Complete Solutions to Miscellaneous Exercises 5

1. We are given the transformation $T\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 + 5x_2 \\ 0 \\ 2x_1 - 3x_2 \end{bmatrix}$. *How do we show that this is linear?*

Need to show that both $T(\mathbf{u}+\mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$ and $T(k\mathbf{u}) = kT(\mathbf{u})$, where k is a scalar, are satisfied. This is definition (5-2).

Let $\mathbf{u} = \begin{bmatrix} a \\ b \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} c \\ d \end{bmatrix}$. Then by applying the given transformation we have

$$T(\mathbf{u}) = T\left(\begin{bmatrix} a \\ b \end{bmatrix}\right) = \begin{bmatrix} a+5b \\ 0 \\ 2a-3b \end{bmatrix} \text{ and } T(\mathbf{v}) = T\left(\begin{bmatrix} c \\ d \end{bmatrix}\right) = \begin{bmatrix} c+5d \\ 0 \\ 2c-3d \end{bmatrix}$$

<u>Checking</u> $T(\mathbf{u}+\mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$:

$$\begin{aligned} T(\mathbf{u}+\mathbf{v}) &= T\left(\begin{bmatrix} a \\ b \end{bmatrix} + \begin{bmatrix} c \\ d \end{bmatrix}\right) \\ &= T\left(\begin{bmatrix} a+c \\ b+d \end{bmatrix}\right) \\ &= \begin{bmatrix} (a+c)+5(b+d) \\ 0 \\ 2(a+c)-3(b+d) \end{bmatrix} \qquad \left[\text{Because } T\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1+5x_2 \\ 0 \\ 2x_1-3x_2 \end{bmatrix}\right] \\ &= \begin{bmatrix} a+5b+c+5d \\ 0 \\ 2a-3b+2c-3d \end{bmatrix} = \underbrace{\begin{bmatrix} a+5b \\ 0 \\ 2a-3b \end{bmatrix}}_{=T(\mathbf{u})} + \underbrace{\begin{bmatrix} c+5d \\ 0 \\ 2c-3d \end{bmatrix}}_{=T(\mathbf{v})} = T(\mathbf{u}) + T(\mathbf{v}) \end{aligned}$$

Thus we have $T(\mathbf{u}+\mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$.

<u>Checking</u> $T(k\mathbf{u}) = kT(\mathbf{u})$:

$$\begin{aligned} T(k\mathbf{u}) &= T\left(k\begin{bmatrix} a \\ b \end{bmatrix}\right) \\ &= T\left(\begin{bmatrix} ka \\ kb \end{bmatrix}\right) \underset{\substack{\text{Applying the} \\ \text{given transformation}}}{=} \begin{bmatrix} ka+5kb \\ 0 \\ 2ka-3kb \end{bmatrix} = k\underbrace{\begin{bmatrix} a+5b \\ 0 \\ 2a-3b \end{bmatrix}}_{=T(\mathbf{u})} = kT(\mathbf{u}) \end{aligned}$$

This means we have $T(k\mathbf{u}) = kT(\mathbf{u})$.

Hence the given transformation is linear because **both** conditions

$$T(\mathbf{u}+\mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v}) \text{ and } T(k\mathbf{u}) = kT(\mathbf{u})$$

are satisfied.

2. By taking the transpose we have

$$T\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2x_1 - 3x_2 + 4x_3 \\ -x_1 + x_2 \end{bmatrix}$$

How do we show T is a linear map (transformation)?
By Definition (5-2) we need to show both the following conditions:

$$T(\mathbf{u}+\mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v}) \text{ and } T(k\mathbf{u}) = kT(\mathbf{u}) \text{ where } k \text{ is a scalar}$$

Let $\mathbf{u} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$. Then by applying the given transformation we have

$$T(\mathbf{u}) = T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} 2x_1 - 3x_2 + 4x_3 \\ -x_1 + x_2 \end{bmatrix} \text{ and } T(\mathbf{v}) = T\left(\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}\right) = \begin{bmatrix} 2y_1 - 3y_2 + 4y_3 \\ -y_1 + y_2 \end{bmatrix}$$

We have

$$\begin{aligned}
T(\mathbf{u}+\mathbf{v}) &= T\left(\begin{bmatrix} x_1 + y_1 \\ x_2 + y_2 \\ x_3 + y_3 \end{bmatrix}\right) \\
&\underset{\text{Applying the given linear map}}{=} \begin{bmatrix} 2(x_1+y_1) - 3(x_2+y_2) + 4(x_3+y_3) \\ -(x_1+y_1) + (x_2+y_2) \end{bmatrix} \\
&= \begin{bmatrix} 2x_1 - 3x_2 + 4x_3 + 2y_1 - 3y_2 + 4y_3 \\ -x_1 + x_2 - y_1 + y_2 \end{bmatrix} \\
&= \begin{bmatrix} 2x_1 - 3x_2 + 4x_3 \\ -x_1 + x_2 \end{bmatrix} + \begin{bmatrix} 2y_1 - 3y_2 + 4y_3 \\ -y_1 + y_2 \end{bmatrix} \\
&= T(\mathbf{u}) + T(\mathbf{v})
\end{aligned}$$

