Complete Solutions to **Miscellaneous Exercises 2**

1. *How do we determine whether the given vectors are linearly **independent**?*
For linear **dependence** one of the vectors must be a multiple of the others. Since the first and last vectors in $\left\{\begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}1\\1\\1\end{bmatrix}, \begin{bmatrix}1\\0\\0\end{bmatrix}\right\}$ are identical therefore the vectors are linearly dependent.

2. This is impossible because Proposition (2-20) says that:

Every basis of $\mathbb{R}^n$ contains exactly n vectors.

This means that we need 3 vectors to span $\mathbb{R}^3$. Thus two vectors **cannot** span $\mathbb{R}^3$.

3. We are given $\mathbf{v}_1 = (1, \quad 1, \quad 1)$, $\mathbf{v}_2 = (1, \quad 0, \quad -1)$ and $\mathbf{v}_3 = (0, \quad 1, \quad 1)$.

(a) *How do we check for linear independence?*
By definition (2-12) which is

$$k_1\mathbf{v}_1 + k_2\mathbf{v}_2 + k_3\mathbf{v}_3 + \cdots + k_n\mathbf{v}_n = \mathbf{O} \quad \Rightarrow \quad k_1 = k_2 = k_3 = \cdots = k_n = 0$$

The vectors $\mathbf{v}_1 = (1, \quad 1, \quad 1)$, $\mathbf{v}_2 = (1, \quad 0, \quad -1)$ are linearly independent because

$$k_1\begin{pmatrix}1\\1\\1\end{pmatrix} + k_2\begin{pmatrix}1\\0\\-1\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix} \quad \text{gives} \quad k_1 = k_2 = 0$$

(b) *Is* $(3, \quad 2, \quad 1)$ *in the span of* $\mathbf{v}_1$ *and* $\mathbf{v}_2$*?*
Let c_1 and c_2 be scalars such that

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 = c_1\begin{pmatrix}1\\1\\1\end{pmatrix} + c_2\begin{pmatrix}1\\0\\-1\end{pmatrix} = \begin{pmatrix}3\\2\\1\end{pmatrix}$$

From the middle row we have $c_1 = 2$. From the top row we have

$$c_1 + c_2 = 3$$

Substituting $c_1 = 2$ into this gives $2 + c_2 = 3 \quad \Rightarrow \quad c_2 = 1$.
Since $c_1 = 2$ and $c_2 = 1$ satisfies the bottom row as well therefore the given vector $(3, \quad 2, \quad 1)$ is in the span of $\mathbf{v}_1$ and $\mathbf{v}_2$ because $2\mathbf{v}_1 + \mathbf{v}_2 = (3, \quad 2, \quad 1)$.

(c) By Proposition (2-20) which says that:

Every basis of $\mathbb{R}^n$ contains exactly n vectors.

We only need three vectors to be a basis for $\mathbb{R}^3$.
Also:

n linearly independent vectors in $\mathbb{R}^n$ form a basis for $\mathbb{R}^n$.

We only need to show that $\mathbf{v}_1$, $\mathbf{v}_2$ and $\mathbf{v}_3$ are linearly independent for them to be a basis for $\mathbb{R}^3$.
Let k_1, k_2 and k_3 be scalars such that

$$k_1\begin{pmatrix}1\\1\\1\end{pmatrix}+k_2\begin{pmatrix}1\\0\\-1\end{pmatrix}+k_3\begin{pmatrix}0\\1\\1\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix}$$

Solving these equations

$$\left.\begin{array}{ccccccc} k_1 & + & k_2 & & & = & 0\\ k_1 & & & + & k_3 & = & 0\\ k_1 & - & k_2 & + & k_3 & = & 0\end{array}\right\}\quad \text{gives } k_1=k_2=k_3=0$$

This means that the given vectors $\mathbf{v}_1=(1,\ 1,\ 1)$, $\mathbf{v}_2=(1,\ 0,\ -1)$ and $\mathbf{v}_3=(0,\ 1,\ 1)$ are linearly independent and we conclude that $\{\mathbf{v}_1,\ \mathbf{v}_2,\ \mathbf{v}_3\}$ is a basis for $\mathbb{R}^3$.

4. Let k_1, k_2 and k_3 be scalars such that

$$k_1\begin{pmatrix}2\\5\\-1\end{pmatrix}+k_2\begin{pmatrix}1\\6\\0\end{pmatrix}+k_3\begin{pmatrix}5\\2\\-4\end{pmatrix}=\begin{pmatrix}3\\1\\1\end{pmatrix}$$

We need to find scalars k_1, k_2 and k_3. *How?*

We can use elementary row operations on the augmented matrix whose columns are given by the column vectors in the above:

$$\begin{array}{c} R_1\\ R_2\\ R_3\end{array}\left(\begin{array}{ccc|c} 2 & 1 & 5 & 3\\ 5 & 6 & 2 & 1\\ -1 & 0 & -4 & 1\end{array}\right)$$

Multiplying the bottom row by -1 and interchanging with the top row we have

$$\begin{array}{c} R_1^\dagger=-R_3\\ R_2\\ R_3^\dagger=R_1\end{array}\left(\begin{array}{ccc|c} 1 & 0 & 4 & -1\\ 5 & 6 & 2 & 1\\ 2 & 1 & 5 & 3\end{array}\right)$$

Executing the elementary row operations $R_2-5R_1^\dagger$ and $R_3^\dagger-2R_1^\dagger$:

$$\begin{array}{c} R_1^\dagger\\ R_2^*=R_2-5R_1^\dagger\\ R_3^*=R_3^\dagger-2R_1^\dagger\end{array}\left(\begin{array}{ccc|c} 1 & 0 & 4 & -1\\ 0 & 6 & -18 & 6\\ 0 & 1 & -3 & 5\end{array}\right)$$

Carrying out the row operation $R_2^*-6R_3^*$ gives

$$\begin{array}{c} R_1^\dagger\\ R_2^{**}=R_2^*-6R_3^*\\ R_3^*\end{array}\left(\begin{array}{ccc|c} 1 & 0 & 4 & -1\\ 0 & 0 & 0 & -24\\ 0 & 1 & -3 & 3\end{array}\right)$$

The middle row shows that we have an inconsistent system so there are **no** k's such that

$$k_1\begin{pmatrix}2\\5\\-1\end{pmatrix}+k_2\begin{pmatrix}1\\6\\0\end{pmatrix}+k_3\begin{pmatrix}5\\2\\-4\end{pmatrix}=\begin{pmatrix}3\\1\\1\end{pmatrix}$$

This means that the given vector $(3, \quad 1, \quad 1)$ **cannot** be expressed as a linear combination of the vectors $(2, \quad 5, \quad -1)$, $(1, \quad 6, \quad 0)$, $(5, \quad 2, \quad -4)$.

