Complete Solutions to Exercises 7.4

Provide solutions for questions 10 and 11.

1. (a) We are given the matrix $\mathbf{A}=\begin{pmatrix}1&0\\0&2\end{pmatrix}$ and we need to find an orthogonal matrix $\mathbf{Q}$ which diagonlizes this matrix. The eigenvalues and eigenvectors of matrix $\mathbf{A}$ are:

$$\lambda_1=1,\ \mathbf{u}=\begin{pmatrix}1\\0\end{pmatrix} \text{ and } \lambda_2=2,\ \mathbf{v}=\begin{pmatrix}0\\1\end{pmatrix}$$

The eigenvectors $\mathbf{u}$ and $\mathbf{v}$ are orthogonal and are also have a norm of 1 so we do **not** need to normalize them. Thus

$$\mathbf{Q}=\left(\mathbf{u}\ \vdots\ \mathbf{v}\right)=\begin{pmatrix}1&0\\0&1\end{pmatrix}=\mathbf{I}$$

Check that $\mathbf{Q}^T\mathbf{AQ}=\mathbf{D}$ where $\mathbf{D}=\begin{pmatrix}1&0\\0&2\end{pmatrix}=\mathbf{A}$:

$$\mathbf{Q}^T\mathbf{AQ}=\mathbf{IAI}=\mathbf{A}$$

Note that matrix $\mathbf{A}$ is already a diagonal matrix so no surprise that the orthogonal diagonalizing matrix is the identity matrix.

(b) We are given the matrix $\mathbf{A}=\begin{pmatrix}1&1\\1&1\end{pmatrix}$. Need to find the eigenvalues and associated eigenvectors:

$$\lambda_1=0,\ \mathbf{u}=\begin{pmatrix}1\\-1\end{pmatrix} \text{ and } \lambda_2=2,\ \mathbf{v}=\begin{pmatrix}1\\1\end{pmatrix}$$

Since $\mathbf{A}$ is a symmetric matrix therefore eigenvectors $\mathbf{u}$ and $\mathbf{v}$ are orthogonal. *Is the orthogonal matrix* $\mathbf{Q}=\left(\mathbf{u}\quad\mathbf{v}\right)$ *?*

No because $\mathbf{u}$ and $\mathbf{v}$ are not normalized. We need to normalize these eigenvectors. *How?* By dividing by the norm (length). *What is the norm of* $\mathbf{u}$ *equal to?*

$$\|\mathbf{u}\|^2=\left\|\begin{pmatrix}1\\-1\end{pmatrix}\right\|^2=\begin{pmatrix}1\\-1\end{pmatrix}\cdot\begin{pmatrix}1\\-1\end{pmatrix}=1^2+(-1)^2=2$$

This is the norm squared $\|\mathbf{u}\|^2=2$, so taking the square root of both sides gives $\|\mathbf{u}\|=\sqrt{2}$. Similarly we have

$$\|\mathbf{v}\|^2=\begin{pmatrix}1\\1\end{pmatrix}\cdot\begin{pmatrix}1\\1\end{pmatrix}=1^2+1^2=2$$

Thus $\|\mathbf{v}\|=\sqrt{2}$. *What are our perpendicular unit eigenvectors* $\mathbf{u}$ *and* $\mathbf{v}$ *equal to?*

$$\mathbf{u}=\frac{1}{\|\mathbf{u}\|}\mathbf{u}=\frac{1}{\sqrt{2}}\begin{pmatrix}1\\-1\end{pmatrix} \text{ and } \mathbf{v}=\frac{1}{\|\mathbf{v}\|}\mathbf{v}=\frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix}$$

Hence our orthogonal matrix $\mathbf{Q}=\left(\mathbf{u}\quad\mathbf{v}\right)=\frac{1}{\sqrt{2}}\begin{pmatrix}1&1\\-1&1\end{pmatrix}$. Checking that $\mathbf{AQ}=\mathbf{QD}$:

$$\mathbf{AQ}=\frac{1}{\sqrt{2}}\begin{pmatrix}1&1\\1&1\end{pmatrix}\begin{pmatrix}1&1\\-1&1\end{pmatrix}=\frac{1}{\sqrt{2}}\begin{pmatrix}0&2\\0&2\end{pmatrix}$$

$$\mathbf{QD}=\frac{1}{\sqrt{2}}\begin{pmatrix}1&1\\-1&1\end{pmatrix}\begin{pmatrix}0&0\\0&2\end{pmatrix}=\frac{1}{\sqrt{2}}\begin{pmatrix}0&2\\0&2\end{pmatrix}$$

(c) Similarly we are given $\mathbf{A}=\begin{pmatrix}2 & 1\\ 1 & 2\end{pmatrix}$ and we need to find the eigenvalues and corresponding eigenvectors:

$$\lambda_1=1,\ \mathbf{u}=\begin{pmatrix}1\\ -1\end{pmatrix} \text{ and } \lambda_2=3,\ \mathbf{v}=\begin{pmatrix}1\\ 1\end{pmatrix}$$

Again we need to normalize the eigenvectors:

$$\|\mathbf{u}\|^2=\begin{pmatrix}1\\ -1\end{pmatrix}\cdot\begin{pmatrix}1\\ -1\end{pmatrix}=1^2+(-1)^2=2$$

Thus $\|\mathbf{u}\|=\sqrt{2}$ and similarly we have $\|\mathbf{v}\|=\sqrt{2}$. Therefore our orthonormal eigenvectors are

$$\mathbf{u}=\frac{1}{\|\mathbf{u}\|}\mathbf{u}=\frac{1}{\sqrt{2}}\begin{pmatrix}1\\ -1\end{pmatrix} \text{ and } \mathbf{v}=\frac{1}{\|\mathbf{v}\|}\mathbf{v}=\frac{1}{\sqrt{2}}\begin{pmatrix}1\\ 1\end{pmatrix}$$

Hence our orthogonal matrix $\mathbf{Q}=\begin{pmatrix}\mathbf{u} & \mathbf{v}\end{pmatrix}=\frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\ -1 & 1\end{pmatrix}$. Checking that $\mathbf{AQ}=\mathbf{QD}$ with $\mathbf{A}=\begin{pmatrix}2 & 1\\ 1 & 2\end{pmatrix}$ and $\mathbf{Q}=\frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\ -1 & 1\end{pmatrix}$:

$$\mathbf{AQ}=\frac{1}{\sqrt{2}}\begin{pmatrix}2 & 1\\ 1 & 2\end{pmatrix}\begin{pmatrix}1 & 1\\ -1 & 1\end{pmatrix}=\frac{1}{\sqrt{2}}\begin{pmatrix}1 & 3\\ -1 & 3\end{pmatrix}$$

$$\mathbf{QD}=\frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\ -1 & 1\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & 3\end{pmatrix}=\frac{1}{\sqrt{2}}\begin{pmatrix}1 & 3\\ -1 & 3\end{pmatrix}$$

(d) We need to find an orthogonal matrix **Q** that diagonalizes $\mathbf{A}=\begin{pmatrix}5 & 12\\ 12 & -5\end{pmatrix}$. First we determine the eigenvalues and eigenvectors:

$$\lambda_1=13,\ \mathbf{u}=\begin{pmatrix}3\\ 2\end{pmatrix} \text{ and } \lambda_2=-13,\ \mathbf{v}=\begin{pmatrix}-2\\ 3\end{pmatrix}$$

Normalising these eigenvectors gives

$$\|\mathbf{u}\|^2=\begin{pmatrix}3\\ 2\end{pmatrix}\cdot\begin{pmatrix}3\\ 2\end{pmatrix}=3^2+2^2=13$$

Hence $\|\mathbf{u}\|=\sqrt{13}$. Similarly we have $\|\mathbf{v}\|=\sqrt{13}$. Our orthonormal eigenvectors are

$$\mathbf{u}=\frac{1}{\sqrt{13}}\begin{pmatrix}3\\ 2\end{pmatrix} \text{ and } \mathbf{v}=\frac{1}{\sqrt{13}}\begin{pmatrix}-2\\ 3\end{pmatrix}$$