Let k be a scalar. For T to be linear we also need to show $T(k\mathbf{u}) = kT(\mathbf{u})$:

$$\begin{aligned}
T(k\mathbf{u}) &= T\left(k\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) \\
&= T\left(\begin{bmatrix} kx_1 \\ kx_2 \\ kx_3 \end{bmatrix}\right) \\
&\underset{\text{Applying the given linear map}}{=} \begin{bmatrix} 2kx_1 - 3kx_2 + 4kx_3 \\ -kx_1 + kx_2 \end{bmatrix} = k\begin{bmatrix} 2x_1 - 3x_2 + 4x_3 \\ -x_1 + x_2 \end{bmatrix} = kT(\mathbf{u})
\end{aligned}$$

Hence T is a linear map.
The standard matrix $\boldsymbol{S}$ is given by the coefficients of x_1, x_2 and x_3:

$$\begin{array}{c} \begin{array}{ccc} x_1 & x_2 & x_3 \end{array} \\ \mathbf{S}=\begin{bmatrix} 2 & -3 & 4 \\ -1 & 1 & 0 \end{bmatrix} \end{array} \qquad \left[\text{Because } T\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}=\begin{bmatrix} 2x_1-3x_2+4x_3 \\ -x_1+x_2 \end{bmatrix}\right]$$

3. An example of a non-linear transformation $f:\mathbb{R}^2\to\mathbb{R}^2$ is

$$f\left(\begin{bmatrix} x \\ y \end{bmatrix}\right)=\begin{pmatrix} x^2 \\ xy \end{pmatrix}$$

This does **not** satisfy $f(k\mathbf{u})=kf(\mathbf{u})$ where $\mathbf{u}=\begin{bmatrix} x \\ y \end{bmatrix}$ because

$$\begin{aligned} f(k\mathbf{u}) &= f\left(k\begin{bmatrix} x \\ y \end{bmatrix}\right) \\ &= f\left(\begin{bmatrix} kx \\ ky \end{bmatrix}\right)=\begin{bmatrix} k^2x^2 \\ k^2xy \end{bmatrix}=k^2\begin{bmatrix} x^2 \\ xy \end{bmatrix}=k^2 f\left(\begin{bmatrix} x \\ y \end{bmatrix}\right)=k^2 f(\mathbf{u}) \end{aligned}$$

We have $f(k\mathbf{u})=k^2 f(\mathbf{u})\neq kf(\mathbf{u})$ which means that f is **not** a linear transformation.

4. (a) The standard matrix for $T(x_1,\ x_2,\ x_3)=(3x_2+2x_3,\ 3x_1-4x_2)$ is determined by evaluating $T(\mathbf{e}_1)$, $T(\mathbf{e}_2)$ and $T(\mathbf{e}_3)$ where $\mathbf{e}_1=(1,\ 0,\ 0)^T$, $\mathbf{e}_2=(0,\ 1,\ 0)^T$ and $\mathbf{e}_3=(0,\ 0,\ 1)^T$ [or reading off the coefficients of x_1, x_2 and x_3]:

$$T(\mathbf{e}_1)=T\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}=\begin{pmatrix} 0 \\ 3 \end{pmatrix},\quad T(\mathbf{e}_2)=T\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}=\begin{pmatrix} 3 \\ -4 \end{pmatrix} \text{ and } T(\mathbf{e}_3)=T\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}=\begin{pmatrix} 2 \\ 0 \end{pmatrix}$$

Thus the standard matrix **S** is given by

$$\mathbf{S}=\left(T(\mathbf{e}_1)\ \middle|\ T(\mathbf{e}_2)\ \middle|\ T(\mathbf{e}_3)\right)=\begin{pmatrix} 0 & 3 & 2 \\ 3 & -4 & 0 \end{pmatrix}$$

(b) By Proposition (5-7) T is one-to-one $\Leftrightarrow \ker(T)=\{\mathbf{O}\}$. *How can we find* $\ker(T)$*?*
By finding x_1, x_2 and x_3 such that $T(x_1,\ x_2,\ x_3)=(3x_2+2x_3,\ 3x_1-4x_2)=\mathbf{O}$:

$$\begin{aligned} 3x_2 + 2x_3 &= 0 \qquad (*) \\ 3x_1 - 4x_2 \qquad &= 0 \qquad (**) \end{aligned}$$

From (*) we have $3x_2=-2x_3 \Rightarrow x_2=-\frac{2}{3}x_3$. Let $x_3=3t$ where t is any real number.
Then $x_2=-2t$. Substituting $x_2=-2t$ into the bottom equation (**) we have

$$3x_1-4(-2t)=0 \quad\Rightarrow\quad x_1=-\frac{8}{3}t$$

We have non-zero solutions because $x_1 = -\frac{8}{3}t$, $x_2 = -2t$ and $x_3 = 3t$ where $t \in \mathbb{R}$. This means that $\ker(T) = \left\{ \begin{pmatrix} -8t/3 \\ -2t \\ 3t \end{pmatrix} \;\middle|\; t \in \mathbb{R} \right\} \neq \mathbf{O}$.