5. (a) Let $c_1, \ c_2, \ \cdots, \ c_k$ be scalars. The given vectors $\{\mathbf{v}_1, \ \mathbf{v}_2, \ \cdots, \ \mathbf{v}_k\}$ are linearly dependent means that these scalars $c_1, \ c_2, \ \cdots, \ c_k$ are **not all** zero but they satisfy

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_k\mathbf{v}_k = \mathbf{O}$$

(b) Let $c_1, \ c_2, \ c_3$ be scalars satisfying

$$c_1\begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} + c_2\begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix} + c_3\begin{bmatrix} 1 \\ 7 \\ -6 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Writing these out as equations we have

$$\begin{array}{rcrcrcll} c_1 & + & 2c_2 & + & c_3 & = & 0 & \quad (1) \\ 2c_1 & - & c_2 & + & 7c_3 & = & 0 & \quad (2) \\ -c_1 & + & 3c_2 & - & 6c_3 & = & 0 & \quad (3) \end{array}$$

Adding the top (1) and bottom (3) equations gives

$$5c_2 - 5c_3 = 0 \quad \Rightarrow \quad c_2 = c_3$$

Let $c_3 = 1$ then $c_2 = c_3 = 1$. Substituting $c_2 = c_3 = 1$ into the top equation (1) yields

$$c_1 + 2 + 1 = 0 \quad \Rightarrow \quad c_1 = -3$$

Hence by substituting $c_1 = -3, \ c_2 = 1$ and $c_3 = 1$ into the above we have

$$-3\begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} + \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix} + \begin{bmatrix} 1 \\ 7 \\ -6 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

This means that the given vectors are linearly dependent because we have non-zero scalars.

6. (a) Two different bases of $\mathbb{R}^2$ are $\left\{\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}\right\}$ and $\left\{\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix}\right\}$.

(b) (i) The given set $\{(0, \quad 0, \quad 1), \ (0, \quad 2, \quad 1), \ (3, \quad 2, \quad 1), \ (4, \quad 5, \quad 6)\}$ **cannot** form a basis for $\mathbb{R}^3$ because Proposition (2-20) says that:

Every basis of $\mathbb{R}^n$ contains exactly n vectors.

which means that we need 3 vectors for $\mathbb{R}^3$ but we are given 4.

<u>Checking Linear Independence</u>:

We have $\{(0, \quad 0, \quad 1), \ (0, \quad 2, \quad 1), \ (3, \quad 2, \quad 1), \ (4, \quad 5, \quad 6)\}$ is linearly dependent.

Why?

Because by

Proposition $(2\text{-}13)$. If $\mathbf{v}_1, \ \mathbf{v}_2, \ \cdots, \ \mathbf{v}_m$ are distinct vectors in $\mathbb{R}^n$ where $m > n$ then these vectors are linearly dependent.

Since $\{(0, \quad 0, \quad 1), \ (0, \quad 2, \quad 1), \ (3, \quad 2, \quad 1), \ (4, \quad 5, \quad 6)\}$ is a set of 4 distinct vectors in $\mathbb{R}^3$ therefore by Proposition (2-13) these vectors are linearly dependent.

<u>Checking Span</u>:

Consider the first three vectors $\{(0,\ 0,\ 1),\ (0,\ 2,\ 1),\ (3,\ 2,\ 1)\}$:

Let k_1, k_2 and k_3 be scalars such that

$$k_1\begin{pmatrix}0\\0\\1\end{pmatrix}+k_2\begin{pmatrix}0\\2\\1\end{pmatrix}+k_3\begin{pmatrix}3\\2\\1\end{pmatrix}=\begin{pmatrix}a\\b\\c\end{pmatrix}\qquad\left[\begin{pmatrix}a\\b\\c\end{pmatrix}\text{ is an arbitrary vector in }\mathbb{R}^3\right]$$

From the first row we have

$$3k_3 = a \text{ which gives } k_3 = \frac{a}{3}$$

From the middle row we have

$$2k_2 + 2k_3 = b$$

Substituting $k_3 = \frac{a}{3}$ into this yields

$$2k_2 + 2\left(\frac{a}{3}\right) = b \quad\Rightarrow\quad 2k_2 = b - \frac{2a}{3} \quad\Rightarrow\quad k_2 = \frac{3b-2a}{6}$$

By the bottom row we have

$$k_1 + k_2 + k_3 = c$$

Substituting $k_2 = \frac{3b-2a}{6}$ and $k_3 = \frac{a}{3}$ into this

$$k_1 + \frac{3b-2a}{6} + \frac{a}{3} = c \;\Rightarrow\; k_1 = c - \frac{3b-2a}{6} - \frac{a}{3} = c - \frac{b}{2}$$

Since with the scalars $k_1 = c - \frac{b}{2}$, $k_2 = \frac{3b-2a}{6}$ and $k_3 = \frac{a}{3}$ we have

$$k_1\begin{pmatrix}0\\0\\1\end{pmatrix}+k_2\begin{pmatrix}0\\2\\1\end{pmatrix}+k_3\begin{pmatrix}3\\2\\1\end{pmatrix}=\begin{pmatrix}a\\b\\c\end{pmatrix}$$

therefore these first three vectors span $\mathbb{R}^3$ which means that the given four vectors

$$\{(0,\ 0,\ 1),\ (0,\ 2,\ 1),\ (3,\ 2,\ 1),\ (4,\ 5,\ 6)\}$$

span $\mathbb{R}^3$.

ii. Let k_1, k_2 and k_3 be scalars such that

$$k_1\begin{pmatrix}1\\0\\1\end{pmatrix}+k_2\begin{pmatrix}4\\3\\-1\end{pmatrix}+k_3\begin{pmatrix}5\\3\\-4\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix}$$

These equations are

$$\begin{array}{rcrcrcl} k_1 & + & 4k_2 & + & 5k_3 & = & 0 \qquad (1)\\ & & 3k_2 & + & 3k_3 & = & 0 \qquad (2)\\ k_1 & - & k_2 & - & 4k_3 & = & 0 \qquad (3)\end{array}$$

From the middle equation (2) we have

$$3k_2 + 3k_3 = 0 \text{ gives } k_2 = -k_3$$

Substituting $k_2 = -k_3$ into the top equation (1) yields

$$k_1 - 4k_3 + 5k_3 = k_1 + k_3 = 0 \text{ which means that } k_1 = -k_3$$

We have $k_1 = k_2 = -k_3$. Substituting this into the bottom equation (3):

$$-k_3 \quad - \quad (-k_3) \quad - \quad 4k_3 \quad = \quad 0 \quad \Rightarrow \ -4k_3 = 0 \ \Rightarrow \ k_3 = 0$$

Hence we have $k_1 = k_2 = -k_3 = 0$. Since $k_1 = k_2 = k_3 = 0$ therefore the given vectors $\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 4 \\ 3 \\ -1 \end{pmatrix}, \begin{pmatrix} 5 \\ 3 \\ -4 \end{pmatrix}$ are linearly independent so

n linearly independent vectors in $\mathbb{R}^n$ form a basis for $\mathbb{R}^n$.

We conclude that the given vectors are a basis for $\mathbb{R}^3$. Since they form a basis therefore they span $\mathbb{R}^3$.

