Complete Solutions to Exercises 6.4

1. (a) We have

$$\begin{array}{c} R_1 \\ R_2 \\ R_3 \end{array}\begin{pmatrix} 1 & 2 & 2 \\ 3 & -3 & -2 \\ 4 & -1 & -5 \end{pmatrix} \Rightarrow \begin{array}{c} R_1 \\ R_2^{\dagger} = R_2 - 3R_1 \\ R_3^{\dagger} = R_3 - 4R_1 \end{array}\begin{pmatrix} 1 & 2 & 2 \\ 0 & -9 & -8 \\ 0 & -9 & -13 \end{pmatrix} \Rightarrow \begin{array}{c} R_1 \\ R_2^{\dagger} \\ R_3^{\dagger} - R_2^{\dagger} \end{array}\begin{pmatrix} 1 & 2 & 2 \\ 0 & -9 & -8 \\ 0 & 0 & -5 \end{pmatrix} = \mathbf{U}$$

This is our upper triangular matrix. By using negative multipliers we have

$$\begin{pmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 4 & 1 & 1 \end{pmatrix} = \mathbf{L}$$

First we solve $\mathbf{Ly} = \mathbf{b}$ where **L** is the above lower triangular matrix and **b** is the given vector. We have

$$\mathbf{Ly} = \begin{pmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 4 & 1 & 1 \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} 5 \\ 0 \\ -10 \end{pmatrix} = \mathbf{b}$$

From the top row we have $y_1 = 5$. Using the middle row with $y_1 = 5$:

$$3y_1 + y_2 = 0 \;\Rightarrow\; 3(5) + y_2 = 0 \;\Rightarrow\; y_2 = -15$$

Expanding the bottom row gives

$$4y_1 + y_2 + y_3 = -10 \;\Rightarrow\; 4(5) + (-15) + y_3 = -10 \;\Rightarrow\; y_3 = -15$$

We have $y_1 = 5,\ y_2 = -15$ and $y_3 = -15$.

Now we solve $\mathbf{Ux} = \mathbf{y}$ with these y values and the above upper triangular matrix **U**.

$$\mathbf{Ux} = \begin{pmatrix} 1 & 2 & 2 \\ 0 & -9 & -8 \\ 0 & 0 & -5 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 5 \\ -15 \\ -15 \end{pmatrix}$$

Expanding the bottom row we have $x_3 = 3$.

From the middle row

$$-9x_2 - 8x_3 = -15 \;\Rightarrow\; -9x_2 - 8(3) = -15 \;\Rightarrow\; x_2 = -1$$

Expanding the top row we have

$$x_1 + 2x_2 + 2x_3 = 5 \;\Rightarrow\; x_1 + 2(-1) + 2(3) = 5 \;\Rightarrow\; x_1 = 1$$

Hence our solution is $x_1 = 1,\ x_2 = -1$ and $x_3 = 3$.

(b) Similarly for $\mathbf{A} = \begin{pmatrix} 1 & 5 & 6 \\ 2 & 11 & 19 \\ 3 & 19 & 47 \end{pmatrix}$, $\mathbf{b} = \begin{pmatrix} 1 \\ -4 \\ -22 \end{pmatrix}$ we have

$$\mathbf{L} = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 4 & 1 \end{pmatrix},\quad \mathbf{U} = \begin{pmatrix} 1 & 5 & 6 \\ 0 & 1 & 7 \\ 0 & 0 & 1 \end{pmatrix}$$

Solving $\mathbf{Ly} = \mathbf{b}$ for the given **b** vector and above lower triangular matrix **L**:

$$\mathbf{Ly} = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 4 & 1 \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} 1 \\ -4 \\ -22 \end{pmatrix} = \mathbf{b} \quad \text{gives} \quad \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} 1 \\ -6 \\ -1 \end{pmatrix}$$

Now we solve $\mathbf{Ux} = \mathbf{y}$. Hence

$$\mathbf{Ux} = \begin{pmatrix} 1 & 5 & 6 \\ 0 & 1 & 7 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} 1 \\ -6 \\ -1 \end{pmatrix} \quad \text{gives} \quad \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix}$$