Hence T is **not** one-to-one.

(c) From part (b) we have $\dim(\ker(T)) = 1$ because there is only one free variable (t).

Using the dimension theorem (5-5) which says:

$$\dim(\ker(T)) + \dim(range(T)) = n \qquad (\dagger)$$

where n is the dimension of the domain. In this case the domain is $\mathbb{R}^3$ because we are given $T(x_1,\ x_2,\ x_3) = (3x_2 + 2x_3,\ 3x_1 - 4x_2)$ which means $T: \mathbb{R}^3 \to \mathbb{R}^2$.

What is the dimension of $\mathbb{R}^3$*?*

It is 3. Substituting $n = 3$ and $\dim(\ker(T)) = 1$ into (†) gives

$$1 + \dim(range(T)) = 3 \quad \Rightarrow \quad \dim(range(T)) = 2$$

This means that $range(T) = \mathbb{R}^2$ and since we have $T: \mathbb{R}^3 \to \mathbb{R}^2$ therefore T is **onto**.

5. (a) *How do we check that the given set S is a subspace of* $\mathbb{R}^2$*?*

By using

Proposition (3-5). A nonempty subset S is a subspace of a vector space $V \Leftrightarrow$

(a) $\mathbf{O} \in S$ [Zero vector is in S].

(b) If $\mathbf{u}$ and $\mathbf{v}$ are vectors in S then any linear combination $k\mathbf{u} + c\mathbf{v}$ is also in S.

Clearly the zero vector $\vec{0}$ is in S because $\vec{0} \cdot \vec{u} = 0$.

Let $\vec{v}$ and $\vec{w}$ be vectors in S and k and c be scalars. Need to show that $k\vec{v} + c\vec{w}$ is also in S for S to be a subspace of $\mathbb{R}^2$:

$$\begin{aligned} (k\vec{v} + c\vec{w}) \cdot \vec{u} &= k(\vec{v} \cdot \vec{u}) + c(\vec{w} \cdot \vec{u}) \\ &= k(0) + c(0) = 0 \qquad \left[\vec{v} \cdot \vec{u} = \vec{w} \cdot \vec{u} = 0 \text{ because } \vec{v},\ \vec{w} \in S\right] \end{aligned}$$

Since **both** conditions of Proposition (3-5) are satisfied therefore S is a subspace of $\mathbb{R}^2$.

(b) (i) We are given

$$T\left(\begin{bmatrix} 1 \\ -1 \end{bmatrix}\right) = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \quad \text{and} \quad T\left(\begin{bmatrix} 0 \\ -1 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

We have to write $\vec{e}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\vec{e}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ in terms of $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ -1 \end{pmatrix}$:

$$\vec{e}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} = a\begin{pmatrix} 1 \\ -1 \end{pmatrix} + b\begin{pmatrix} 0 \\ -1 \end{pmatrix} \qquad \text{gives } a = 1,\ b = -1$$

$$\vec{e}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} = c\begin{pmatrix} 1 \\ -1 \end{pmatrix} + d\begin{pmatrix} 0 \\ -1 \end{pmatrix} \qquad \text{gives } c = 0,\ d = -1$$

We can find $T(\vec{e}_1)$ and $T(\vec{e}_2)$ by using the above:

$$T\left(\vec{e}_1\right)=T\left(\begin{bmatrix}1\\0\end{bmatrix}\right)=T\left(\begin{pmatrix}1\\-1\end{pmatrix}-\begin{pmatrix}0\\-1\end{pmatrix}\right)\qquad\left[\text{Substituting } a=1 \text{ and } b=-1\right]$$

$$=T\begin{pmatrix}1\\-1\end{pmatrix}-T\begin{pmatrix}0\\-1\end{pmatrix}=\begin{bmatrix}2\\1\end{bmatrix}-\begin{bmatrix}1\\3\end{bmatrix}=\begin{bmatrix}1\\-2\end{bmatrix}$$

$$T\left(\vec{e}_2\right)=T\left(\begin{bmatrix}0\\1\end{bmatrix}\right)=T\left(0\begin{pmatrix}1\\-1\end{pmatrix}-\begin{pmatrix}0\\-1\end{pmatrix}\right)\qquad\left[\text{Substituting } c=0 \text{ and } d=-1\right]$$

$$=0T\begin{pmatrix}1\\-1\end{pmatrix}-T\begin{pmatrix}0\\-1\end{pmatrix}=0\begin{bmatrix}2\\1\end{bmatrix}-\begin{bmatrix}1\\3\end{bmatrix}=\begin{bmatrix}-1\\-3\end{bmatrix}$$