7. A basis for $V = \mathbb{R}^5$, $S = \left\{ \begin{bmatrix} a+b \\ b \\ c \\ 0 \\ c+b \end{bmatrix} \ \middle| \ a,\ b,\ c \in \mathbb{R} \right\}$ is $\left\{ \begin{bmatrix} a \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} b \\ b \\ 0 \\ 0 \\ b \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ c \\ 0 \\ c \end{bmatrix} \right\}$.

8. The first 3 vectors are linearly independent because

$$k_1 \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix} + k_2 \begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{bmatrix} + k_3 \begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad \text{gives} \quad k_1 = k_2 = k_3 = 0$$

We can make the fourth vector out of these by

$$k_1 \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix} + k_2 \begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{bmatrix} + k_3 \begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ -1 \\ 0 \end{bmatrix} \quad \text{with} \quad k_1 = -1,\ k_2 = 1 \ \text{and} \ k_3 = 0$$

Similarly we have

$$k_1 \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix} + k_2 \begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{bmatrix} + k_3 \begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ -1 \end{bmatrix} \quad \text{with} \quad k_1 = -1,\ k_2 = 0 \ \text{and} \ k_3 = 1$$

$$k_1 \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix} + k_2 \begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{bmatrix} + k_3 \begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \end{bmatrix} \quad \text{with} \quad k_1 = 0,\ k_2 = -1 \ \text{and} \ k_3 = 1$$

The largest number of linearly independent vectors is 3 and they are the first 3 given vectors.

9. (a) *How do we show the given vectors are linearly independent?*

We need to show that:

(2-1) $\qquad k_1\mathbf{v}_1 + k_2\mathbf{v}_2 + k_3\mathbf{v}_3 + \cdots + k_n\mathbf{v}_n = \mathbf{O} \ \Rightarrow \ k_1 = k_2 = k_3 = \cdots = k_n = 0$

Let k_1, k_2 and k_3 be scalars such that

$$k_1\begin{pmatrix}3\\-1\\1\\0\end{pmatrix}+k_2\begin{pmatrix}4\\-7\\3\\2\end{pmatrix}+k_3\begin{pmatrix}3\\7\\-2\\-6\end{pmatrix}=\begin{pmatrix}0\\0\\0\\0\end{pmatrix}$$

Writing these out as equations we have

$$\begin{array}{rcrcrcll} 3k_1 & + & 4k_2 & + & 3k_3 & = & 0 & (1)\\ -k_1 & - & 7k_2 & + & 7k_3 & = & 0 & (2)\\ k_1 & + & 3k_2 & - & 2k_3 & = & 0 & (3)\\ & & 2k_2 & - & 6k_3 & = & 0 & (4) \end{array}$$

From the bottom equation (4) we have

$$2k_2 = 6k_3 \quad \Rightarrow \quad k_2 = 3k_3$$

Adding equations (2) and (3) gives

$$-4k_2 + 5k_3 = 0 \quad \Rightarrow \quad 4k_2 = 5k_3$$

Substituting the above $k_2 = 3k_3$ into this $4k_2 = 5k_3$ yields

$$4(3k_3) = 5k_3 \quad \Rightarrow \quad 12k_3 = 5k_3 \quad \Rightarrow \quad 7k_3 = 0 \quad \Rightarrow \quad k_3 = 0$$

Substituting $k_3 = 0$ into $k_2 = 3k_3$ gives $k_2 = 0$. Substituting $k_2 = 0$ and $k_3 = 0$ into the any of the first three equations (1), (2) and (3) gives $k_1 = 0$.

Since the only solution is with **all** the scalars equal to zero, $k_1 = 0$, $k_2 = 0$ and $k_3 = 0$, therefore the given vectors are linearly independent.

(b) It is enough to find λ for which the given vectors are linearly independent:

$$k_1\begin{pmatrix}1\\1\\2\end{pmatrix}+k_2\begin{pmatrix}1\\0\\-1\end{pmatrix}+k_3\begin{pmatrix}1\\-1\\\lambda\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix}$$

Writing these out as equations we have

$$\begin{array}{rcrcrcll} k_1 & + & k_2 & + & k_3 & = & 0 & (1)\\ k_1 & & & - & k_3 & = & 0 & (2)\\ 2k_1 & - & k_2 & + & \lambda k_3 & = & 0 & (3) \end{array}$$

From the middle equation (2) we have

$$k_1 - k_3 = 0 \text{ gives } k_1 = k_3$$

Substituting this $k_1 = k_3$ into the bottom equation (3) yields

$$2k_3 - k_2 + \lambda k_3 = 0 \quad \Rightarrow \quad (2+\lambda)k_3 = k_2$$

Putting this $(2+\lambda)k_3 = k_2$ and $k_1 = k_3$ into the top equation (1):

$$k_3 + (2+\lambda)k_3 + k_3 = 0$$

$$(4+\lambda)k_3 = 0 \quad \Rightarrow \quad \lambda \neq -4 \qquad [\text{Because } k_3 = 0]$$

Thus the given vectors are linearly independent provided that $\lambda \neq -4$. Hence we conclude that these vectors form a basis of $\mathbb{R}^3$ for all real values of λ providing $\lambda \neq -4$.

10. We need to find values of *c* so that the given vectors are linearly dependent.

Remember if vectors are linearly dependent then they **cannot** form a basis.

The vectors in the set $S = \{\mathbf{v}_1, \ \mathbf{v}_2, \ \mathbf{v}_3, \ \cdots, \ \mathbf{v}_n\}$ in $\mathbb{R}^m$ are linearly dependent $\Leftrightarrow$ one of these vectors, say $\mathbf{v}_k$, is a linear combination of the preceding vectors, that is $\mathbf{v}_k = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3 + \cdots + c_{k-1}\mathbf{v}_{k-1}$.

We only need to show that one of the vectors is a linear combination of the others. Let k_1 and k_2 be scalars such that

$$k_1\begin{pmatrix}1\\2\\4\end{pmatrix} + k_2\begin{pmatrix}2\\3\\5\end{pmatrix} = \begin{pmatrix}2\\7\\c\end{pmatrix}$$

Writing out these equations we have

$$\begin{array}{rcrcrl} k_1 & + & 2k_2 & = & 2 & \quad (1) \\ 2k_1 & + & 3k_2 & = & 7 & \quad (2) \\ 4k_1 & + & 5k_2 & = & c & \quad (3) \end{array}$$

Multiplying equation (1) by 2 and then subtracting equation (2):

$$\begin{array}{rrcrcr} & 2k_1 & + & 4k_2 & = & 4 \\ - & 2k_1 & + & 3k_2 & = & 7 \\ \hline & 0 & + & k_2 & = & -3 \end{array}$$

Substituting $k_2 = -3$ into equation (1):

$$k_1 - 6 = 2 \text{ implies } k_1 = 8$$

Substituting $k_1 = 8$ and $k_2 = -3$ into the bottom equation (3) gives

$$4(8) + 5(-3) = 17 = c$$

For $c = 17$ the given vectors are linearly dependent which means that they **cannot** form a basis for $\mathbb{R}^3$.

