-0 & 1-1 & 0-(-1) \\ 0 & 0 & 1 & -1 \end{array}\right)$$

Simplifying the arithmetic gives

$$\begin{array}{r} r_1 \\ r_2'' \\ r_3^{\$} \end{array} \left(\begin{array}{ccc|c} 1 & 1 & 0 & -1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -1 \end{array}\right)$$

The matrix is still **not** in reduced row echelon form. *Why not?*
The second 1 in the first row needs to be 0 for the matrix to be in reduced row echelon form. *How do we get a 0 in that position?*
Carry out the row operation $r_1 - r_2''$:

$$\begin{array}{r} r_1'' = r_1 - r_2'' \\ r_2'' \\ r_3^{\$} \end{array} \left(\begin{array}{ccc|c} 1-0 & 1-1 & 0-0 & -1-(1) \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -1 \end{array}\right)$$

We have

$$\begin{array}{r} \\ r_1'' \\ r_2'' \\ r_3^{\$} \end{array} \begin{array}{c} \begin{array}{cccc} x & y & z & \end{array} \\ \left(\begin{array}{ccc|c} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -1 \end{array}\right) \end{array}$$

The matrix is now in reduced row echelon form and we can read off the values of x, y and z directly. Hence $x = -2$, $y = 1$ and $z = -1$.
(c) The augmented matrix is

$$\begin{array}{r} R_1 \\ R_2 \\ R_3 \end{array} \left(\begin{array}{ccc|c} 2 & 1 & -1 & 2 \\ 4 & 3 & 2 & -3 \\ 6 & -5 & 3 & -14 \end{array}\right)$$

We execute the row operations $R_2 - 2R_1$ and $R_3 - 3R_1$ to get 0's in place of 4 and 6 respectively:

$$\begin{array}{r} R_1 \\ r_2 = R_2 - 2R_1 \\ r_3 = R_3 - 3R_1 \end{array} \left(\begin{array}{ccc|c} 2 & 1 & -1 & 2 \\ 4-2(2) & 3-2(1) & 2-2(-1) & -3-2(2) \\ 6-3(2) & -5-3(1) & 3-3(-1) & -14-3(2) \end{array}\right)$$

Simplifying the entries gives

$$\begin{array}{r} R_1 \\ r_2 \\ r_3 \end{array} \left(\begin{array}{ccc|c} 2 & 1 & -1 & 2 \\ 0 & 1 & 4 & -7 \\ 0 & -8 & 6 & -20 \end{array}\right)$$

We need a 0 in place of -8 in the bottom row so we execute $r_3 + 8r_2$:

$$\begin{array}{r} R_1 \\ r_2 \\ r_3^* = r_3 + 8r_2 \end{array} \left(\begin{array}{ccc|c} 2 & 1 & -1 & 2 \\ 0 & 1 & 4 & -7 \\ 0 & -8+8(1) & 6+8(4) & -20+8(-7) \end{array}\right)$$

Simplifying the last row we have

$$\begin{array}{c} R_1 \\ r_2 \\ r_3* \end{array}\left(\begin{array}{ccc|c} 2 & 1 & -1 & 2 \\ 0 & 1 & 4 & -7 \\ 0 & 0 & 38 & -76 \end{array}\right)$$

Divide the last row by 38:

$$\begin{array}{c} R_1 \\ r_2 \\ r_3** = r_3*/38 \end{array}\left(\begin{array}{ccc|c} 2 & 1 & -1 & 2 \\ 0 & 1 & 4 & -7 \\ 0 & 0 & 1 & -2 \end{array}\right)$$

The augmented matrix is **not** in reduced row echelon form. To get 0's in place of -1 and 4 in the third column we execute the following:

$$\begin{array}{c} R_1* = R_1 + r_3** \\ r_2* = r_2 - 4r_3** \\ r_3** \end{array}\left(\begin{array}{ccc|c} 2+0 & 1+0 & -1+1 & 2-2 \\ 0-4(0) & 1-4(0) & 4-4(1) & -7-4(-2) \\ 0 & 0 & 1 & -2 \end{array}\right)$$

Simplifying gives

$$\begin{array}{c} R_1* \\ r_2* \\ r_3** \end{array}\left(\begin{array}{ccc|c} 2 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -2 \end{array}\right)$$

Subtract the first two rows:

$$\begin{array}{c} r_1 = R_1* - r_2* \\ r_2* \\ r_3** \end{array}\left(\begin{array}{ccc|c} 2-0 & 1-1 & 0-0 & 0-1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -2 \end{array}\right)$$

Simplifying the first row we have

$$\begin{array}{c} r_1 \\ r_2* \\ r_3** \end{array}\left(\begin{array}{ccc|c} 2 & 0 & 0 & -1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -2 \end{array}\right)$$

Dividing the first row by 2 gives

$$\begin{array}{cc} & \begin{array}{ccc} x & y & z \end{array} \\ \begin{array}{c} r_1/2 \\ r_2* \\ r_3** \end{array} & \left(\begin{array}{ccc|c} 1 & 0 & 0 & -1/2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -2 \end{array}\right) \end{array}$$

Our solution is $x = -\dfrac{1}{2}$, $y = 1$ and $z = -2$.