(ii) The matrix representation **A** with respect to the standard basis is

$$\mathbf{A}=\left[T\left(\vec{e}_1\right)\;\middle|\;T\left(\vec{e}_2\right)\right]=\begin{bmatrix}1&-1\\-2&-3\end{bmatrix}\text{ where } T\begin{pmatrix}x\\y\end{pmatrix}=\mathbf{A}\begin{pmatrix}x\\y\end{pmatrix}$$

(iii) Let $\mathbf{w}=\begin{pmatrix}w_1\\w_2\end{pmatrix}$ then

$$T\left(\begin{bmatrix}w_1\\w_2\end{bmatrix}\right)=\begin{bmatrix}1&-1\\-2&-3\end{bmatrix}\begin{bmatrix}w_1\\w_2\end{bmatrix}=\begin{bmatrix}1\\8\end{bmatrix}$$

$$\begin{bmatrix}w_1\\w_2\end{bmatrix}=\begin{bmatrix}1&-1\\-2&-3\end{bmatrix}^{-1}\begin{bmatrix}1\\8\end{bmatrix}\qquad\begin{bmatrix}\text{Taking the}\\\text{inverse matrix}\end{bmatrix}$$

$$=\frac{1}{-3-2}\begin{bmatrix}-3&1\\2&1\end{bmatrix}\begin{bmatrix}1\\8\end{bmatrix}$$

$$=-\frac{1}{5}\begin{bmatrix}5\\10\end{bmatrix}=\begin{bmatrix}-1\\-2\end{bmatrix}$$

That is $\mathbf{w}=\begin{pmatrix}w_1\\w_2\end{pmatrix}=\begin{bmatrix}-1\\-2\end{bmatrix}$.

6. (a) The given transformation S is

$$S\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}=\begin{pmatrix}x_1-4x_2+2x_3\\2x_1+7x_2-x_3\\-x_1-8x_2+2x_3\\2x_1+x_2+x_3\end{pmatrix}$$

Remember the standard matrix is given by the coefficients of x_1, x_2 and x_3. This means that the standard matrix, call it **A**, of operator S is

$$\mathbf{A}=\begin{pmatrix}1&-4&2\\2&7&-1\\-1&-8&2\\2&1&1\end{pmatrix}$$

(b) To find a basis for the range of T we take the transpose of matrix **A** and then place this into row echelon form:

$$\mathbf{A}^T = \begin{pmatrix} 1 & -4 & 2 \\ 2 & 7 & -1 \\ -1 & -8 & 2 \\ 2 & 1 & 1 \end{pmatrix}^T = \begin{pmatrix} 1 & 2 & -1 & 2 \\ -4 & 7 & -8 & 1 \\ 2 & -1 & 2 & 1 \end{pmatrix}$$

Labelling the rows of $\mathbf{A}^{\mathrm{T}}$:

$$\begin{array}{c} R_1 \\ R_2 \\ R_3 \end{array} \begin{pmatrix} 1 & 2 & -1 & 2 \\ -4 & 7 & -8 & 1 \\ 2 & -1 & 2 & 1 \end{pmatrix}$$

Carrying out the row operations $R_2 + 4R_1$ and $R_3 - 2R_1$:

$$\begin{array}{c} R_1 \\ R_2' = R_2 + 4R_1 \\ R_3' = R_3 - 2R_1 \end{array} \begin{pmatrix} 1 & 2 & -1 & 2 \\ 0 & 15 & -12 & 9 \\ 0 & -5 & 4 & -3 \end{pmatrix}$$

Executing $R_2'/3$:

$$\begin{array}{c} R_1 \\ R_2^* = R_2'/3 \\ R_3' \end{array} \begin{pmatrix} 1 & 2 & -1 & 2 \\ 0 & 5 & -4 & 3 \\ 0 & -5 & 4 & -3 \end{pmatrix}$$

Executing $R_3' + R_2^*$:

$$\begin{array}{c} R_1 \\ R_2^* \\ R_3^* = R_3' + R_2^* \end{array} \begin{pmatrix} 1 & 2 & -1 & 2 \\ 0 & 5 & -4 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

A basis for the range are the non-zero rows of the last matrix, that is $\left\{ \begin{pmatrix} 1 \\ 2 \\ -1 \\ 2 \end{pmatrix}, \begin{pmatrix} 0 \\ 5 \\ -4 \\ 3 \end{pmatrix} \right\}$.

7. (a) $T: V \to W$ is a linear transformation if both the following conditions are satisfied:

$$T(\mathbf{u} + \mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v}) \text{ and } T(k\mathbf{u}) = kT(\mathbf{u})$$

for all vectors **u** and **v** in V and any scalar k.

(b) The kernel of T, ker T, is the set of vectors **v** in V of $T: V \to W$ such that $T(\mathbf{v}) = \mathbf{O}$.

(c) We need to prove that:

$$T \text{ is injective, this means one to one, } \Leftrightarrow \quad \ker T = \{\mathbf{O}\}$$

Proof.