Our orthogonal vector is $\mathbf{Q}=\begin{pmatrix}\mathbf{u} & \mathbf{v}\end{pmatrix}=\frac{1}{\sqrt{13}}\begin{pmatrix}3 & -2\\ 2 & 3\end{pmatrix}$. Checking that $\mathbf{AQ}=\mathbf{QD}$:

$$\mathbf{AQ}=\frac{1}{\sqrt{13}}\begin{pmatrix}5 & 12\\ 12 & -5\end{pmatrix}\begin{pmatrix}3 & -2\\ 2 & 3\end{pmatrix}=\frac{1}{\sqrt{13}}\begin{pmatrix}39 & 26\\ 26 & -39\end{pmatrix}$$

$$\mathbf{QD}=\frac{1}{\sqrt{13}}\begin{pmatrix}3 & -2\\ 2 & 3\end{pmatrix}\begin{pmatrix}13 & 0\\ 0 & -13\end{pmatrix}=\frac{1}{\sqrt{13}}\begin{pmatrix}39 & 26\\ 26 & -39\end{pmatrix}$$

2. The working out is very similar to question 1.

(a) We are given matrix $\mathbf{A}=\begin{pmatrix}9 & 3\\ 3 & 1\end{pmatrix}$. Determining the eigenvalues and eigenvectors gives

$$\lambda_1=0,\ \mathbf{u}=\begin{pmatrix}-1\\ 3\end{pmatrix} \text{ and } \lambda_2=10,\ \mathbf{v}=\begin{pmatrix}3\\ 1\end{pmatrix}$$

Normalising the eigenvector **u** and **v** yields:

$$\|\mathbf{u}\|^2=\begin{pmatrix}-1\\ 3\end{pmatrix}\cdot\begin{pmatrix}-1\\ 3\end{pmatrix}=(-1)^2+3^2=10$$

$$\|\mathbf{v}\|^2=\begin{pmatrix}3\\ 1\end{pmatrix}\cdot\begin{pmatrix}3\\ 1\end{pmatrix}=3^2+1^2=10$$

Taking the square root of both the norm squares gives $\|\mathbf{u}\|=\sqrt{10}$ and $\|\mathbf{v}\|=\sqrt{10}$. *What are the orthonormal (perpendicular unit) eigenvectors?*

$$\mathbf{u}=\frac{1}{\|\mathbf{u}\|}\mathbf{u}=\frac{1}{\sqrt{10}}\begin{pmatrix}-1\\ 3\end{pmatrix} \text{ and } \mathbf{v}=\frac{1}{\|\mathbf{v}\|}\mathbf{v}=\frac{1}{\sqrt{10}}\begin{pmatrix}3\\ 1\end{pmatrix}$$

Thus $\mathbf{Q}=\begin{pmatrix}\mathbf{u} & \mathbf{v}\end{pmatrix}=\frac{1}{\sqrt{10}}\begin{pmatrix}-1 & 3\\ 3 & 1\end{pmatrix}$. We can check $\mathbf{AQ}=\mathbf{QD}$:

$$\mathbf{AQ}=\frac{1}{\sqrt{10}}\begin{pmatrix}9 & 3\\ 3 & 1\end{pmatrix}\begin{pmatrix}-1 & 3\\ 3 & 1\end{pmatrix}=\frac{1}{\sqrt{10}}\begin{pmatrix}0 & 30\\ 0 & 10\end{pmatrix}$$

$$\mathbf{QD}=\frac{1}{\sqrt{10}}\begin{pmatrix}-1 & 3\\ 3 & 1\end{pmatrix}\begin{pmatrix}0 & 0\\ 0 & 10\end{pmatrix}=\frac{1}{\sqrt{10}}\begin{pmatrix}0 & 30\\ 0 & 10\end{pmatrix}$$

(b) Similarly for $\mathbf{A}=\begin{pmatrix}3 & \sqrt{2}\\ \sqrt{2} & 2\end{pmatrix}$ we have

$$\lambda_1=4,\ \mathbf{u}=\begin{pmatrix}2\\ \sqrt{2}\end{pmatrix} \text{ and } \lambda_2=1,\ \mathbf{v}=\begin{pmatrix}\sqrt{2}\\ -2\end{pmatrix}$$

What else do we need to do?
Normalize the eigenvectors **u** and **v**:

$$\|\mathbf{u}\|^2=\begin{pmatrix}2\\ \sqrt{2}\end{pmatrix}\cdot\begin{pmatrix}2\\ \sqrt{2}\end{pmatrix}=2^2+\left(\sqrt{2}\right)^2=6$$

$$\|\mathbf{v}\|^2=\begin{pmatrix}\sqrt{2}\\ -2\end{pmatrix}\cdot\begin{pmatrix}\sqrt{2}\\ -2\end{pmatrix}=\left(\sqrt{2}\right)^2+(-2)^2=6$$

Taking the square root of both sides gives $\|\mathbf{u}\|=\|\mathbf{v}\|=\sqrt{6}$. *What are the perpendicular unit eigenvectors?*

$$\mathbf{u}=\frac{1}{\|\mathbf{u}\|}\mathbf{u}=\frac{1}{\sqrt{6}}\begin{pmatrix}2\\ \sqrt{2}\end{pmatrix} \text{ and } \mathbf{v}=\frac{1}{\|\mathbf{v}\|}\mathbf{v}=\frac{1}{\sqrt{6}}\begin{pmatrix}\sqrt{2}\\ -2\end{pmatrix}$$

Our orthogonal matrix is given by $\mathbf{Q}=\begin{pmatrix}\mathbf{u} & \mathbf{v}\end{pmatrix}=\frac{1}{\sqrt{6}}\begin{pmatrix}2 & \sqrt{2}\\ \sqrt{2} & -2\end{pmatrix}$. Checking $\mathbf{AQ}=\mathbf{QD}$:

$$\mathbf{AQ}=\frac{1}{\sqrt{6}}\begin{pmatrix}3 & \sqrt{2}\\ \sqrt{2} & 2\end{pmatrix}\begin{pmatrix}2 & \sqrt{2}\\ \sqrt{2} & -2\end{pmatrix}=\frac{1}{\sqrt{6}}\begin{pmatrix}8 & \sqrt{2}\\ 4\sqrt{2} & -2\end{pmatrix}$$

$$\mathbf{QD}=\frac{1}{\sqrt{6}}\begin{pmatrix}2 & \sqrt{2}\\ \sqrt{2} & -2\end{pmatrix}\begin{pmatrix}4 & 0\\ 0 & 1\end{pmatrix}=\frac{1}{\sqrt{6}}\begin{pmatrix}8 & \sqrt{2}\\ 4\sqrt{2} & -2\end{pmatrix}$$

(c) We need to find an orthogonal matrix which diagonalizes $\mathbf{A}=\begin{pmatrix}-5 & \sqrt{3}\\ \sqrt{3} & -3\end{pmatrix}$. The eigenvectors and eigenvalues of this matrix are:

$$\lambda_1=-6,\ \mathbf{u}=\begin{pmatrix}-3\\ \sqrt{3}\end{pmatrix} \text{ and } \lambda_2=-2,\ \mathbf{v}=\begin{pmatrix}\sqrt{3}\\ 3\end{pmatrix}$$

We need to normalize the eigenvectors. *How?*
By dividing each vector by its norm:

$$\|\mathbf{u}\|^2=\begin{pmatrix}-3\\ \sqrt{3}\end{pmatrix}\cdot\begin{pmatrix}-3\\ \sqrt{3}\end{pmatrix}=(-3)^2+\left(\sqrt{3}\right)^2=12$$

$$\|\mathbf{v}\|^2=\begin{pmatrix}\sqrt{3}\\ 3\end{pmatrix}\cdot\begin{pmatrix}\sqrt{3}\\ 3\end{pmatrix}=\left(\sqrt{3}\right)^2+3^2=12$$