11. *Proof.*

Let c_1 and c_2 be scalars so that

$$c_1\mathbf{v}_1 + c_2(\mathbf{v}_1 + \mathbf{v}_2) = \mathbf{O}$$

$$(c_1 + c_2)\mathbf{v}_1 + c_2\mathbf{v}_2 = \mathbf{O}$$

Since we are given that $\mathbf{v}_1$ and $\mathbf{v}_2$ are linearly independent therefore

$$c_1 + c_2 = 0, \quad c_2 = 0$$

Substituting $c_2 = 0$ into $c_1 + c_2 = 0$ gives $c_1 = 0$. Both our scalars are zero, that is $c_1 = 0$ and $c_2 = 0$ is the only solution of $c_1\mathbf{v}_1 + c_2(\mathbf{v}_1 + \mathbf{v}_2) = \mathbf{O}$ therefore $\{\mathbf{v}_1, \ \mathbf{v}_1 + \mathbf{v}_2\}$ is linearly independent.

■

12. (a) The given vectors $\left\{\begin{pmatrix}1\\2\\3\end{pmatrix}, \begin{pmatrix}2\\-1\\5\end{pmatrix}\right\}$ do **not** form a basis for $\mathbb{R}^3$ because we only have two vectors in this set whilst every basis for $\mathbb{R}^3$ should have three vectors. This is because of Proposition (2-20) which says that:

Every basis of $\mathbb{R}^n$ contains exactly n vectors.

(b) Clearly it can be observed that the given vectors $\left\{\begin{pmatrix}1\\2\\3\end{pmatrix}, \begin{pmatrix}4\\5\\6\end{pmatrix}, \begin{pmatrix}6\\9\\12\end{pmatrix}\right\}$ are linearly dependent because

$$2\begin{pmatrix}1\\2\\3\end{pmatrix}+\begin{pmatrix}4\\5\\6\end{pmatrix}=\begin{pmatrix}6\\9\\12\end{pmatrix}$$

Since the given vectors are linearly dependent therefore they **cannot** form a basis for $\mathbb{R}^3$.

(c) The given vectors $\left\{\begin{pmatrix}1\\2\\3\end{pmatrix}, \begin{pmatrix}4\\5\\6\end{pmatrix}, \begin{pmatrix}7\\8\\9\end{pmatrix}, \begin{pmatrix}10\\11\\12\end{pmatrix}\right\}$ do **not** form a basis for $\mathbb{R}^3$. *Why not?*

Because by Proposition (2-20) which says that:

Every basis of $\mathbb{R}^n$ contains exactly n vectors.

we need exactly three vectors for a basis of $\mathbb{R}^3$. Since we are given four vectors therefore they **cannot** form a basis for $\mathbb{R}^3$.

13. (a) We need to check whether $\mathbf{u}=\begin{pmatrix}1, & 3, & 1\end{pmatrix}$ is in the span of $\mathbf{v}=\begin{pmatrix}4, & 2, & -1\end{pmatrix}$ and $\mathbf{w}=\begin{pmatrix}-3, & 1, & 2\end{pmatrix}$. *What does this mean?*

<u>Check</u>:

Check that the vector **u** is a linear combination of the vectors **v** and **w**.

A simple check is:

$$\begin{pmatrix}4\\2\\-1\end{pmatrix}+\begin{pmatrix}-3\\1\\2\end{pmatrix}=\begin{pmatrix}1\\3\\1\end{pmatrix} \quad \text{or} \quad \mathbf{v}+\mathbf{w}=\mathbf{u}$$

<u>Another Approach</u>:

Let k and c be scalars such that

$$k\mathbf{v}+c\mathbf{w}=k\begin{pmatrix}4\\2\\-1\end{pmatrix}+c\begin{pmatrix}-3\\1\\2\end{pmatrix}=\begin{pmatrix}1\\3\\1\end{pmatrix}$$

Writing these out as equations we have

$$\left.\begin{array}{rcrcr}4k & - & 3c & = & 1\\ 2k & + & c & = & 3\\ -k & + & 2c & = & 1\end{array}\right\} \quad \text{gives} \quad k=c=1$$

Since we have values for scalars k and c therefore the vector **u** is in the span of **v** and **w**.

(b) The vectors **u**, **v** and **w** are linearly dependent because

$$\mathbf{v}+\mathbf{w}=\mathbf{u} \quad \text{or} \quad \mathbf{v}+\mathbf{w}-\mathbf{u}=\mathbf{O}$$

14. (a) To check to see whether $\{\mathbf{u}_1,\ \mathbf{u}_2,\ \mathbf{u}_3\}$ is orthogonal we need to evaluate the inner product between these vectors:

$$\mathbf{u}_1 \cdot \mathbf{u}_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} -1 \\ 4 \\ 1 \end{bmatrix} = \left[1 \times (-1)\right] + (0 \times 4) + (1 \times 1) = 0$$

$$\mathbf{u}_1 \cdot \mathbf{u}_3 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix} = 2 + 0 - 2 = 0$$

$$\mathbf{u}_2 \cdot \mathbf{u}_3 = \begin{bmatrix} -1 \\ 4 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix} = -2 + 4 - 2 = 0$$

Since the inner (dot) product in each case is zero therefore $\{\mathbf{u}_1,\ \mathbf{u}_2,\ \mathbf{u}_3\}$ is a set of orthogonal vectors.

(b) By Proposition (2-20) which says that:

Every basis of $\mathbb{R}^n$ contains exactly n vectors.

We conclude that we need 3 vectors to be a basis for $\mathbb{R}^3$. Since $\{\mathbf{u}_1,\ \mathbf{u}_2,\ \mathbf{u}_3\}$ is a set of orthogonal vectors therefore they are linearly independent so are a basis for $\mathbb{R}^3$.

Let $k_1,\ k_2$ and k_3 be scalars such that

$$k_1 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + k_2 \begin{bmatrix} -1 \\ 4 \\ 1 \end{bmatrix} + k_3 \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix} = \begin{bmatrix} 8 \\ -4 \\ -3 \end{bmatrix} \qquad \left[\text{or } k_1\mathbf{u}_1 + k_2\mathbf{u}_2 + k_3\mathbf{u}_3 = \mathbf{x}\right]$$

Writing out these equations

$$\begin{array}{ccccccccc} k_1 & & -k_2 & + & 2k_3 & = & 8 & & (\dagger) \\ & & 4k_2 & + & k_3 & = & -4 & & (\dagger\dagger) \\ k_1 & + & k_2 & - & 2k_3 & = & -3 & & (\dagger\dagger\dagger) \end{array}$$

Adding the top (†) and bottom (†††) equations gives

$$2k_1 = 5 \text{ implies } k_1 = \frac{5}{2}$$

Multiply the middle equation (††) by 2 and adding the bottom equation (†††) to this

$$\begin{array}{ccccccc} & & 8k_2 & + & 2k_3 & = & -8 \\ k_1 & + & k_2 & - & 2k_3 & = & -3 \\ \hline k_1 & + & 9k_2 & + & 0 & = & -11 \end{array}$$

We have $k_1 + 9k_2 = -11$. Substituting the above $k_1 = \frac{5}{2}$ into this

$$\frac{5}{2} + 9k_2 = -11 \quad \Rightarrow \quad 9k_2 = -11 - \frac{5}{2} = -\frac{27}{2} \quad \Rightarrow \quad k_2 = -\frac{27}{18} = -\frac{3}{2}$$