($\Rightarrow$). We assume T is one to one. By Proposition (5-1) we have $T(\mathbf{O}) = \mathbf{O}$. Since T is one to one therefore there can be **no** other vector in V which is transformed to the zero vector under T. Hence $\ker T = \{\mathbf{O}\}$.

$(\Leftarrow)$. We assume $\ker T = \{\mathbf{O}\}$. *What do we need to prove?*

T is one to one (injective). *How?*

By Definition (5-7) which says:

Transformation T is one to one $\Leftrightarrow$ $\mathbf{u} \neq \mathbf{v}$ implies $T(\mathbf{u}) \neq T(\mathbf{v})$.

Let **u** and **v** be in V such that $\mathbf{u} \neq \mathbf{v}$. We have

$$T(\mathbf{u}-\mathbf{v}) \underset{\text{because } T \text{ is linear}}{=} T(\mathbf{u}) - T(\mathbf{v}) \neq \mathbf{O}$$

Because if $T(\mathbf{u}) - T(\mathbf{v}) = \mathbf{O}$ then we have

$$T(\mathbf{u}) - T(\mathbf{v}) = T(\mathbf{u}-\mathbf{v}) = \mathbf{O} \Rightarrow \mathbf{u}-\mathbf{v} = \mathbf{O} \Rightarrow \mathbf{u} = \mathbf{v}$$

This is contradiction because we had $\mathbf{u} \neq \mathbf{v}$ therefore

$$T(\mathbf{u}) - T(\mathbf{v}) \neq \mathbf{O}$$

Hence $T(\mathbf{u}) \neq T(\mathbf{v})$. By Definition (5-7) we conclude that T is one to one (injective).

■

(d) We need to find ker T which means we need to find x, y and z such that

$$T\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x+y \\ x+y+z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

We have to solve the simultaneous equations

$$\begin{aligned} x + y \quad &= 0 \qquad (*) \\ x + y + z &= 0 \qquad (**) \end{aligned}$$

From the top equation (*) we have $x = -y$. Let $y = t$ where t is any real number then $x = -t$. Substituting these, $x = -t$ and $y = t$, into the bottom equation (**) gives $z = 0$. Thus ker T is given by

$$\ker T = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -t \\ t \\ 0 \end{pmatrix} = t\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} \text{ where } t \in \mathbb{R}$$

We conclude that $\ker(T) = span\left\{(-1, \quad 1, \quad 0)^T\right\}$.

8. a. To find a basis for the image (range) of T we need to transpose the given matrix and we can call this new matrix **A**:

$$\begin{pmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & -1 & 1 \\ 0 & 0 & 2 & 1 \end{pmatrix}^T = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & -1 & 2 \\ 0 & 1 & 1 \end{pmatrix} = \mathbf{A}$$

How do we find a basis for the image of T?

It is the non-zero rows of the (reduced) row echelon form of matrix **A**:

$$\begin{array}{c} R_1 \\ R_2 \\ R_3 \\ R_4 \end{array} \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & -1 & 2 \\ 0 & 1 & 1 \end{pmatrix}$$

Carrying out the row operations $R_2 - R_1$ and $R_3 - R_1$:

$$\begin{array}{r} R_1 \\ R_2{*} = R_2 - R_1 \\ R_3{*} = R_3 - R_1 \\ R_4 \end{array} \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & -2 & 2 \\ 0 & 1 & 1 \end{pmatrix}$$

Carrying out the row operation $R_3{*} + 2R_4$:

$$\begin{array}{r} R_1 \\ R_2{*} \\ R_3{'} = R_3{*} + 2R_4 \\ R_4 \end{array} \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 4 \\ 0 & 1 & 1 \end{pmatrix}$$

Dividing the third row by 4 and interchanging rows $R_2{*}$ and R_4:

$$\begin{array}{r} R_1 \\ R_2{'} = R_4 \\ R_3{''} = R_3{'}/4 \\ R_4 = R_2{*} \end{array} \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

Executing $R_2{'} - R_3{''}$:

$$\begin{array}{r} R_1 \\ R_2{''} = R_2{'} - R_3{*} \\ R_3{*} \\ R_4 \end{array} \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

Finally executing $R_1 - R_2{''}$ gives us a matrix in reduced row echelon form:

$$\begin{array}{r} R_1{'} = R_1 - R_2{''} \\ R_2{''} \\ R_3{*} \\ R_4 \end{array} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

A basis B for the image of T are the non-zero rows of this last matrix, that is

$$B = \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}$$

The columns of the matrix associated with T does span $\mathbb{R}^3$ because a basis for this is the set B above which is the standard basis for $\mathbb{R}^3$.

b. The transformation T is onto because the basis for the image of T is the set B given in part a which is the standard basis for $\mathbb{R}^3$ which means that the range of T is $\mathbb{R}^3$ and we are given $T: \mathbb{R}^4 \to \mathbb{R}^3$.