By taking the square root we have $\|\mathbf{u}\|=\|\mathbf{v}\|=\sqrt{12}$. *What are the normalized eigenvectors* $\mathbf{u}$ *and* $\mathbf{v}$ *equal to?*

$$\hat{\mathbf{u}}=\frac{1}{\|\mathbf{u}\|}\mathbf{u}=\frac{1}{\sqrt{12}}\begin{pmatrix}-3\\ \sqrt{3}\end{pmatrix} \text{ and } \hat{\mathbf{v}}=\frac{1}{\|\mathbf{v}\|}\mathbf{v}=\frac{1}{\sqrt{12}}\begin{pmatrix}\sqrt{3}\\ 3\end{pmatrix}$$

Hence the orthogonal vector is $\mathbf{Q}=\begin{pmatrix}\hat{\mathbf{u}} & \hat{\mathbf{v}}\end{pmatrix}=\frac{1}{\sqrt{12}}\begin{pmatrix}-3 & \sqrt{3}\\ \sqrt{3} & 3\end{pmatrix}$. We can check $\mathbf{AQ}=\mathbf{QD}$:

$$\mathbf{AQ}=\frac{1}{\sqrt{12}}\begin{pmatrix}-5 & \sqrt{3}\\ \sqrt{3} & -3\end{pmatrix}\begin{pmatrix}-3 & \sqrt{3}\\ \sqrt{3} & 3\end{pmatrix}=\frac{1}{\sqrt{12}}\begin{pmatrix}18 & -2\sqrt{3}\\ -6\sqrt{3} & -6\end{pmatrix}$$

$$\mathbf{QD}=\frac{1}{\sqrt{12}}\begin{pmatrix}-3 & \sqrt{3}\\ \sqrt{3} & 3\end{pmatrix}\begin{pmatrix}-6 & 0\\ 0 & -2\end{pmatrix}=\frac{1}{\sqrt{12}}\begin{pmatrix}18 & -2\sqrt{3}\\ -6\sqrt{3} & -6\end{pmatrix}$$

(d) For the given matrix $\mathbf{A}=\begin{pmatrix}5 & \sqrt{12}\\ \sqrt{12} & 1\end{pmatrix}$ we need to find an orthogonal matrix which diagonalizes **A**. First we find the eigenvalues and eigenvectors:

$$\lambda_1=-1,\ \mathbf{u}=\begin{pmatrix}-2\\ \sqrt{12}\end{pmatrix} \text{ and } \lambda_2=7,\ \mathbf{v}=\begin{pmatrix}\sqrt{12}\\ 2\end{pmatrix}$$

Normalising the eigenvectors gives:

$$\|\mathbf{u}\|^2=\begin{pmatrix}-2\\ \sqrt{12}\end{pmatrix}\cdot\begin{pmatrix}-2\\ \sqrt{12}\end{pmatrix}=(-2)^2+\left(\sqrt{12}\right)^2=16$$

We have $\|\mathbf{u}\|^2=16$. Taking the square root gives $\|\mathbf{u}\|=4$. Similarly we have $\|\mathbf{v}\|=4$.
Normalized eigenvectors are

$$\mathbf{u} = \frac{1}{4}\begin{pmatrix} -2 \\ \sqrt{12} \end{pmatrix} \text{ and } \mathbf{v} = \frac{1}{4}\begin{pmatrix} \sqrt{12} \\ 2 \end{pmatrix}$$

What is the orthogonal matrix $\mathbf{Q}$ *equal to?*

$$\mathbf{Q} = \begin{pmatrix} \mathbf{u} & \mathbf{v} \end{pmatrix} = \frac{1}{4}\begin{pmatrix} -2 & \sqrt{12} \\ \sqrt{12} & 2 \end{pmatrix}$$

$$= \frac{1}{4}\begin{pmatrix} -2 & 2\sqrt{3} \\ 2\sqrt{3} & 2 \end{pmatrix} \qquad \left[\text{Because } \sqrt{12} = \sqrt{4\times 3} = 2\sqrt{3}\right]$$

$$= \frac{1}{2}\begin{pmatrix} -1 & \sqrt{3} \\ \sqrt{3} & 1 \end{pmatrix} \qquad \left[\begin{array}{l}\text{Taking out 2 and} \\ \text{writing } 2/4 = 1/2\end{array}\right]$$

Checking that $\mathbf{AQ} = \mathbf{QD}$ with $\mathbf{A} = \begin{pmatrix} 5 & \sqrt{12} \\ \sqrt{12} & 1 \end{pmatrix} = \begin{pmatrix} 5 & 2\sqrt{3} \\ 2\sqrt{3} & 1 \end{pmatrix}$:

$$\mathbf{AQ} = \frac{1}{2}\begin{pmatrix} 5 & 2\sqrt{3} \\ 2\sqrt{3} & 1 \end{pmatrix}\begin{pmatrix} -1 & \sqrt{3} \\ \sqrt{3} & 1 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 & 7\sqrt{3} \\ -\sqrt{3} & 7 \end{pmatrix}$$

$$\mathbf{QD} = \frac{1}{2}\begin{pmatrix} -1 & \sqrt{3} \\ \sqrt{3} & 1 \end{pmatrix}\begin{pmatrix} -1 & 0 \\ 0 & 7 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 & 7\sqrt{3} \\ -\sqrt{3} & 7 \end{pmatrix}$$

3. (a) We are given the matrix $\mathbf{A} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}$. Since $\mathbf{A}$ is a diagonal matrix so the orthogonal matrix is the identity matrix $\mathbf{I}$. The matrix has eigenvalues 1, 2 and 3. Thus the orthogonal matrix is given by

$$\mathbf{Q} = \mathbf{I} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Checking that $\mathbf{Q}^T\mathbf{AQ} = \mathbf{I}^T\mathbf{AI} = \mathbf{A}$. The diagonal matrix is the given matrix $\mathbf{A}$.

(b) <u>Steps 1 and 2</u>: We are given the matrix $\mathbf{A} = \begin{pmatrix} 2 & 2 & 2 \\ 2 & 2 & 2 \\ 2 & 2 & 2 \end{pmatrix}$. The eigenvalues and corresponding eigenvectors are

$$\lambda_1 = 0,\ \mathbf{u} = \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix},\ \lambda_2 = 0,\ \mathbf{v} = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} \text{ and } \lambda_3 = 6,\ \mathbf{w} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

<u>Step 3</u>: We need to check that the eigenvectors $\mathbf{u}$ and $\mathbf{v}$ are orthogonal since they belong to the same eigenvalue $\lambda_1 = 0$ and $\lambda_2 = 0$. We have

$$\mathbf{u}\cdot\mathbf{v} = \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix}\cdot\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} = (1\times 1) + (1\times(-1)) + (-2\times 0) = 0$$

Because $\mathbf{u}\cdot\mathbf{v}=0$ therefore **u** and **v** are orthogonal.
<u>Step 4</u>: We need to normalize these eigenvectors:

$$\mathbf{u}=\frac{1}{\sqrt{1^2+1^2+(-2)^2}}\begin{pmatrix}1\\1\\-2\end{pmatrix}=\frac{1}{\sqrt{6}}\begin{pmatrix}1\\1\\-2\end{pmatrix}$$

$$\mathbf{v}=\frac{1}{\sqrt{1^2+(-1)^2+0^2}}\begin{pmatrix}1\\-1\\0\end{pmatrix}=\frac{1}{\sqrt{2}}\begin{pmatrix}1\\-1\\0\end{pmatrix}$$

$$\mathbf{w}=\frac{1}{\sqrt{1^2+1^2+1^2}}\begin{pmatrix}1\\1\\1\end{pmatrix}=\frac{1}{\sqrt{3}}\begin{pmatrix}1\\1\\1\end{pmatrix}$$