(d) The augmented matrix is given by

$$\begin{array}{c} R_1 \\ R_2 \\ R_3 \end{array}\left(\begin{array}{ccc|c} -2 & 3 & -2 & 8 \\ -1 & 2 & -10 & 0 \\ 5 & -7 & 4 & -20 \end{array}\right)$$

First we multiply the middle row by -1 and interchange this with the first row:

$$\begin{array}{r} r_1 = -R_2 \\ r_2 = R_1 \\ R_3 \end{array}\left(\begin{array}{ccc|c} 1 & -2 & 10 & 0 \\ -2 & 3 & -2 & 8 \\ 5 & -7 & 4 & -20 \end{array}\right)$$

We need 0's in the bottom left hand corner of the matrix so we execute the following row operations:

$$\begin{array}{r} r_1 \\ r_2^{\dagger} = r_2 + 2r_1 \\ r_3 = R_3 - 5r_1 \end{array}\left(\begin{array}{ccc|c} 1 & -2 & 10 & 0 \\ -2+2(1) & 3+2(-2) & -2+2(10) & 8+2(0) \\ 5-5(1) & -7-5(-2) & 4-5(10) & -20-5(0) \end{array}\right)$$

Simplifying the entries gives

$$\begin{array}{r} r_1 \\ r_2^{\dagger} \\ r_3 \end{array}\left(\begin{array}{ccc|c} 1 & -2 & 10 & 0 \\ 0 & -1 & 18 & 8 \\ 0 & 3 & -46 & -20 \end{array}\right)$$

How do we get a 0 in place of 3 in the bottom row?

By the row operation $r_3 + 3r_2^{\dagger}$:

$$\begin{array}{r} r_1 \\ r_2^{\dagger} \\ r_3^{\dagger} = r_3 + 3r_2^{\dagger} \end{array}\left(\begin{array}{ccc|c} 1 & -2 & 10 & 0 \\ 0 & -1 & 18 & 8 \\ 0+3(0) & 3+3(-1) & -46+3(18) & -20+3(8) \end{array}\right)$$

Simplifying the entries in the bottom row we have

$$\begin{array}{r} r_1 \\ r_2^{\dagger} \\ r_3^{\dagger} \end{array}\left(\begin{array}{ccc|c} 1 & -2 & 10 & 0 \\ 0 & -1 & 18 & 8 \\ 0 & 0 & 8 & 4 \end{array}\right)$$

Multiply the last row by 1/8 (or divide by 8) and multiply the second row by -1:

$$\begin{array}{r} r_1 \\ r_2^{\dagger\dagger} = -r_2^{\dagger} \\ r_3^{\dagger\dagger} = r_3^{\dagger}/8 \end{array}\left(\begin{array}{ccc|c} 1 & -2 & 10 & 0 \\ 0 & 1 & -18 & -8 \\ 0 & 0 & 1 & 1/2 \end{array}\right)$$

The matrix is now in row echelon form but we need to place it into reduced row echelon form which means that we need 0's in the top right hand corner of the matrix. We carry out the following row operations:

$$\begin{array}{r} r_1^{\dagger} = r_1 - 10r_3^{\dagger\dagger} \\ r_2^{*} = r_2^{\dagger\dagger} + 18r_3^{\dagger\dagger} \\ r_3^{\dagger\dagger} \end{array}\left(\begin{array}{ccc|c} 1-10(0) & -2-10(0) & 10-10(1) & 0-10(1/2) \\ 0+18(0) & 1+18(0) & -18+18(1) & -8+18(1/2) \\ 0 & 0 & 1 & 1/2 \end{array}\right)$$

Simplifying the entries gives:

$$\begin{array}{r} r_1^{\dagger} \\ r_2^{*} \\ r_3^{\dagger\dagger} \end{array}\left(\begin{array}{ccc|c} 1 & -2 & 0 & -5 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1/2 \end{array}\right)$$

Need a 0 in place of -2 in the top row. We execute the following row operation:

$$\begin{array}{r} r_1^{\dagger\dagger} = r_1^{\dagger} + 2r_2^{*} \\ r_2^{*} \\ r_3^{\dagger\dagger} \end{array}\left(\begin{array}{ccc|c} 1 & -2+2(1) & 0 & -5+2(1) \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1/2 \end{array}\right)$$

Simplifying the entries gives

$$\begin{array}{c c} & \begin{array}{ccc} x & y & z \end{array} \\ \begin{array}{c} r_1^{\dagger\dagger} \\ r_2^{*} \\ r_3^{\dagger\dagger} \end{array} & \left(\begin{array}{ccc|c} 1 & 0 & 0 & -3 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1/2 \end{array}\right) \end{array}$$

Hence our solution is $x = -3$, $y = 1$ and $z = \frac{1}{2}$.