c. A basis for the null space of T can be found by placing the given matrix into reduced row echelon form and solving the resulting equations $\mathbf{Rx} = \mathbf{O}$:

$$\begin{array}{r} R_1 \\ R_2 \\ R_3 \end{array} \begin{pmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & -1 & 1 \\ 0 & 0 & 2 & 1 \end{pmatrix}$$

Carrying out the row operation $R_2 - R_1$:

$$\begin{array}{c} R_1 \\ R_2{*} = R_2 - R_1 \\ R_3 \end{array} \begin{pmatrix} 1 & 1 & 1 & 0 \\ 0 & 0 & -2 & 1 \\ 0 & 0 & 2 & 1 \end{pmatrix}$$

Executing $R_3 + R_2{*}$:

$$\begin{array}{c} R_1 \\ R_2{*} \\ R_3{*} = R_3 + R_2{*} \end{array} \begin{pmatrix} 1 & 1 & 1 & 0 \\ 0 & 0 & -2 & 1 \\ 0 & 0 & 0 & 2 \end{pmatrix}$$

Dividing the bottom row by 2 gives

$$\begin{array}{c} R_1 \\ R_2{*} \\ R_3' = R_3{*}/2 \end{array} \begin{pmatrix} 1 & 1 & 1 & 0 \\ 0 & 0 & -2 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Executing $R_2{*} - R_3'$:

$$\begin{array}{c} R_1 \\ R_2' = R_2{*} - R_3' \\ R_3' \end{array} \begin{pmatrix} 1 & 1 & 1 & 0 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Dividing the middle row by -2 gives

$$\begin{array}{c} R_1 \\ R_2'' = R_2'/(-2) \\ R_3' \end{array} \begin{pmatrix} 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Carrying out the row operation $R_1 - R_2''$ gives us the reduced row echelon form matrix **R**:

$$\begin{array}{c} R_1' = R_1 - R_2'' \\ R_2'' \\ R_3' \end{array} \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \mathbf{R}$$

Null space is found by solving $\mathbf{Rx} = \mathbf{O}$ which is

$$\begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \qquad \text{gives } x_1 = -x_2,\ x_3 = x_4 = 0$$

A basis *B'* for the null space is $B' = \left\{ \begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \end{pmatrix} \right\}$.

d. By part c we have one vector in the basis of the null space so the dimension of the null space is 1 which means that $nullity(T) = 1$.

Since $nullity(T) = 1$ therefore the given transformation *T* is **not** one to one because Proposition (5-8) in the main text says:

$$T \text{ is one to one} \Leftrightarrow nullity(T) = 0.$$

9. We are given that

$$T\begin{bmatrix}3\\-5\end{bmatrix}=\begin{bmatrix}1\\-1\\2\end{bmatrix} \text{ and } T\begin{bmatrix}-1\\2\end{bmatrix}=\begin{bmatrix}3\\0\\-2\end{bmatrix}$$

However we need to find $T\begin{bmatrix}7\\-11\end{bmatrix}$. *How?*

We need to write $\begin{bmatrix}7\\-11\end{bmatrix}$ in terms of $\begin{bmatrix}3\\-5\end{bmatrix}$ and $\begin{bmatrix}-1\\2\end{bmatrix}$. Let a and b be the scalars such that

$$a\begin{bmatrix}3\\-5\end{bmatrix}+b\begin{bmatrix}-1\\2\end{bmatrix}=\begin{bmatrix}7\\-11\end{bmatrix}$$

Writing these out as equations and solving

$$\left.\begin{aligned}3a-b&=7\\-5a+2b&=-11\end{aligned}\right\} \text{ gives } a=3 \text{ and } b=2$$

Thus we have $3\begin{bmatrix}3\\-5\end{bmatrix}+2\begin{bmatrix}-1\\2\end{bmatrix}=\begin{bmatrix}7\\-11\end{bmatrix}$. To determine $T\begin{bmatrix}7\\-11\end{bmatrix}$ we have

$$\begin{aligned}T\begin{bmatrix}7\\-11\end{bmatrix}&=T\left(3\begin{bmatrix}3\\-5\end{bmatrix}+2\begin{bmatrix}-1\\2\end{bmatrix}\right) && \left[\text{Substituting } \begin{bmatrix}7\\-11\end{bmatrix}=3\begin{bmatrix}3\\-5\end{bmatrix}+2\begin{bmatrix}-1\\2\end{bmatrix}\right]\\
&=3T\begin{bmatrix}3\\-5\end{bmatrix}+2T\begin{bmatrix}-1\\2\end{bmatrix} && [\text{Because } T \text{ is linear}]\\
&=3\begin{bmatrix}1\\-1\\2\end{bmatrix}+2\begin{bmatrix}3\\0\\-2\end{bmatrix} && \left[\text{Using } T\begin{bmatrix}3\\-5\end{bmatrix}=\begin{bmatrix}1\\-1\\2\end{bmatrix},\ T\begin{bmatrix}-1\\2\end{bmatrix}=\begin{bmatrix}3\\0\\-2\end{bmatrix}\right]\\
&=\begin{bmatrix}3+6\\-3+0\\6-4\end{bmatrix}=\begin{bmatrix}9\\-3\\2\end{bmatrix}\end{aligned}$$

Thus $T\begin{bmatrix}7\\-11\end{bmatrix}=\begin{bmatrix}9\\-3\\2\end{bmatrix}$.