<u>Step 5</u>: Since the eigenvectors $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ are orthonormal vectors therefore

$$\mathbf{Q}=\begin{pmatrix}\mathbf{u} & \mathbf{v} & \mathbf{w}\end{pmatrix}=\begin{pmatrix}1/\sqrt{6} & 1/\sqrt{2} & 1/\sqrt{3}\\ 1/\sqrt{6} & -1/\sqrt{2} & 1/\sqrt{3}\\ -2/\sqrt{6} & 0 & 1/\sqrt{3}\end{pmatrix}$$

<u>Step 6</u>: By using MATLAB or otherwise you can check that we have

$$\mathbf{Q}^T\mathbf{A}\mathbf{Q}=\mathbf{D}=\begin{pmatrix}0&0&0\\0&0&0\\0&0&6\end{pmatrix}$$

(c) <u>Steps 1 and 2</u>: For the given matrix $\mathbf{A}=\begin{pmatrix}0&0&0\\0&1&1\\0&1&1\end{pmatrix}$ we have

$$\lambda_1=0,\ \mathbf{u}=\begin{pmatrix}1\\0\\0\end{pmatrix},\ \lambda_2=0,\ \mathbf{v}=\begin{pmatrix}0\\1\\-1\end{pmatrix}\ \text{ and }\ \lambda_3=2,\ \mathbf{w}=\begin{pmatrix}0\\1\\1\end{pmatrix}$$

<u>Step 3</u>: We need to check that the eigenvectors **u** and **v** are orthogonal since they belong to the same eigenvalue $\lambda_1=0$ and $\lambda_2=0$. We have

$$\mathbf{u}\cdot\mathbf{v}=\begin{pmatrix}1\\0\\0\end{pmatrix}\cdot\begin{pmatrix}0\\1\\-1\end{pmatrix}=(1\times 0)+(0\times 1)+(0\times(-1))=0$$

Because $\mathbf{u}\cdot\mathbf{v}=0$ therefore the eigenvectors **u** and **v** are orthogonal.
<u>Step 4</u>: The eigenvector **u** is already normalized but we need to normalize **v** and **w**:

$$\|\mathbf{v}\|^2=\begin{pmatrix}0\\1\\-1\end{pmatrix}\cdot\begin{pmatrix}0\\1\\-1\end{pmatrix}=0^2+1^2+(-1)^2=2$$

$$\|\mathbf{w}\|^2=\begin{pmatrix}0\\1\\1\end{pmatrix}\cdot\begin{pmatrix}0\\1\\1\end{pmatrix}=0^2+1^2+1^2=2$$

Taking the square root gives $\|\mathbf{v}\| = \sqrt{2}$ and $\|\mathbf{w}\| = \sqrt{2}$. Normalized eigenvectors are

$$\mathbf{v} = \frac{1}{\|\mathbf{v}\|}\mathbf{v} = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} \quad \text{and} \quad \mathbf{w} = \frac{1}{\|\mathbf{w}\|}\mathbf{w} = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$$

Step 5:The orthogonal matrix $\mathbf{Q}$ is given by

$$\mathbf{Q} = \begin{pmatrix} \mathbf{u} & \mathbf{v} & \mathbf{w} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1/\sqrt{2} & 1/\sqrt{2} \\ 0 & -1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix}$$

$$= \frac{1}{\sqrt{2}}\begin{pmatrix} \sqrt{2} & 0 & 0 \\ 0 & 1 & 1 \\ 0 & -1 & 1 \end{pmatrix} \qquad \left[\begin{array}{l} \text{Taking out } \dfrac{1}{\sqrt{2}} \text{ and} \\ \text{rewriting } 1 = \dfrac{1}{\sqrt{2}}\sqrt{2} \end{array}\right]$$

Step 6:You may like to check that $\mathbf{Q}^T\mathbf{AQ} = \mathbf{D}$ where $\mathbf{D}$ is the diagonal matrix given by

$$\mathbf{D} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

4. (a) Steps1 and 2:

We are given the symmetric matrix $\mathbf{A} = \begin{pmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{pmatrix}$ and we first find the eigenvalues and the corresponding eigenvectors:

$$\lambda_1 = -1,\ \mathbf{u} = \begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix},\ \lambda_2 = -1,\ \mathbf{v} = \begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix} \quad \text{and} \quad \lambda_3 = 5,\ \mathbf{w} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

Step 3: This time the eigenvectors $\mathbf{u}$ and $\mathbf{v}$ are **not** orthogonal because

$$\mathbf{u}\cdot\mathbf{v} = \begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix}\cdot\begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix} = (-2\times 2) + (1\times(-3)) + (1\times 1) = -6 \qquad (*)$$

We need to convert $\mathbf{u}$ and $\mathbf{v}$, eigenvectors belonging to the same eigenvalue $\lambda_1 = \lambda_2 = -1$, into an orthogonal set of vectors. *How?*
By applying the Gram Schmidt Process (4-10) which is

$$\mathbf{q}_1 = \mathbf{u} = \begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix}$$

$$\mathbf{q}_2 = \mathbf{v} - \frac{\mathbf{v}\cdot\mathbf{q}_1}{\|\mathbf{q}_1\|^2}\mathbf{q}_1 \qquad (**)$$

What is $\mathbf{v}\cdot\mathbf{q}_1$ and $\|\mathbf{q}_1\|^2$ *equal to?*

Since $\mathbf{q}_1 = \mathbf{u}$ therefore $\mathbf{v}\cdot\mathbf{q}_1 = \mathbf{v}\cdot\mathbf{u} = \mathbf{u}\cdot\mathbf{v} \underset{\text{By }(*)}{=} -6$. *What is* $\|\mathbf{q}_1\|^2$ *equal to?*

$$\|\mathbf{q}_1\|^2 = \mathbf{q}_1\cdot\mathbf{q}_1 = \mathbf{u}\cdot\mathbf{u} = \begin{pmatrix}-2\\1\\1\end{pmatrix}\cdot\begin{pmatrix}-2\\1\\1\end{pmatrix} = (-2)^2+1^2+1^2 = 6 \qquad (\dagger)$$

Substituting $\mathbf{v}\cdot\mathbf{q}_1 = -6$ and $\|\mathbf{q}_1\|^2 = 6$ into $\mathbf{q}_2 = \mathbf{v} - \frac{\mathbf{v}\cdot\mathbf{q}_1}{\|\mathbf{q}_1\|^2}\mathbf{q}_1$ gives

$$\mathbf{q}_2 = \mathbf{v} - \frac{\mathbf{v}\cdot\mathbf{q}_1}{\|\mathbf{q}_1\|^2}\mathbf{q}_1 = \begin{pmatrix}2\\-3\\1\end{pmatrix} - \frac{-6}{6}\begin{pmatrix}-2\\1\\1\end{pmatrix} = \begin{pmatrix}2+(-2)\\-3+1\\1+1\end{pmatrix} = \begin{pmatrix}0\\-2\\2\end{pmatrix} = 2\begin{pmatrix}0\\-1\\1\end{pmatrix}$$

We can ignore the scalar 2 in $\mathbf{q}_2$ and normalize $\mathbf{q}_2^* = \begin{pmatrix}0\\-1\\1\end{pmatrix}$.