(c) Similarly we have

$$\mathbf{L} = \begin{pmatrix} 1 & 0 & 0 \\ 2 & -3 & 0 \\ 5 & 7 & 1 \end{pmatrix}, \ \mathbf{U} = \begin{pmatrix} 2 & 1 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 5 \end{pmatrix}$$

Solving $\mathbf{Ly} = \mathbf{b}$ for the given **b** vector and above lower triangular matrix **L**:

$$\mathbf{Ly} = \begin{pmatrix} 1 & 0 & 0 \\ 2 & -3 & 0 \\ 5 & 7 & 1 \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} 5 \\ -11 \\ 84 \end{pmatrix} \quad \text{gives} \quad \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} 5 \\ 7 \\ 10 \end{pmatrix}$$

Now we solve $\mathbf{Ux} = \mathbf{y}$. Hence

$$\mathbf{Ux} = \begin{pmatrix} 2 & 1 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 5 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 5 \\ 7 \\ 10 \end{pmatrix} \quad \text{gives} \quad \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -2 \\ 3 \\ 2 \end{pmatrix}$$

(d) Repeating the above process we have

$$\mathbf{L} = \begin{pmatrix} 1 & 0 & 0 \\ 3 & -3 & 0 \\ -1 & 2 & 1 \end{pmatrix}, \ \mathbf{U} = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 5 & 7 \\ 0 & 0 & 7 \end{pmatrix}$$

Solving $\mathbf{Ly} = \mathbf{b}$ for the given **b** vector and above lower triangular matrix **L**:

$$\mathbf{Ly} = \begin{pmatrix} 1 & 0 & 0 \\ 3 & -3 & 0 \\ -1 & 2 & 1 \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} -5 \\ 9 \\ -4 \end{pmatrix} \quad \text{gives} \quad \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} -5 \\ -8 \\ 7 \end{pmatrix}$$

Now we solve $\mathbf{Ux} = \mathbf{y}$. Hence

$$\mathbf{Ux} = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 5 & 7 \\ 0 & 0 & 7 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -5 \\ -8 \\ 7 \end{pmatrix} \quad \text{gives} \quad \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -2 \\ -3 \\ 1 \end{pmatrix}$$

2. Placing the given matrix **A** into an upper triangular matrix:

$$\begin{array}{c} R_1 \\ R_2 \\ R_3 \\ R_4 \end{array} \begin{pmatrix} 1 & 2 & 3 & 4 \\ 17 & 22 & 27 & 8 \\ 77 & 44 & 47 & -494 \\ -10 & 1 & 7 & 63 \end{pmatrix}$$

Carrying out the row operations $R_2 - 17R_1$, $R_3 - 77R_1$ and $R_4 + 10R_1$ on matrix **A** and the *negative multipliers* on the identity gives:

$$\begin{array}{r} R_1 \\ R_2^* = R_2 - 17R_1 \\ R_3^* = R_3 - 77R_1 \\ R_4^* = R_4 + 10R_1 \end{array}\left(\begin{array}{cccc} 1 & 2 & 3 & 4 \\ 0 & -12 & -24 & -60 \\ 0 & -110 & -184 & -802 \\ 0 & 21 & 37 & 103 \end{array}\right)\begin{array}{l} \\ \leftarrow \text{Multiplier} = -17 \\ \leftarrow \text{Multiplier} = -77 \\ \leftarrow \text{Multiplier} = +10 \end{array}\qquad \left(\begin{array}{cccc} 1 & 0 & 0 & 0 \\ 17 & 1 & 0 & 0 \\ 77 & 0 & 1 & 0 \\ -10 & 0 & 0 & 1 \end{array}\right)$$