10. One definition of mathematics is the science of patterns. *What pattern do you notice about the given vector* $\begin{bmatrix}1\\1\\-1\end{bmatrix}$*?*

$$\begin{bmatrix}1\\1\\-1\end{bmatrix}=-1\begin{bmatrix}-1\\-1\\1\end{bmatrix}$$

Since L is a linear transformation therefore

$$L\left(\begin{bmatrix}1\\1\\-1\end{bmatrix}\right)=L\left(-1\begin{bmatrix}-1\\-1\\1\end{bmatrix}\right)=-L\left(\begin{bmatrix}-1\\-1\\1\end{bmatrix}\right)\underset{\substack{\text{Given in the}\\\text{question}}}{=}-\begin{bmatrix}-1\\2\\4\end{bmatrix}=\begin{bmatrix}1\\-2\\-4\end{bmatrix}$$

11. (a) We are given that matrix **A** is of size 6×5. *What does this mean?*
Means the matrix **A** has 6 rows and 5 columns. We are also given $T(\mathbf{x})=\mathbf{A}\mathbf{x}$ which means we have

$$6\text{ rows}\left\}\overbrace{\begin{pmatrix}a_{11} & \cdots & a_{15}\\ \vdots & \ddots & \vdots\\ a_{61} & \cdots & a_{65}\end{pmatrix}}^{5\text{ columns}}\right.\begin{pmatrix}x_1\\ \vdots\\ x_5\end{pmatrix}=\begin{pmatrix}a_{11}x_1 & \cdots & a_{15}x_5\\ \vdots & \ddots & \vdots\\ a_{61}x_1 & \cdots & a_{65}x_5\end{pmatrix}$$

The given transformation $T:\mathbb{R}^5\to\mathbb{R}^6$. We have $m=6$ and $n=5$.
(b) Since the range of T is a subset of $\mathbb{R}^6$ therefore the maximum number of linearly independent vectors in the range is 6.
(c) We are given that nullity of **A** is 0. By the dimension theorem (5-5) we have

$$nullity(T)+rank(T)=n$$

Substituting $nullity(T)=0$ and $n=5$ gives $rank(T)=5$. Since $rank(T)=5$ therefore range of T **cannot** equal $\mathbb{R}^6$ because $\dim\left(\mathbb{R}^6\right)=6$ so T is **not** onto.
(d) Since $nullity(T)=0$ therefore T is one-to-one because Proposition (5-8) says:

$$T:\ V\to W\ \text{ is one to one }\Leftrightarrow\ nullity(T)=0$$

12. Let **A** be the matrix representing the transformation $T:\mathbb{R}^2\to\mathbb{R}^3$. We have $\mathbf{A}=\left[T(\mathbf{e}_1)\ \middle|\ T(\mathbf{e}_2)\right]$. We are given

$$T(\mathbf{e}_1)=\begin{bmatrix}2\\1\\h\end{bmatrix}\text{ and }T(\mathbf{e}_2)=\begin{bmatrix}3\\k\\0\end{bmatrix}$$

Thus the matrix **A** is given by $\mathbf{A}=\begin{pmatrix}2 & 3\\1 & k\\h & 0\end{pmatrix}$. *For what values of h and k is the transformation one-to-one?*
By Proposition (5-7) we have T is one to one $\Leftrightarrow\ \ker(T)=\mathbf{O}$ where **O** is the zero vector. *How do we find the kernel of T ?*
Let $\mathbf{x}=\begin{bmatrix}x\\y\end{bmatrix}$ then $\ker(T)=\{\mathbf{x}\mid\mathbf{A}\mathbf{x}=\mathbf{O}\}$. For one to one we need $\mathbf{x}=\begin{bmatrix}x\\y\end{bmatrix}=\mathbf{O}$ which means the only solution to $\mathbf{A}\mathbf{x}=\mathbf{O}$ is $x=0$ and $y=0$.
By expanding $\mathbf{A}\mathbf{x}=\mathbf{O}$ we have

$$\mathbf{A}\mathbf{x}=\begin{pmatrix}2 & 3\\1 & k\\h & 0\end{pmatrix}\begin{bmatrix}x\\y\end{bmatrix}=\begin{bmatrix}2x+3y\\x+ky\\hx\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$$

From the bottom row we have

$$hx=0 \quad \Rightarrow \quad h\neq 0$$

Why do we need $h\neq 0$*?*

Because if $h=0$ then $x\neq 0$ will satisfy the bottom row. Remember the only solution to the above equations is $x=y=0$.