Now $\mathbf{q}_1$, $\mathbf{q}_2^*$ and $\mathbf{w}$ are orthogonal. (Check by finding the dot product.)
What else do we need to do in order to find the orthogonal matrix $\mathbf{Q}$*?*
<u>Step 4</u>: Need to normalize $\mathbf{q}_1$, $\mathbf{q}_2^*$ and $\mathbf{w}$:

$$\mathbf{q}_1 = \frac{1}{\|\mathbf{q}_1\|}\mathbf{q}_1 = \frac{1}{\sqrt{6}}\begin{pmatrix}-2\\1\\1\end{pmatrix} \qquad \left[\begin{array}{l}\text{By } (\dagger)\ \|\mathbf{q}_1\|^2 = 6\\ \text{and } \|\mathbf{q}_1\| = \sqrt{6}\end{array}\right]$$

$$\mathbf{q}_2^* = \frac{1}{\|\mathbf{q}_2\|}\mathbf{q}_2^* = \frac{1}{\sqrt{(-1)^2+1^2}}\begin{pmatrix}0\\-1\\1\end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix}0\\-1\\1\end{pmatrix}$$

$$\mathbf{w} = \frac{1}{\|\mathbf{w}\|}\mathbf{w} = \frac{1}{\sqrt{1^2+1^2+1^2}}\begin{pmatrix}1\\1\\1\end{pmatrix} = \frac{1}{\sqrt{3}}\begin{pmatrix}1\\1\\1\end{pmatrix}$$

<u>Step 5</u>: Our orthonormal vectors are $\mathbf{q}_1$, $\mathbf{q}_2^*$ and $\mathbf{w}$. Thus

$$\mathbf{Q} = \begin{pmatrix}\mathbf{q}_1 & \mathbf{q}_2^* & \mathbf{w}\end{pmatrix} = \begin{pmatrix}-2/\sqrt{6} & 0 & 1/\sqrt{3}\\ 1/\sqrt{6} & -1/\sqrt{2} & 1/\sqrt{3}\\ 1/\sqrt{6} & 1/\sqrt{2} & 1/\sqrt{3}\end{pmatrix}$$

<u>Step 6</u>: You can check that $\mathbf{Q}^T\mathbf{A}\mathbf{Q} = \mathbf{D}$ where $\mathbf{D}$ is a diagonal matrix by using MATLAB. MATLAB instructions are as follows:

```
A=[1 2 2; 2 1 2; 2 2 1]

A =

   1   2   2
   2   1   2
   2   2   1
```

```
 Q=[-2/sqrt(6) 0 1/sqrt(3); 1/sqrt(6) -1/sqrt(2) 1/sqrt(3); 1/sqrt(6) 1/sqrt(2) 1/sqrt(3)]
Q =
  -0.8165         0    0.5774
   0.4082   -0.7071    0.5774
   0.4082    0.7071    0.5774
>> Q'*A*Q
ans =
  -1.0000   -0.0000    0.0000
        0   -1.0000         0
        0         0    5.0000
```

Thus we have the diagonal matrix **D** with the entries on the leading diagonal as the eigenvalues of the matrix **A**.

(b) Steps1 and 2:

We are given the symmetric matrix $\mathbf{A}=\begin{pmatrix}2&1&1\\1&2&1\\1&1&2\end{pmatrix}$ which is matrix in part (a) but with the 1's and 2's interchanged. The eigenvalues and eigenvectors of this matrix are

$$\lambda_1=1,\ \mathbf{u}=\begin{pmatrix}-1\\1\\0\end{pmatrix},\ \lambda_2=1,\ \mathbf{v}=\begin{pmatrix}-1\\0\\1\end{pmatrix}\ \text{ and }\ \lambda_3=4,\ \mathbf{w}=\begin{pmatrix}1\\1\\1\end{pmatrix}$$

Step 3: Again the eigenvectors **u** and **v** belonging to the same eigenvalue are **not** orthogonal because

$$\mathbf{u}\cdot\mathbf{v}=\begin{pmatrix}-1\\1\\0\end{pmatrix}\cdot\begin{pmatrix}-1\\0\\1\end{pmatrix}=(-1)^2+0+0=1 \qquad (\dagger)$$

Need to use the Gram Schmidt Process to convert these **u** and **v** into orthogonal vectors:

$$\mathbf{q}_1=\mathbf{u}=\begin{pmatrix}-1\\1\\0\end{pmatrix}$$

$$\mathbf{q}_2=\mathbf{v}-\frac{\mathbf{v}\cdot\mathbf{q}_1}{\|\mathbf{q}_1\|^2}\mathbf{q}_1$$

Since $\mathbf{q}_1=\mathbf{u}$ therefore $\mathbf{v}\cdot\mathbf{q}_1=\mathbf{v}\cdot\mathbf{u}=\mathbf{u}\cdot\mathbf{v}\underset{\text{By }(\dagger)}{=}1$. Also

$$\|\mathbf{q}_1\|^2=\mathbf{q}_1\cdot\mathbf{q}_1=\mathbf{u}\cdot\mathbf{u}=\begin{pmatrix}-1\\1\\0\end{pmatrix}\cdot\begin{pmatrix}-1\\1\\0\end{pmatrix}=(-1)^2+1^2+0^2=2$$

Substituting $\mathbf{v}\cdot\mathbf{q}_1=1$ and $\|\mathbf{q}_1\|^2=2$ into $\mathbf{q}_2=\mathbf{v}-\frac{\mathbf{v}\cdot\mathbf{q}_1}{\|\mathbf{q}_1\|^2}\mathbf{q}_1$ gives

$$\mathbf{q}_2 = \mathbf{v} - \frac{\mathbf{v}\cdot\mathbf{q}_1}{\|\mathbf{q}_1\|^2}\mathbf{q}_1$$

$$= \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} - \frac{1}{2}\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -1+1/2 \\ 0-1/2 \\ 1 \end{pmatrix} = \begin{pmatrix} -1/2 \\ -1/2 \\ 1 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} -1 \\ -1 \\ 2 \end{pmatrix}$$

Now $\mathbf{q}_1$, $\mathbf{q}_2$ and $\mathbf{w}$ are orthogonal. (You may check by showing the dot product is zero.)

<u>Step 4</u>: Need to normalize $\mathbf{q}_1$, $\mathbf{q}_2$ and $\mathbf{w}$:

$$\mathbf{q}_1 = \frac{1}{\|\mathbf{q}_1\|}\mathbf{q}_1 = \frac{1}{\sqrt{2}}\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} \qquad \left[\text{Because } \|\mathbf{q}_1\|^2 = 2\right]$$

Remember for normalising we can ignore the fraction. For normalising $\mathbf{q}_2$ we ignore the ½:

$$\mathbf{q}_2^* = \frac{1}{\|\mathbf{q}_2^*\|}\mathbf{q}_2^* = \frac{1}{\sqrt{(-1)^2+(-1)^2+2^2}}\begin{pmatrix} -1 \\ -1 \\ 2 \end{pmatrix} = \frac{1}{\sqrt{6}}\begin{pmatrix} -1 \\ -1 \\ 2 \end{pmatrix}$$

$$\mathbf{w} = \frac{1}{\|\mathbf{w}\|}\mathbf{w} = \frac{1}{\sqrt{1^2+1^2+1^2}}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \frac{1}{\sqrt{3}}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

<u>Step 5</u>: Our orthonormal vectors are $\mathbf{q}_1$, $\mathbf{q}_2^*$ and $\mathbf{w}$. Thus

$$\mathbf{Q} = \begin{pmatrix} \mathbf{q}_1 & \mathbf{q}_2^* & \mathbf{w} \end{pmatrix} = \begin{pmatrix} -1/\sqrt{2} & -1/\sqrt{6} & 1/\sqrt{3} \\ 1/\sqrt{2} & -1/\sqrt{6} & 1/\sqrt{3} \\ 0 & 2/\sqrt{6} & 1/\sqrt{3} \end{pmatrix}$$

<u>Step 6</u>: You can check that $\mathbf{Q}^T\mathbf{A}\mathbf{Q} = \mathbf{D}$ where $\mathbf{D}$ is a diagonal matrix by using MATLAB. MATLAB instructions are as follows:

```
A=[2 1 1; 1 2 1; 1 1 2]
A =
   2   1   1
   1   2   1
   1   1   2
>> Q=[-1/sqrt(2) -1/sqrt(6) 1/sqrt(3); 1/sqrt(2) -1/sqrt(6) 1/sqrt(3); 0 2/sqrt(6) 1/sqrt(3)]
Q =
  -0.7071  -0.4082   0.5774
   0.7071  -0.4082   0.5774
        0   0.8165   0.5774
>> Q'*A*Q
ans =
   1.0000        0        0
        0   1.0000   0.0000
        0        0   4.0000
```