Substituting $k_2 = -\frac{3}{2}$ into the middle equation (††) above yields

$$4\left(-\frac{3}{2}\right) + k_3 = -4 \quad \Rightarrow \quad k_3 = -4 + 6 = 2$$

We have scalar values $k_1 = \frac{5}{2}$, $k_2 = -\frac{3}{2}$ and $k_3 = 2$ such that

$$\frac{5}{2}\mathbf{u}_1 - \frac{3}{2}\mathbf{u}_2 + 2\mathbf{u}_3 = \mathbf{x}$$

15. (a) We have

$$\mathbf{u}\cdot\mathbf{u} = \begin{bmatrix}2\\0\\3\end{bmatrix}\cdot\begin{bmatrix}2\\0\\3\end{bmatrix} = 2^2 + 0^2 + 3^2 = 13$$

(b) The norm is given by the square root of the result in (a), that is

$$\|\mathbf{u}\| = \sqrt{\mathbf{u}\cdot\mathbf{u}} = \sqrt{13}$$

(c) A unit vector in the direction of **u** is given by

$$\mathbf{u} = \frac{1}{\|\mathbf{u}\|}\mathbf{u} = \frac{1}{\sqrt{13}}\begin{bmatrix}2\\0\\3\end{bmatrix}$$

(d) The inner product of **u** and **v** is given by

$$\mathbf{u}\cdot\mathbf{v} = \begin{bmatrix}2\\0\\3\end{bmatrix}\cdot\begin{bmatrix}6\\2\\-4\end{bmatrix} = (2\times 6) + (0\times 2) + (3\times(-4)) = 0$$

(e) The given vectors **u** and **v** are orthogonal because their inner product is zero as shown in part (d) above.

(f) The distance function $d(\mathbf{v},\ \mathbf{x})$ is defined by

$$d(\mathbf{v},\ \mathbf{x}) = \|\mathbf{v}-\mathbf{x}\|$$

We have

$$\begin{aligned} d(\mathbf{v},\ \mathbf{x}) &= \|\mathbf{v}-\mathbf{x}\| \\ &= \left\|\begin{bmatrix}6\\2\\-4\end{bmatrix} - \begin{bmatrix}2\\4\\0\end{bmatrix}\right\| = \left\|\begin{bmatrix}4\\-2\\-4\end{bmatrix}\right\| = \sqrt{4^2 + (-2)^2 + (-4)^2} = 6 \end{aligned}$$

16. (a) The scalar (or dot) product of $\mathbf{x} = (x_1,\ \cdots,\ x_n)$ and $\mathbf{y} = (y_1,\ \cdots,\ y_n)$ is defined as

$$\begin{aligned} \mathbf{x}\cdot\mathbf{y} &= \mathbf{x}^T\mathbf{y} \\ &= \begin{pmatrix}x_1 & \cdots & x_n\end{pmatrix}\begin{pmatrix}y_1\\ \vdots \\ y_n\end{pmatrix} = x_1y_1 + x_2y_2 + \cdots + x_ny_n \end{aligned}$$

(b) The norm of $\mathbf{x} = (x_1,\ \cdots,\ x_n)$ in $\mathbb{R}^n$ is defined as

$$\|\mathbf{x}\| = \sqrt{(\mathbf{x}\cdot\mathbf{x})} = \sqrt{x_1^{\,2} + x_2^{\,2} + \cdots + x_n^{\,2}}$$

(c) The angle θ between two vectors **u** and **v** is given by

$$\cos(\theta) = \frac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{u}\|\|\mathbf{v}\|} \qquad (*)$$

We can use (*) to find the angle θ between the given vectors:

$$\mathbf{u} = (-1,\ 1,\ 1,\ -1,\ 0),\ \mathbf{v} = (0,\ 2,\ 1,\ 0,\ 2)$$

We need to find each component of $\mathbf{u}\cdot\mathbf{v}$, $\|\mathbf{u}\|$, $\|\mathbf{v}\|$:

$$\begin{aligned}\mathbf{u}\cdot\mathbf{v} &= (-1\times 0)+(1\times 2)+(1\times 1)+(-1\times 0)+(0\times 2)\\ &= 3\end{aligned}$$

$$\|\mathbf{u}\| = \sqrt{(-1)^2+1^2+1^2+(-1)^2+0^2} = \sqrt{4} = 2$$

$$\|\mathbf{v}\| = \sqrt{0^2+2^2+1^2+0^2+2^2} = \sqrt{9} = 3$$

Substituting these $\mathbf{u}\cdot\mathbf{v}=3$, $\|\mathbf{u}\|=2$, $\|\mathbf{v}\|=3$ into (*) yields

$$\cos(\theta) = \frac{3}{2\times 3} = \frac{1}{2} \quad \Rightarrow \quad \theta = \cos^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{3}$$

17. (a) The standard inner product of two vectors $\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$ and $\mathbf{y} = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}$ in $\mathbb{R}^3$ is defined as

$$\langle \mathbf{x},\ \mathbf{y}\rangle = \mathbf{x}\cdot\mathbf{y} = \mathbf{x}^T\mathbf{y} = x_1y_1 + x_2y_2 + x_3y_3$$

And the norm is defined as

$$\|\mathbf{x}\| = \sqrt{\mathbf{x}\cdot\mathbf{x}} = \sqrt{x_1^{\ 2}+x_2^{\ 2}+x_3^{\ 2}}$$

(b) Orthogonal means that the inner product between any two vectors is equal to zero. An orthonormal set of vectors are vectors which are orthogonal to each other and each vector has a norm of 1.

(c) Let $\mathbf{v} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ and for this vector to be orthogonal to $\mathbf{u} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ we have

$$\mathbf{u}\cdot\mathbf{v} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}\cdot\begin{pmatrix} x \\ y \\ z \end{pmatrix} = x+y+z = 0$$

The condition for the general vector **v** to be orthogonal to the given vector **u** is

$$x+y+z=0 \quad \Rightarrow \quad x=-y-z$$

Let $y=t$ and $z=s$ where $s,\ t\in\mathbb{R}$ then substituting these into $x=-y-z$ gives $x=-t-s$. Thus the general vector is given by

$$\mathbf{v} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -t-s \\ t \\ s \end{pmatrix} = s\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} + t\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$$

Let $s=t=1$ then a vector $\mathbf{v} = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} + \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix}$.