Carrying out $R_2^*/(-12)$ yields

$$\begin{array}{r} R_1 \\ R_2^{\dagger} = R_2^*/(-12) \\ R_3^* \\ R_4^* \end{array}\left(\begin{array}{cccc} 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 5 \\ 0 & -110 & -184 & -802 \\ 0 & 21 & 37 & 103 \end{array}\right)\begin{array}{l} \\ \leftarrow \text{Multiplier} = -\dfrac{1}{12} \\ = -1/12 \\ \\ \end{array}\qquad \left(\begin{array}{cccc} 1 & 0 & 0 & 0 \\ 17 & -12 & 0 & 0 \\ 77 & 0 & 1 & 0 \\ -10 & 0 & 0 & 1 \end{array}\right)$$

Executing $R_3^* + 110R_2^{\dagger}$ and $R_4^* - 21R_2^{\dagger}$ gives

$$\begin{array}{r} R_1 \\ R_2^{\dagger} \\ R_3^{\dagger} = R_3^* + 110R_2^{\dagger} \\ R_4^{\dagger} = R_4^* - 21R_2^{\dagger} \end{array}\left(\begin{array}{cccc} 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 5 \\ 0 & 0 & 36 & -252 \\ 0 & 0 & -5 & -2 \end{array}\right)\begin{array}{l} \\ \\ \leftarrow \text{Multiplier} = +110 \\ \leftarrow \text{Multiplier} = -21 \end{array}\qquad \left(\begin{array}{cccc} 1 & 0 & 0 & 0 \\ 17 & -12 & 0 & 0 \\ 77 & -110 & 1 & 0 \\ -10 & 21 & 0 & 1 \end{array}\right)$$

Executing $R_3^{\dagger}/36$ gives

$$\begin{array}{r} R_1 \\ R_2^{\dagger} \\ R_3^{\dagger\dagger} = R_3^{\dagger}/36 \\ R_4^{\dagger} \end{array}\left(\begin{array}{cccc} 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 5 \\ 0 & 0 & 1 & -7 \\ 0 & 0 & -5 & -2 \end{array}\right)\begin{array}{l} \\ \\ \leftarrow \text{Multiplier} = 1/36 \\ \\ \end{array}\qquad \left(\begin{array}{cccc} 1 & 0 & 0 & 0 \\ 17 & -12 & 0 & 0 \\ 77 & -110 & 36 & 0 \\ -10 & 21 & 0 & 1 \end{array}\right)$$

Carrying out the row operation $R_4^{\dagger} + 5R_3^{\dagger\dagger}$ gives

$$\begin{array}{r} R_1 \\ R_2^{\dagger} \\ R_3^{\dagger\dagger} \\ R_4^{\dagger} + 5R_3^{\dagger\dagger} \end{array}\left(\begin{array}{cccc} 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 5 \\ 0 & 0 & 1 & -7 \\ 0 & 0 & 0 & -37 \end{array}\right) = \mathbf{U} \begin{array}{l} \\ \\ \\ \leftarrow \text{Multiplier} = +5 \end{array}\qquad \left(\begin{array}{cccc} 1 & 0 & 0 & 0 \\ 17 & -12 & 0 & 0 \\ 77 & -110 & 36 & 0 \\ -10 & 21 & -5 & 1 \end{array}\right) = \mathbf{L}$$

Solving $\mathbf{Ly} = \mathbf{b}$ for the given **b** vector and above lower triangular matrix **L**:

$$\mathbf{Ly} = \left(\begin{array}{cccc} 1 & 0 & 0 & 0 \\ 17 & -12 & 0 & 0 \\ 77 & -110 & 36 & 0 \\ -10 & 21 & -5 & 1 \end{array}\right)\left(\begin{array}{c} y_1 \\ y_2 \\ y_3 \\ y_4 \end{array}\right) = \left(\begin{array}{c} -10 \\ 22 \\ 2106 \\ -243 \end{array}\right) \quad \text{gives} \quad \left(\begin{array}{c} y_1 \\ y_2 \\ y_3 \\ y_4 \end{array}\right) = \left(\begin{array}{c} -10 \\ -16 \\ 31 \\ 148 \end{array}\right)$$