(c) Steps1 and 2:

We are given the matrix $\mathbf{A}=\begin{pmatrix}-5 & 4 & 2\\ 4 & -5 & 2\\ 2 & 2 & -8\end{pmatrix}$. The eigenvalues and corresponding eigenvectors are

$$\lambda_1=-9,\ \mathbf{u}=\begin{pmatrix}-1\\ 1\\ 0\end{pmatrix},\ \lambda_2=-9,\ \mathbf{v}=\begin{pmatrix}1\\ 0\\ -2\end{pmatrix}\ \text{and}\ \lambda_3=0,\ \mathbf{w}=\begin{pmatrix}2\\ 2\\ 1\end{pmatrix}$$

Step 3:

Check the eigenvectors **u** and **v** belonging to the same eigenvalue $\lambda_1=\lambda_2=-9$ are orthogonal:

$$\mathbf{u}\cdot\mathbf{v}=\begin{pmatrix}-1\\ 1\\ 0\end{pmatrix}\cdot\begin{pmatrix}1\\ 0\\ -2\end{pmatrix}=(-1\times 1)+(1\times 0)+(0\times(-2))=-1 \qquad (☺)$$

Thus **u** and **v** are **not** orthogonal so we need to use the Gram Schmidt Process to place these into an orthogonal set $\mathbf{q}_1$ and $\mathbf{q}_2$ say:

$$\mathbf{u}=\mathbf{q}_1$$

$$\mathbf{q}_2=\mathbf{v}-\frac{\mathbf{v}\cdot\mathbf{q}_1}{\|\mathbf{q}_1\|^2}\mathbf{q}_1 \qquad (*)$$

Similar to solutions to parts (a) and (b) above we have

$$\mathbf{v}\cdot\mathbf{q}_1=\mathbf{v}\cdot\mathbf{u}=\mathbf{u}\cdot\mathbf{v}=-1 \qquad [\text{By } (☺)]$$

$$\|\mathbf{q}_1\|^2=\begin{pmatrix}-1\\ 1\\ 0\end{pmatrix}\cdot\begin{pmatrix}-1\\ 1\\ 0\end{pmatrix}=(-1)^2+1^2+0^2=2$$

Substituting $\mathbf{v}\cdot\mathbf{q}_1=-1$ and $\|\mathbf{q}_1\|^2=2$ into (*) gives

$$\mathbf{q}_2=\mathbf{v}-\frac{\mathbf{v}\cdot\mathbf{q}_1}{\|\mathbf{q}_1\|^2}\mathbf{q}_1=\begin{pmatrix}1\\ 0\\ -2\end{pmatrix}-\frac{-1}{2}\begin{pmatrix}-1\\ 1\\ 0\end{pmatrix}$$

$$=\begin{pmatrix}1\\ 0\\ -2\end{pmatrix}+\frac{1}{2}\begin{pmatrix}-1\\ 1\\ 0\end{pmatrix}=\begin{pmatrix}1-1/2\\ 0+1/2\\ -2+0\end{pmatrix}=\begin{pmatrix}1/2\\ 1/2\\ -2\end{pmatrix}=\frac{1}{2}\begin{pmatrix}1\\ 1\\ -4\end{pmatrix}$$

Note that the vectors $\mathbf{q}_1$ and $\mathbf{q}_2$ are a basis for the eigenspace E_{-9}.

Step 4:

We need to normalize our orthogonal set of vectors $\mathbf{q}_1$, $\mathbf{q}_2$ and **w**.

$$\mathbf{q}_1=\frac{1}{\|\mathbf{q}_1\|}\mathbf{q}_1=\frac{1}{\sqrt{2}}\begin{pmatrix}-1\\ 1\\ 0\end{pmatrix}$$

Remember to normalize $\mathbf{q}_2$ we can remove the fraction and normalize $\mathbf{q}_2^*$ where $\mathbf{q}_2^*$ is the vector $\mathbf{q}_2$ but without the ½. *What is the norm of* $\mathbf{q}_2^*$*?*

$$\left\|\mathbf{q}_2^*\right\|^2 = \begin{pmatrix} 1 \\ 1 \\ -4 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \\ -4 \end{pmatrix} = 1^2 + 1^2 + (-4)^2 = 18$$

Taking the square root of both sides gives $\left\|\mathbf{q}_2^*\right\| = \sqrt{18} = 3\sqrt{2}$. We have

$$\mathbf{q}_2^* = \frac{1}{\left\|\mathbf{q}_2^*\right\|}\mathbf{q}_2^* = \frac{1}{3\sqrt{2}} \begin{pmatrix} 1 \\ 1 \\ -4 \end{pmatrix}$$

The norm squared of **w** is

$$\left\|\mathbf{w}\right\|^2 = \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix} = 2^2 + 2^2 + 1^2 = 9$$

Taking the square root gives $\left\|\mathbf{w}\right\| = \sqrt{9} = 3$. The normalized vector is

$$\mathbf{w} = \frac{1}{\left\|\mathbf{w}\right\|}\mathbf{w} = \frac{1}{3}\begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix}$$

<u>Step 5</u>: Form the matrix **Q** whose columns are the orthonormal vectors $\mathbf{q}_1 = \frac{1}{\sqrt{2}}\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$,

$\mathbf{q}_2^* = \frac{1}{3\sqrt{2}}\begin{pmatrix} 1 \\ 1 \\ -4 \end{pmatrix}$ and $\mathbf{w} = \frac{1}{3}\begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix}$:

$$\mathbf{Q} = \begin{pmatrix} \mathbf{q}_1 & \mathbf{q}_2^* & \mathbf{w} \end{pmatrix} = \begin{pmatrix} -1/\sqrt{2} & 1/3\sqrt{2} & 2/3 \\ 1/\sqrt{2} & 1/3\sqrt{2} & 2/3 \\ 0 & -4/3\sqrt{2} & 1/3 \end{pmatrix}$$

<u>Step 6</u>: Check that $\mathbf{Q}^{-1}\mathbf{A}\mathbf{Q} = \mathbf{Q}^T\mathbf{A}\mathbf{Q} = \mathbf{D}$ by using MATLAB.

```
>> A=[-5 4 2; 4 -5 2; 2 2 -8]
A =
   -5    4    2
    4   -5    2
    2    2   -8
>> Q=[-1/sqrt(2) 1/(3*sqrt(2)) 2/3; 1/sqrt(2) 1/(3*sqrt(2)) 2/3; 0 -4/(3*sqrt(2)) 1/3]
Q =
  -0.7071    0.2357    0.6667
   0.7071    0.2357    0.6667
        0   -0.9428    0.3333
>> Q'*A*Q
ans =
  -9.0000         0         0
        0   -9.0000         0
        0    0.0000    0.0000
```

5. The matrix $\mathbf{A}=\begin{pmatrix}1 & 1\\ 1 & 1\end{pmatrix}$ is identical to the matrix of question 1(b).

Our orthogonal matrix **Q** was given by $\mathbf{Q}=\frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\ -1 & 1\end{pmatrix}$.