Let $\mathbf{w} = \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix}$ then this vector satisfies both the following conditions:

$$\mathbf{u}\cdot\mathbf{w}=\begin{pmatrix}1\\1\\1\end{pmatrix}\cdot\begin{pmatrix}w_1\\w_2\\w_3\end{pmatrix}=w_1+w_2+w_3=0$$

$$\mathbf{v}\cdot\mathbf{w}=\begin{pmatrix}-2\\1\\1\end{pmatrix}\cdot\begin{pmatrix}w_1\\w_2\\w_3\end{pmatrix}=-2w_1+w_2+w_3=0$$

From both these equations we have

$$\left.\begin{aligned}w_2+w_3&=-w_1\\w_2+w_3&=2w_1\end{aligned}\right\}\text{ implies that } w_1=0$$

Thus $w_2=-w_3$. Let $w_3=1$ then $w_2=-1$ and we have $\mathbf{w}=\begin{pmatrix}0\\-1\\1\end{pmatrix}$. You may check that this vector **w** is orthogonal to the vectors **u** and **v**, that is both $\mathbf{u}\cdot\mathbf{w}=0$ and $\mathbf{v}\cdot\mathbf{w}=0$.
For an orthonormal set we need to normalise each of the vectors **u**, **v** and **w**:

$$\mathbf{u}=\frac{1}{\sqrt{3}}\begin{pmatrix}1\\1\\1\end{pmatrix},\quad \hat{\mathbf{v}}=\frac{1}{\sqrt{6}}\begin{pmatrix}-2\\1\\1\end{pmatrix}\text{ and }\mathbf{w}=\frac{1}{\sqrt{2}}\begin{pmatrix}0\\-1\\1\end{pmatrix}$$

These $\{\mathbf{u},\ \mathbf{v},\mathbf{w}\}$ is an orthonormal set of vectors in $\mathbb{R}^3$.

18. (a) The dot product $\mathbf{u}\cdot\mathbf{v}$ is given by

$$\mathbf{u}\cdot\mathbf{v}=\begin{pmatrix}2\\2\\-1\\3\end{pmatrix}\cdot\begin{pmatrix}1\\3\\2\\-2\end{pmatrix}=(2\times1)+(2\times3)+(-1\times2)+(3\times(-2))=0\qquad(\dagger)$$

Simplifying $(\mathbf{u}-\mathbf{v})\cdot(\mathbf{u}-\mathbf{w})$ gives

$$(\mathbf{u}-\mathbf{v})\cdot(\mathbf{u}-\mathbf{w})=(\mathbf{u}\cdot\mathbf{u})-(\mathbf{u}\cdot\mathbf{w})-(\mathbf{v}\cdot\mathbf{u})+(\mathbf{v}\cdot\mathbf{w})\qquad(*)$$

Finding each of these inner products:

$$\mathbf{u}\cdot\mathbf{u}=\begin{pmatrix}2\\2\\-1\\3\end{pmatrix}\cdot\begin{pmatrix}2\\2\\-1\\3\end{pmatrix}=2^2+2^2+(-1)^2+3^2=18$$

$$\mathbf{u}\cdot\mathbf{w}=\begin{pmatrix}2\\2\\-1\\3\end{pmatrix}\cdot\begin{pmatrix}3\\1\\3\\1\end{pmatrix}=(2\times3)+(2\times1)+(-1\times3)+(3\times1)=8$$

$$\mathbf{v}\cdot\mathbf{u}=\underbrace{\mathbf{u}\cdot\mathbf{v}\ =\ 0}_{\text{By }(\dagger)\text{ Above}}$$

$$\mathbf{v}\cdot\mathbf{w}=\begin{pmatrix}1\\3\\2\\-2\end{pmatrix}\cdot\begin{pmatrix}3\\1\\3\\1\end{pmatrix}=(1\times3)+(3\times1)+(2\times3)+(-2\times1)=10$$

Substituting $\mathbf{u}\cdot\mathbf{u}=18$, $\mathbf{u}\cdot\mathbf{w}=8$, $\mathbf{v}\cdot\mathbf{u}=0$ and $\mathbf{v}\cdot\mathbf{w}=10$ into (*) gives

$$\begin{aligned}(\mathbf{u}-\mathbf{v})\cdot(\mathbf{u}-\mathbf{w})&=(\mathbf{u}\cdot\mathbf{u})-(\mathbf{u}\cdot\mathbf{w})-(\mathbf{v}\cdot\mathbf{u})+(\mathbf{v}\cdot\mathbf{w})\\&=18-8-0+10=20\end{aligned}$$

Simplifying

$$(2\mathbf{u}-3\mathbf{v})\cdot(\mathbf{u}+4\mathbf{w})=2(\mathbf{u}\cdot\mathbf{u})+8(\mathbf{u}\cdot\mathbf{w})-3(\mathbf{v}\cdot\mathbf{u})-12(\mathbf{v}\cdot\mathbf{w})$$

Substituting $\mathbf{u}\cdot\mathbf{u}=18$, $\mathbf{u}\cdot\mathbf{w}=8$, $\mathbf{v}\cdot\mathbf{u}=0$ and $\mathbf{v}\cdot\mathbf{w}=10$ into this gives

$$(2\mathbf{u}-3\mathbf{v})\cdot(\mathbf{u}+4\mathbf{w})=2(18)+8(8)-3(0)-12(10)=-20$$

(b) The Cauchy-Schwarz Inequality is defined as $|\mathbf{u}\cdot\mathbf{v}|\le\|\mathbf{u}\|\|\mathbf{v}\|$. We are given that $\mathbf{u}=(1,\ \ 1,\ \ 1)$ and $\mathbf{v}=(-1,\ \ 1,\ \ 2)$ so

$$|\mathbf{u}\cdot\mathbf{v}|=\left|\begin{pmatrix}1\\1\\1\end{pmatrix}\cdot\begin{pmatrix}-1\\1\\2\end{pmatrix}\right|=|-1+1+2|=2$$

Also

$$\|\mathbf{u}\|=\|(1,\ \ 1,\ \ 1)\|=\sqrt{1^2+1^2+1^2}=\sqrt{3}$$

$$\|\mathbf{v}\|=\|(-1,\ \ 1,\ \ 2)\|=\sqrt{(-1)^2+1^2+2^2}=\sqrt{6}$$

We have $\|\mathbf{u}\|\|\mathbf{v}\|=\sqrt{3}\sqrt{6}=\sqrt{18}=3\sqrt{2}$. Hence we conclude that the Cauchy-Schwarz inequality holds, that is

$$|\mathbf{u}\cdot\mathbf{v}|=2\le3\sqrt{2}=\|\mathbf{u}\|\|\mathbf{v}\|$$

19. Required to prove that if $\mathbf{v}_1,\ \mathbf{v}_2,\ \mathbf{v}_3\in\mathbb{R}^3$ are an orthogonal set of non-zero vectors then they are linearly independent. *How?*
Consider the linear combination

$$k_1\mathbf{v}_1+k_2\mathbf{v}_2+k_3\mathbf{v}_3=\mathbf{O}$$

where k_1, k_2 and k_3 are scalars and show that $k_1=k_2=k_3=0$.