Now we solve $\mathbf{Ux} = \mathbf{y}$. Hence

$$\mathbf{Ux} = \left(\begin{array}{cccc} 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 5 \\ 0 & 0 & 1 & -7 \\ 0 & 0 & 0 & -37 \end{array}\right)\left(\begin{array}{c} x_1 \\ x_2 \\ x_3 \\ x_4 \end{array}\right) = \left(\begin{array}{c} -10 \\ -16 \\ 31 \\ 148 \end{array}\right) \quad \text{gives} \quad \left(\begin{array}{c} x_1 \\ x_2 \\ x_3 \\ x_4 \end{array}\right) = \left(\begin{array}{c} 1 \\ -2 \\ 3 \\ -4 \end{array}\right)$$

3. In question 2 we found that

$$\mathbf{A}=\mathbf{LU}=\begin{pmatrix}1&0&0&0\\17&-12&0&0\\77&-110&36&0\\-10&21&-5&1\end{pmatrix}\begin{pmatrix}1&2&3&4\\0&1&2&5\\0&0&1&-7\\0&0&0&-37\end{pmatrix}$$

This means the determinant of matrix **A** is given by

$$\det(\mathbf{A})=\det(\mathbf{LU})=\det(\mathbf{L})\times\det(\mathbf{U})$$

$$=(1\times(-12)\times 36\times 1)\times(1\times 1\times 1\times(-37))$$

Determinant of triangular matrices is the product of the entries on diagonal.

$$=15\,984$$

4. The given matrix **A** is an upper triangular matrix so we have

$$\mathbf{A}=\mathbf{IA}=\mathbf{IU}=\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}\begin{pmatrix}1&2&3\\0&4&5\\0&0&6\end{pmatrix}$$

5. First we need to convert the given matrix $\mathbf{A}=\begin{pmatrix}1&2&3\\3&7&14\\4&13&38\end{pmatrix}$ into **LU**:

$$\begin{array}{c}R_1\\R_2\\R_3\end{array}\begin{pmatrix}1&2&3\\3&7&14\\4&13&38\end{pmatrix}$$

Carrying out the row operations R_2-3R_1 and R_3-4R_1 on matrix **A** and the *negative multipliers* on the identity gives:

$$\begin{array}{r}R_1\\R_2^*=R_2-3R_1\\R_3^*=R_3-4R_1\end{array}\begin{pmatrix}1&2&3\\0&1&5\\0&5&26\end{pmatrix}\begin{array}{l}\\ \leftarrow \text{Multiplier }=-3\\ \leftarrow \text{Multiplier }=-4\end{array}\qquad\begin{pmatrix}1&0&0\\3&1&0\\4&0&1\end{pmatrix}$$

Executing $R_3^*-5R_2^*$ gives:

$$\begin{array}{r}R_1\\R_2^*\\R_3^*-5R_2^*\end{array}\begin{pmatrix}1&2&3\\0&1&5\\0&0&1\end{pmatrix}=\mathbf{U}\quad\begin{array}{l}\\ \\ \leftarrow \text{Multiplier }=-5\end{array}\qquad\begin{pmatrix}1&0&0\\3&1&0\\4&5&1\end{pmatrix}=\mathbf{L}$$

We have $\mathbf{A}=\begin{pmatrix}1&2&3\\3&7&14\\4&13&38\end{pmatrix}=\mathbf{LU}=\begin{pmatrix}1&0&0\\3&1&0\\4&5&1\end{pmatrix}\begin{pmatrix}1&2&3\\0&1&5\\0&0&1\end{pmatrix}$.