To find the powers of matrix **A** we use the formula $\mathbf{A}^m=\mathbf{Q}\mathbf{D}^m\mathbf{Q}^T$ where **D** is the diagonal matrix whose leading diagonal entries are the eigenvalues of **A**, that is $\mathbf{D}=\begin{pmatrix}0 & 0\\ 0 & 2\end{pmatrix}$.

$$\begin{aligned}
\mathbf{A}^{10}=\mathbf{Q}\mathbf{D}^{10}\mathbf{Q}^T &= \frac{1}{\sqrt{2}}\frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\ -1 & 1\end{pmatrix}\begin{pmatrix}0 & 0\\ 0 & 2\end{pmatrix}^{10}\begin{pmatrix}1 & -1\\ 1 & 1\end{pmatrix}\\
&=\frac{1}{2}\begin{pmatrix}1 & 1\\ -1 & 1\end{pmatrix}\begin{pmatrix}0 & 0\\ 0 & 2^{10}\end{pmatrix}\begin{pmatrix}1 & -1\\ 1 & 1\end{pmatrix}\\
&=\frac{2^{10}}{2}\begin{pmatrix}1 & 1\\ -1 & 1\end{pmatrix}\begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}1 & -1\\ 1 & 1\end{pmatrix}\\
&=2^9\begin{pmatrix}0 & 1\\ 0 & 1\end{pmatrix}\begin{pmatrix}1 & -1\\ 1 & 1\end{pmatrix}\\
&=2^9\begin{pmatrix}1 & 1\\ 1 & 1\end{pmatrix}=2^9\mathbf{A}
\end{aligned}$$

To prove that $\mathbf{A}^m=2^{m-1}\mathbf{A}$ we use the above with m in place of 10.

$$\begin{aligned}
\mathbf{A}^{m}=\mathbf{Q}\mathbf{D}^{m}\mathbf{Q}^T &= \frac{1}{2}\begin{pmatrix}1 & 1\\ -1 & 1\end{pmatrix}\begin{pmatrix}0 & 0\\ 0 & 2^m\end{pmatrix}\begin{pmatrix}1 & -1\\ 1 & 1\end{pmatrix}\\
&=\frac{2^m}{2}\begin{pmatrix}1 & 1\\ -1 & 1\end{pmatrix}\begin{pmatrix}0 & 0\\ 0 & 2^m\end{pmatrix}\begin{pmatrix}1 & -1\\ 1 & 1\end{pmatrix}\\
&=2^{m-1}\begin{pmatrix}1 & 1\\ 1 & 1\end{pmatrix}=2^{m-1}\mathbf{A}
\end{aligned}$$

6. We need to prove that if **A** is a diagonal matrix then orthogonal diagonalizing matrix $\mathbf{Q}=\mathbf{I}$.

Proof.

Let **A** be a diagonal matrix and if $\mathbf{Q}=\mathbf{I}$ then

$$\mathbf{Q}^{-1}\mathbf{A}\mathbf{Q}=\mathbf{I}^{-1}\mathbf{A}\mathbf{I}=\mathbf{I}\mathbf{A}\mathbf{I}=\mathbf{A}$$

Therefore the identity matrix **I** **orthogonally** diagonalizes **A**. Thus our orthogonal matrix $\mathbf{Q}=\mathbf{I}$.

■

7 (a) We need to prove that the zero matrix **O** is orthogonally diagonalizable.

Proof.

The eigenvalues λ of the zero matrix **O** is given by

$$\det(\mathbf{O}-\lambda\mathbf{I})=0 \text{ which gives } \lambda_1=\lambda_2=\lambda_3=\cdots=\lambda_n=0$$

The corresponding eigenvectors for these $\lambda_1=\lambda_2=\lambda_3=\cdots=\lambda_n=0$ are

$$\mathbf{v}_1 = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \ \mathbf{v}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \end{pmatrix}, \ \cdots, \ \mathbf{v}_n = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}$$

The orthogonal matrix is $\mathbf{Q} = \begin{pmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \cdots & \mathbf{v}_n \end{pmatrix} = \mathbf{I}$. Thus

$$\mathbf{Q}^{-1}\mathbf{O}\mathbf{Q} = \mathbf{I}^{-1}\mathbf{O}\mathbf{I} = \mathbf{I}\mathbf{O}\mathbf{I} = \mathbf{O}$$

Thus the zero matrix **O** is diagonalizable.
We could also use the result of question 6 above since **O** is a diagonal matrix.

■

(b) We need to prove that the identity matrix **I** is orthogonally diagonalizable.
Proof.
Since the identity matrix is a diagonal matrix therefore we can use the result of question 6 to deduce that the identity matrix is diagonalizable and the orthogonal matrix $\mathbf{Q} = \mathbf{I}$.

■

8. We need to show that $\mathbf{A} = \begin{pmatrix} a & b \\ b & c \end{pmatrix}$ is diagonalizable and find the orthogonal matrix **Q**.

Since **A** is a symmetric matrix therefore by Theorem (7-20) we conclude that the matrix **A** is orthogonally diagonalizable. The eigenvalues of matrix **A** are given by

$$\begin{aligned} \det(\mathbf{A} - \lambda\mathbf{I}) &= \det\begin{pmatrix} a-\lambda & b \\ b & c-\lambda \end{pmatrix} \\ &= (a-\lambda)(c-\lambda) - b^2 \\ &= \lambda^2 - (a+c)\lambda + ac - b^2 = 0 \qquad (*) \end{aligned}$$

We have a quadratic equation which we can solve by using the quadratic equation formula:

$$\begin{aligned} \lambda &= \frac{(a+c) \pm \sqrt{(a+c)^2 - 4(ac-b^2)}}{2} \\ &= \frac{(a+c) \pm \sqrt{a^2 + c^2 + 2ac - 4ac + 4b^2}}{2} \\ &= \frac{(a+c) \pm \sqrt{a^2 + c^2 - 2ac + 4b^2}}{2} \\ &= \frac{(a+c) \pm \sqrt{(a-c)^2 + 4b^2}}{2} \end{aligned}$$

Thus we have $\lambda_1 = \dfrac{(a+c) + \sqrt{(a-c)^2 + 4b^2}}{2}$ and $\lambda_2 = \dfrac{(a+c) - \sqrt{(a-c)^2 + 4b^2}}{2}$.

What are the eigenvectors **u** *and* **v** *equal to?*
Let **u** be the eigenvector for λ_1 then:

$$(\mathbf{A} - \lambda_1\mathbf{I})\mathbf{u} = \begin{pmatrix} a-\lambda_1 & b \\ b & c-\lambda_1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

We have the simultaneous equations:

$$(a-\lambda_1)x+by=0$$
$$bx+(c-\lambda_1)y=0$$

Solving these for x and y gives $x=b,\ \ y=\lambda_1-a$ because substituting these $x=b,\ \ y=\lambda_1-a$ into the above equations gives

$$\begin{aligned}(a-\lambda_1)x+by &= (a-\lambda_1)b+b(\lambda_1-a)=0\\ bx+(c-\lambda_1)y &= bb+(c-\lambda_1)(\lambda_1-a)\\ &= b^2+c\lambda_1-ca-\lambda_1^2+\lambda_1 a\\ &= -\left(\lambda_1^2-(c+a)\lambda_1+ca-b^2\right) \underset{\text{By }(*)}{=} 0\end{aligned}$$

Thus $\mathbf{u}=\begin{pmatrix} b \\ \lambda_1-a\end{pmatrix}$. Similarly let $\mathbf{v}$ be the eigenvector for λ_2 then

$$(\mathbf{A}-\lambda_2\mathbf{I})\mathbf{u}=\begin{pmatrix} a-\lambda_2 & b \\ b & c-\lambda_2\end{pmatrix}\begin{pmatrix} x \\ y\end{pmatrix}=\begin{pmatrix} 0 \\ 0\end{pmatrix}$$

This gives the simultaneous equations

$$(a-\lambda_2)x+by=0$$
$$bx+(c-\lambda_2)y=0$$

Solving these gives $x=\lambda_2-c,\ \ y=b$ because substituting these $x=\lambda_2-c,\ \ y=b$ into the above equations gives

$$\begin{aligned}(a-\lambda_2)x+by &= (a-\lambda_2)(\lambda_2-c)+bb\\ &= a\lambda_2-ac-\lambda_2^2+\lambda_2 c+b^2\\ &= -\left(\lambda_2^2-(a+c)\lambda_2+ac-b^2\right) \underset{\text{By }(*)}{=} 0\end{aligned}$$

$$b(\lambda_2-c)+(c-\lambda_2)b=0$$

Hence the eigenvector is $\mathbf{v}=\begin{pmatrix} \lambda_2-c \\ b\end{pmatrix}$.