Proof.
Consider the linear combination

$$k_1\mathbf{v}_1+k_2\mathbf{v}_2+k_3\mathbf{v}_3=\mathbf{O}$$

The inner product of $k_1\mathbf{v}_1+k_2\mathbf{v}_2+k_3\mathbf{v}_3$ and $\mathbf{v}_1$ is zero because $\mathbf{v}\cdot\mathbf{O}=\mathbf{O}\cdot\mathbf{v}=0$ and so we have

$$\begin{aligned}\mathbf{O}\cdot\mathbf{v}_1&=(k_1\mathbf{v}_1+k_2\mathbf{v}_2+k_3\mathbf{v}_3)\cdot\mathbf{v}_1 &&\left[\text{Because } k_1\mathbf{v}_1+k_2\mathbf{v}_2+k_3\mathbf{v}_3=\mathbf{O}\right]\\&=k_1(\mathbf{v}_1\cdot\mathbf{v}_1)+k_2\underbrace{(\mathbf{v}_2\cdot\mathbf{v}_1)}_{=0}+k_3\underbrace{(\mathbf{v}_3\cdot\mathbf{v}_1)}_{=0} &&\left[\text{Because } \mathbf{v}_1,\ \mathbf{v}_2,\ \mathbf{v}_3 \text{ orthogonal}\right]\\&=k_1\|\mathbf{v}_1\|=0\end{aligned}$$

Since $\mathbf{v}_1$ is a non-zero vector therefore $k_1\|\mathbf{v}_1\|=0$ implies $k_1=0$.
Similarly we have

$$(k_1\mathbf{v}_1 + k_2\mathbf{v}_2 + k_3\mathbf{v}_3)\cdot \mathbf{v}_2 = k_2\|\mathbf{v}_2\| = 0 \text{ implies } k_2 = 0$$

$$(k_1\mathbf{v}_1 + k_2\mathbf{v}_2 + k_3\mathbf{v}_3)\cdot \mathbf{v}_3 = k_3\|\mathbf{v}_3\| = 0 \text{ implies } k_3 = 0$$

Since $k_1 = k_2 = k_3 = 0$ therefore $\mathbf{v}_1,\ \mathbf{v}_2,\ \mathbf{v}_3$ are linearly independent vectors.

■

20. We need to prove that if $\{\mathbf{u}_1,\ \mathbf{u}_2,\ \cdots,\ \mathbf{u}_n\}$ is a basis for $\mathbb{R}^n$ then $\{\mathbf{Au}_1,\ \mathbf{Au}_2,\ \cdots,\ \mathbf{Au}_n\}$ is also a basis for $\mathbb{R}^n$ [**A** is an invertible $n\times n$ matrix].

How do we prove this result?

Since $\{\mathbf{Au}_1,\ \mathbf{Au}_2,\ \cdots,\ \mathbf{Au}_n\}$ is *n* vectors and Proposition (2-20) says

Every basis of $\mathbb{R}^n$ that: contains exactly *n* vectors

Therefore we have the correct number of vectors.

n linearly independent vectors in $\mathbb{R}^n$ form a basis for $\mathbb{R}^n$.

It is enough to show that $\{\mathbf{Au}_1,\ \mathbf{Au}_2,\ \cdots,\ \mathbf{Au}_n\}$ are linearly independent.

Proof.

Let $k_1, k_2, \cdots, k_n$ be scalars. Consider the linear combination

$$k_1(\mathbf{Au}_1) + k_2(\mathbf{Au}_2) + \cdots + k_n(\mathbf{Au}_n) = \mathbf{O}$$

$$\mathbf{A}(k_1\mathbf{u}_1 + k_2\mathbf{u}_2 + \cdots + k_n\mathbf{u}_n) = \mathbf{O}$$

Since **A** is invertible therefore taking $\mathbf{A}^{-1}$ of both sides gives

$$k_1\mathbf{u}_1 + k_2\mathbf{u}_2 + \cdots + k_n\mathbf{u}_n = \mathbf{O}$$

Since we are given that $\{\mathbf{u}_1,\ \mathbf{u}_2,\ \cdots,\ \mathbf{u}_n\}$ is a basis for $\mathbb{R}^n$ therefore they are linearly independent so all the scalars k 's are zero, that is $k_1 = k_2 = \cdots = k_n = 0$.

Hence $\{\mathbf{Au}_1,\ \mathbf{Au}_2,\ \cdots,\ \mathbf{Au}_n\}$ are linearly independent which means it is a basis for $\mathbb{R}^n$.

■

21. We need to prove that

$$\mathbf{u}\cdot\mathbf{v} = 0 \quad \Leftrightarrow \quad \|\mathbf{u}+\mathbf{v}\| = \|\mathbf{u}-\mathbf{v}\|$$

Proof.

$(\Rightarrow)$. Assume $\mathbf{u}\cdot\mathbf{v} = 0$. Examining the Right Hand Side we have

$$\begin{aligned}\|\mathbf{u}+\mathbf{v}\|^2 &= (\mathbf{u}+\mathbf{v})\cdot(\mathbf{u}+\mathbf{v}) \\ &= (\mathbf{u}\cdot\mathbf{u}) + (\mathbf{u}\cdot\mathbf{v}) + (\mathbf{v}\cdot\mathbf{u}) + (\mathbf{v}\cdot\mathbf{v}) \\ &= \|\mathbf{u}\|^2 + 2\underbrace{(\mathbf{u}\cdot\mathbf{v})}_{=0} + \|\mathbf{v}\|^2 \qquad [\text{Because we are assuming } \mathbf{u}\cdot\mathbf{v} = 0] \\ &= \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2\end{aligned}$$

Similarly we have

$$\begin{aligned}\|\mathbf{u}-\mathbf{v}\|^2 &= \|\mathbf{u}\|^2 - 2(\mathbf{u}\cdot\mathbf{v}) + \|\mathbf{v}\|^2 \\ &= \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2 \qquad [\text{Because } \mathbf{u}\cdot\mathbf{v} = 0]\end{aligned}$$

Therefore $\|\mathbf{u}+\mathbf{v}\|^2 = \|\mathbf{u}-\mathbf{v}\|^2$. Taking the square root of both sides gives us our result, $\|\mathbf{u}+\mathbf{v}\| = \|\mathbf{u}-\mathbf{v}\|$.

$(\Leftarrow)$. Now going the other way we assume $\|\mathbf{u}+\mathbf{v}\|=\|\mathbf{u}-\mathbf{v}\|$. Squaring this yields

$$\|\mathbf{u}+\mathbf{v}\|^2=\|\mathbf{u}-\mathbf{v}\|^2$$

From the above derivations we have

$$\|\mathbf{u}+\mathbf{v}\|^2=\|\mathbf{u}\|^2+2(\mathbf{u}\cdot\mathbf{v})+\|\mathbf{v}\|^2$$
$$\|\mathbf{u}-\mathbf{v}\|^2=\|\mathbf{u}\|^2-2(\mathbf{u}\cdot\mathbf{v})+\|\mathbf{v}\|^2$$

Equating these because we have $\|\mathbf{u}+\mathbf{v}\|^2=\|\mathbf{u}-\mathbf{v}\|^2$ gives

$$\|\mathbf{u}\|^2+2(\mathbf{u}\cdot\mathbf{v})+\|\mathbf{v}\|^2=\|\mathbf{u}\|^2-2(\mathbf{u}\cdot\mathbf{v})+\|\mathbf{v}\|^2$$
$$4(\mathbf{u}\cdot\mathbf{v})=0$$

Hence we have our result $\mathbf{u}\cdot\mathbf{v}=0$.