(b) The inverse is given by

$$\mathbf{A}^{-1}=(\mathbf{LU})^{-1}=\mathbf{U}^{-1}\mathbf{L}^{-1}$$

By using the row operations procedure given in chapter 1 we can find the inverses of **U** and **L**:

$$\left(\mathbf{U}\mid\mathbf{I}\right)=\begin{array}{c}R_1\\R_2\\R_3\end{array}\left(\begin{array}{ccc|ccc}1&2&3&1&0&0\\0&1&5&0&1&0\\0&0&1&0&0&1\end{array}\right)$$

Executing R_1-2R_2 gives:

$$\begin{array}{c}R_1^\dagger=R_1-2R_2\\R_2\\R_3\end{array}\left(\begin{array}{ccc|ccc}1&0&-7&1&-2&0\\0&1&5&0&1&0\\0&0&1&0&0&1\end{array}\right)$$

Carrying out the row operations $R_1^\dagger+7R_3$ and R_2-5R_3

$$\begin{array}{c}R_1^\dagger+7R_3\\R_2-5R_3\\R_3\end{array}\left(\begin{array}{ccc|ccc}1&0&0&1&-2&7\\0&1&0&0&1&-5\\0&0&1&0&0&1\end{array}\right)$$

Hence $\mathbf{U}^{-1}=\begin{pmatrix}1&-2&7\\0&1&-5\\0&0&1\end{pmatrix}$ and similarly we have

$$\left(\mathbf{L}\mid\mathbf{I}\right)=\begin{array}{c}R_1\\R_2\\R_3\end{array}\left(\begin{array}{ccc|ccc}1&0&0&1&0&0\\3&1&0&0&1&0\\4&5&1&0&0&1\end{array}\right)$$

Executing R_2-3R_1 and R_3-4R_1 gives:

$$\begin{array}{c}R_1\\R_2*=R_2-3R_1\\R_3*=R_3-4R_1\end{array}\left(\begin{array}{ccc|ccc}1&0&0&1&0&0\\0&1&0&-3&1&0\\0&5&1&-4&0&1\end{array}\right)$$

Carrying out the row operation R_3*-5R_2* yields:

$$\begin{array}{c}R_1\\R_2*_1\\R_3*-5R_2*\end{array}\left(\begin{array}{ccc|ccc}1&0&0&1&0&0\\0&1&0&-3&1&0\\0&0&1&11&-5&1\end{array}\right)$$

Hence $\mathbf{L}^{-1}=\begin{pmatrix}1&0&0\\-3&1&0\\11&-5&1\end{pmatrix}$. Multiplying these together in the order $\mathbf{U}^{-1}\times\mathbf{L}^{-1}$ gives

$$\mathbf{A}^{-1}=\mathbf{U}^{-1}\mathbf{L}^{-1}=\begin{pmatrix}1&-2&7\\0&1&-5\\0&0&1\end{pmatrix}\begin{pmatrix}1&0&0\\-3&1&0\\11&-5&1\end{pmatrix}=\begin{pmatrix}84&-37&7\\-58&26&-5\\11&-5&1\end{pmatrix}$$

(c) (i) We are given that $\mathbf{b}=\begin{pmatrix}1&2&3\end{pmatrix}^T$. So

$$\mathbf{x}=\mathbf{A}^{-1}\mathbf{b}=\begin{pmatrix}84&-37&7\\-58&26&-5\\11&-5&1\end{pmatrix}\begin{pmatrix}1\\2\\3\end{pmatrix}=\begin{pmatrix}31\\-21\\4\end{pmatrix}\quad\Rightarrow\quad\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}=\begin{pmatrix}31\\-21\\4\end{pmatrix}$$

(ii) Similarly for $\mathbf{b}=\begin{pmatrix}-1&3&1\end{pmatrix}^T$ we have

$$\mathbf{x} = \mathbf{A}^{-1}\mathbf{b} = \begin{pmatrix} 84 & -37 & 7 \\ -58 & 26 & -5 \\ 11 & -5 & 1 \end{pmatrix}\begin{pmatrix} -1 \\ 3 \\ 1 \end{pmatrix} = \begin{pmatrix} -188 \\ 131 \\ -25 \end{pmatrix} \quad \Rightarrow \quad \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -188 \\ 131 \\ -25 \end{pmatrix}$$

6. We cannot convert the following $\begin{pmatrix} 0 & 1 \\ 2 & 3 \end{pmatrix}$ into **LU** factorization because of the first zero.

We would need to carry out the row operation of exchanging rows which is not allowed for **LU** factorization.