We need to show that the eigenvectors are orthogonal because we **may** have a repeated eigenvalue such as $\lambda_1=\lambda_2$:

$$\begin{aligned}\mathbf{u}\cdot\mathbf{v}=\begin{pmatrix} b \\ \lambda_1-a\end{pmatrix}\cdot\begin{pmatrix} \lambda_2-c \\ b\end{pmatrix} &= b(\lambda_2-c)+(\lambda_1-a)b\\ &= b\left(\lambda_2+\lambda_1-(a+c)\right) \qquad (**)\end{aligned}$$

How do we show this is zero?

By using the hint which was

$x^2+px+q=0$ has roots a and b then $a+b=-p$

and (*) from above

$$\lambda^2-(a+c)\lambda+ac-b^2=0 \qquad (*)$$

has the roots λ_1 and λ_2. Therefore $\lambda_1+\lambda_2=a+c$ and substituting this into (**) we have

$$\mathbf{u}\cdot\mathbf{v}=b\left(\lambda_2+\lambda_1-(a+c)\right)=b(0)=0$$

Hence **u** and **v** are orthogonal set of vectors. We need to normalize these eigenvectors:

$$\|\mathbf{u}\|^2 = \left\|\begin{pmatrix} b \\ \lambda_1 - a\end{pmatrix}\right\|^2 = \begin{pmatrix} b \\ \lambda_1 - a\end{pmatrix}\cdot\begin{pmatrix} b \\ \lambda_1 - a\end{pmatrix}$$
$$= b^2 + (\lambda_1 - a)^2$$

Taking the square root gives the norm $\|\mathbf{u}\| = \sqrt{b^2 + (\lambda_1 - a)^2}$. The normalized vector is

$$\mathbf{u} = \frac{1}{\|\mathbf{u}\|}\mathbf{u} = \frac{1}{\sqrt{b^2 + (\lambda_1 - a)^2}}\begin{pmatrix} b \\ \lambda_1 - a\end{pmatrix}$$

Similarly we normalize the eigenvector **v**:

$$\|\mathbf{v}\|^2 = \begin{pmatrix} \lambda_2 - c \\ b\end{pmatrix}\cdot\begin{pmatrix} \lambda_2 - c \\ b\end{pmatrix} = (\lambda_2 - c)^2 + b^2$$

Taking the square root of both sides gives the norm $\|\mathbf{v}\| = \sqrt{(\lambda_2 - c)^2 + b^2}$.

$$\mathbf{v} = \frac{1}{\|\mathbf{v}\|}\mathbf{v} = \frac{1}{\sqrt{(\lambda_2 - c)^2 + b^2}}\begin{pmatrix} \lambda_2 - c \\ b\end{pmatrix}$$

What is our orthogonal vector which diagonalizes the given matrix ***A****?*

$$\mathbf{Q} = \begin{pmatrix}\mathbf{u} & \mathbf{v}\end{pmatrix} = \begin{pmatrix} \dfrac{b}{\sqrt{b^2 + (\lambda_1 - a)^2}} & \dfrac{\lambda_2 - c}{\sqrt{(\lambda_2 - c)^2 + b^2}} \\ \dfrac{\lambda_1 - a}{\sqrt{b^2 + (\lambda_1 - a)^2}} & \dfrac{b}{\sqrt{(\lambda_2 - c)^2 + b^2}} \end{pmatrix}$$

9. We need to prove that if **Q** orthogonally diagonalizes the matrix **A** then **Q** also diagonalizes the matrix $\mathbf{A}^{-1}$.

Proof.

We have $\mathbf{Q}^T\mathbf{A}\mathbf{Q} = \mathbf{D}$ because **Q** diagonalizes the matrix **A**. Taking the inverse of both sides we have

$$\begin{aligned}
\mathbf{D}^{-1} &= \left(\mathbf{Q}^T\mathbf{A}\mathbf{Q}\right)^{-1} \\
&= \mathbf{Q}^{-1}\mathbf{A}^{-1}\left(\mathbf{Q}^T\right)^{-1} && \left[\text{Because } (\mathbf{ABC})^{-1} = \mathbf{C}^{-1}\mathbf{B}^{-1}\mathbf{A}^{-1}\right] \\
&= \mathbf{Q}^{-1}\mathbf{A}^{-1}\left(\mathbf{Q}^{-1}\right)^{-1} && \left[\text{Since } \mathbf{Q} \text{ is orthogonal so } \mathbf{Q}^{-1} = \mathbf{Q}^T\right] \\
&= \mathbf{Q}^{-1}\mathbf{A}^{-1}\mathbf{Q} && \left[\text{Because } \left(\mathbf{Q}^{-1}\right)^{-1} = \mathbf{Q}\right]
\end{aligned}$$

We have $\mathbf{Q}^{-1}\mathbf{A}^{-1}\mathbf{Q} = \mathbf{D}^{-1}$ where $\mathbf{D}^{-1}$ is the diagonal matrix with the leading diagonal entries $\frac{1}{\lambda_1}, \frac{1}{\lambda_2}, \frac{1}{\lambda_3}, \cdots$ and $\frac{1}{\lambda_n}$ which are the eigenvalues of $\mathbf{A}^{-1}$. Thus **Q** orthogonally diagonalizes the matrix $\mathbf{A}^{-1}$.

■

10. We need to prove that (7-17) which claims:

> Let **A** be a symmetric matrix with distinct eigenvalues $\lambda_1, \lambda_2, \cdots, \lambda_n$ and corresponding eigenvectors $\mathbf{v}_1, \mathbf{v}_2, \cdots$ and $\mathbf{v}_n$. Then these eigenvectors are orthogonal.

We prove this result by induction. The 3 steps of induction are:
1. Check the result is correct for some base case $n = k_0$.
2. Assume the result is true for $n = k$.
3. Prove the result for $n = k + 1$ by using steps 1 and 2.

Proof.

Step 1:

For $n = 2$ the result is true because of:

> Proposition (7-16). Let **A** be a symmetric matrix. If λ_1 and λ_2 are *distinct* eigenvalues of matrix **A** then their corresponding eigenvectors **u** and **v** respectively are orthogonal.

Step 2:

Assume the result is true for $n = k$:

The eigenvectors $\mathbf{v}_1,\ \mathbf{v}_2,\ \cdots,\ \mathbf{v}_k$ which belong to $\lambda_1,\ \lambda_2,\ \cdots,\ \lambda_k$ are orthogonal.

Step 3:

Required to prove this result for $n = k + 1$. We need to prove:

> The eigenvectors $\mathbf{v}_1,\ \mathbf{v}_2,\ \cdots,\ \mathbf{v}_{k+1}$ which belong to $\lambda_1,\ \lambda_2,\ \cdots,\ \lambda_{k+1}$ are orthogonal.

Consider $\mathbf{v}_1,\ \mathbf{v}_2,\ \cdots,\ \mathbf{v}_k,\ \mathbf{v}_{k+1}$. By step 2 we know that the eigenvectors $\mathbf{v}_1,\ \mathbf{v}_2,\ \cdots,\ \mathbf{v}_k$ are orthogonal. Since the eigenvalue λ_{k+1} associated with $\mathbf{v}_{k+1}$ is *distinct* from each of the other eigenvalues $\lambda_1,\ \lambda_2,\ \cdots,\ \lambda_k$ so by step 1 the eigenvector $\mathbf{v}_{k+1}$ is orthogonal to each of the eigenvectors $\mathbf{v}_1,\ \mathbf{v}_2,\ \cdots,\ \mathbf{v}_k$. This is our required result.

■