■

22. We need to prove that all non-zero vectors $\mathbf{u}$ in $\mathbb{R}^n$ we have $\mathbf{u}\cdot\mathbf{v}=\mathbf{u}\cdot\mathbf{w}$ then $\mathbf{v}=\mathbf{w}$.

Proof.

From $\mathbf{u}\cdot\mathbf{v}=\mathbf{u}\cdot\mathbf{w}$ we have

$$\mathbf{u}\cdot\mathbf{v}-\mathbf{u}\cdot\mathbf{w}=0$$
$$\mathbf{u}\cdot(\mathbf{v}-\mathbf{w})=0$$

Let $\mathbf{u}=\begin{pmatrix}u_1\\ \vdots\\ u_n\end{pmatrix}$, $\mathbf{v}=\begin{pmatrix}v_1\\ \vdots\\ v_n\end{pmatrix}$ and $\mathbf{w}=\begin{pmatrix}w_1\\ \vdots\\ w_n\end{pmatrix}$ be in $\mathbb{R}^n$ therefore by the last line above $\mathbf{u}\cdot(\mathbf{v}-\mathbf{w})=0$ we have

$$u_1(v_1-w_1)+u_2(v_2-w_2)+\cdots+u_n(v_n-w_n)=0 \qquad (*)$$

Since this is true for all the non-zero vectors **u** therefore (*) is only possible provided

$$v_1-w_1=0,\ v_2-w_2=0,\ \cdots,\ v_n-w_n=0$$
$$v_1=w_1,\quad v_2=w_2,\ \cdots,\ v_n=w_n$$

This means we have $\mathbf{v}=\mathbf{w}$. This is our required result.

■

23. *Proof.*

$(\Leftarrow)$. We first assume $ad-bc\neq 0$ and deduce that $\mathbf{u}=\begin{pmatrix}a\\ b\end{pmatrix}$ and $\mathbf{v}=\begin{pmatrix}c\\ d\end{pmatrix}$ are linearly independent.

Let k_1 and k_2 be scalars so that

$$k_1\mathbf{u}+k_2\mathbf{v}=k_1\begin{pmatrix}a\\ b\end{pmatrix}+k_2\begin{pmatrix}c\\ d\end{pmatrix}=\begin{pmatrix}0\\ 0\end{pmatrix}$$

Writing out the equations gives

$$k_1a+k_2c=0 \qquad (*)$$
$$k_1b+k_2d=0 \qquad (**)$$

Multiplying (*) by *d* and (**) by *c* and then subtracting your resulting equations:

$$\begin{array}{rl} & k_1ad + k_2cd = 0 \\ - & k_1bc + k_2dc = 0 \\ \hline & k_1(ad - bc) = 0 \end{array}$$

Since we are assuming $ad - bc \neq 0$ therefore $k_1 = 0$. Since we are assuming $ad - bc \neq 0$ therefore we **cannot** have $c = d = 0$ which means that by substituting $k_1 = 0$ into (*) and (**) gives $k_2 = 0$. We have both scalars $k_1 = 0$ and $k_2 = 0$ therefore $\mathbf{u} = \begin{pmatrix} a \\ b \end{pmatrix}$ and $\mathbf{v} = \begin{pmatrix} c \\ d \end{pmatrix}$ are linearly independent.

$(\Rightarrow)$. Assume $\mathbf{u} = \begin{pmatrix} a \\ b \end{pmatrix}$ and $\mathbf{v} = \begin{pmatrix} c \\ d \end{pmatrix}$ are linearly independent. Consider the linear combination

$$k_1\begin{pmatrix} a \\ b \end{pmatrix} + k_2\begin{pmatrix} c \\ d \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

The equations are

$$k_1a + k_2c = 0 \qquad (\dagger)$$

$$k_1b + k_2d = 0 \qquad (\dagger\dagger)$$

Multiplying (†) by *d* and (††) by *c* and subtracting yields

$$\begin{array}{rl} & k_1ad + k_2cd = 0 \\ - & k_1bc + k_2dc = 0 \\ \hline & k_1(ad - bc) = 0 \qquad (1) \end{array}$$

Similarly multiplying (†) by *b* and (††) by *a* and subtracting yields

$$\begin{array}{rl} & k_1ab + k_2cb = 0 \\ - & k_1ab + k_2da = 0 \\ \hline & k_2(bc - ad) = 0 \qquad (2) \end{array}$$

Equating equations (1) and (2) gives

$$\begin{aligned} k_1(ad - bc) &= k_2(bc - ad) \\ &= -k_2(ad - bc) \\ k_1(ad - bc) + k_2(ad - bc) &= 0 \\ (k_1 + k_2)(ad - bc) &= 0 \qquad [\text{Factorizing}] \end{aligned}$$

If $ad - bc = 0$ then we can have any values for k_1 and k_2 but this is impossible because vectors **u** and **v** are linearly independent which means that $k_1 = k_2 = 0$. Thus $ad - bc \neq 0$ which is our required result.

■

24. Required to prove that a set of *m* vectors where $m < n$ **cannot** span $\mathbb{R}^n$.

Proof.

Suppose we have a set of m linearly independent vectors which span $\mathbb{R}^n$. Then these vectors form a basis for $\mathbb{R}^n$. We have *m* vectors but a basis of $\mathbb{R}^n$ should have *n* vectors because:

Proposition (2-20). Every basis of $\mathbb{R}^n$ contains exactly n vectors.
Hence m vectors $(m < n)$ **cannot** span $\mathbb{R}^n$.

■

25. We need to prove that any n vectors which span $\mathbb{R}^n$ form a basis for $\mathbb{R}^n$.
Proof.
Let $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \cdots, \mathbf{v}_n\}$ be a set of n vectors which spans $\mathbb{R}^n$. Let $k_1, k_2, \cdots, k_n$ be scalars such that

$$k_1\mathbf{v}_1 + k_2\mathbf{v}_2 + k_3\mathbf{v}_3 + \cdots + k_n\mathbf{v}_n = \mathbf{O}$$

Required to prove $k_1 = k_2 = k_3 = \cdots = k_n = 0$.
Suppose $k_n \neq 0$ [Not Zero] then we can rewrite the above as

$$k_1\mathbf{v}_1 + k_2\mathbf{v}_2 + k_3\mathbf{v}_3 + \cdots + k_{n-1}\mathbf{v}_{n-1} = k_n\mathbf{v}_n$$

$$\frac{k_1}{k_n}\mathbf{v}_1 + \frac{k_2}{k_n}\mathbf{v}_2 + \frac{k_3}{k_n}\mathbf{v}_3 + \cdots + \frac{k_{n-1}}{k_n}\mathbf{v}_{n-1} = \mathbf{v}_n$$

This means that $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \cdots, \mathbf{v}_{n-1}\}$ spans $\mathbb{R}^n$. This is impossible because of the result of question 24 above which says less than n vectors **cannot** span $\mathbb{R}^n$. This means that $k_n = 0$. Similarly **all** the k's must be zero which implies that the set $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \cdots, \mathbf{v}_n\}$ is linearly independent. Hence the n vectors which span $\mathbb{R}^n$ form a basis.

■