The first two rows are the simultaneous equations

$$2x+3y=0 \qquad (1)$$
$$x+ky=0 \qquad (2)$$

Multiplying equation (2) by 2 and subtracting from (1) we have

$$\begin{array}{rl} & 2x+3y=0 \\ - & 2x+2ky=0 \\ \hline & (3-2k)y=0 \end{array}$$

We have $3-2k\neq 0 \quad \Rightarrow \quad k\neq \dfrac{3}{2}$.

This means that the given transformation is one to one for all real values of h and k provided $h\neq 0$ and $k\neq \dfrac{3}{2}$.

13. (a) The zero vector in $C[0,\ 1]$ is the constant function 0.

(b) Let $f,\ g\in C^{1}[0,\ 1]$. *What do we need to show?*

Required to show both the following conditions:

$$T(f+g)=T(f)+T(g) \text{ and } T(kf)=kT(f) \text{ where } k \text{ is a scalar}$$

<u>Checking</u> $T(f+g)=T(f)+T(g)$:

$$\begin{aligned} T(f+g) &= \frac{d(f+g)}{dx}(x)-2(f+g)(x) \\ &= \frac{d}{dx}\big[(f+g)(x)\big]-2(f)(x)-2(g)(x) \\ &= \frac{d}{dx}\big[(f)(x)\big]+\frac{d}{dx}\big[(g)(x)\big]-2(f)(x)-2(g)(x) \\ &= \underbrace{\frac{df}{dx}(x)-2(f)(x)}_{=T(f)}+\underbrace{\frac{dg}{dx}(x)-2(g)(x)}_{T(g)} \\ &= T(f)+T(g) \end{aligned}$$

<u>Checking</u> $T(kf)=kT(f)$:

$$\begin{aligned} T(kf) &= \frac{d(kf)}{dx}(x)-2(kf)(x) \\ &= k\frac{d}{dx}\big[(f)(x)\big]-2k(f)(x) \\ &= k\underbrace{\left[\frac{d}{dx}\big[(f)(x)\big]-2(f)(x)\right]}_{=T(f)}=kT(f) \end{aligned}$$

Hence T is a linear transformation.

(c) The kernel of T is the set of functions $f \in C^1[0, \ 1]$ which satisfies

$$\frac{df}{dx}(x) - 2f(x) = 0$$

Rearranging this we have

$$\frac{d}{dx}\left[f(x)\right] = 2f(x)$$

$$\frac{d\left[f(x)\right]}{f(x)} = 2dx$$

Integrating both sides gives

$$\int \frac{d\left[f(x)\right]}{f(x)} = 2\int dx$$

$$\ln\left(f(x)\right) = 2x + C$$

Taking exponentials of both sides gives

$$f(x) = e^{2x+C} = e^{2x}e^{C} = Ae^{2x} \quad \text{where } A = e^{C} \text{ is a constant}$$

We have $\ker(T) = Ae^{2x}$ which means $\ker(T) = span\left\{e^{x}\right\}$.Therefore there is only one vector in basis of $\ker(T)$ which implies $\dim\left(\ker(T)\right) = 1$.

14. We are given that $T: \ V \to W$ is a linear transformation.

(a) We need to show that $\ker(T)$ is a subspace of V.

How do we show this result?

By using Proposition (3-5) which says:

A non-empty subset S in a vector space V is a **subspace** of $V \Leftrightarrow$

(a) $\mathbf{O} \in S$

(b) $\mathbf{u}, \ \mathbf{v} \in S$ then for any scalars $k, \ c$ we have $k\mathbf{u} + c\mathbf{v} \in S$

We need to show both the conditions, (a) and (b), of Proposition (3-5).

Proof.

Checking condition (a):

Since $T(\mathbf{O}) = \mathbf{O}$ therefore $\mathbf{O} \in \ker(T)$ so condition (a) is satisfied. This also means that $\ker(T)$ is a non-empty subset of V.

Checking condition (b):

Let $\mathbf{u}$ and $\mathbf{v}$ be vectors in $\ker(T)$ and $k, \ c$ be any scalars. Consider the transformation of the linear combination $k\mathbf{u} + c\mathbf{v}$:

$$\begin{aligned} T(k\mathbf{u} + c\mathbf{v}) &= kT(\mathbf{u}) + cT(\mathbf{v}) && \left[\text{Because } T \text{ is linear}\right] \\ &= k\mathbf{O} + c\mathbf{O} && \left[\text{Because } \mathbf{u}, \ \mathbf{v} \in \ker(T) \text{ so } T(\mathbf{u}) = T(\mathbf{v}) = \mathbf{O}\right] \\ &= \mathbf{O} \end{aligned}$$

This means that $k\mathbf{u} + c\mathbf{v} \in \ker(T)$.

Thus both conditions of (3-5) are satisfied so we conclude $\ker(T)$ is a subspace of V.

■

(b) We need to show that $im(T)$ is a subspace of W. *How?*

Again we use Proposition (3-5) as described in part (a) above. Remember image of T is the same as the range of T.

Proof.

<u>Checking condition (a)</u>:

Since $T(\mathbf{O})=\mathbf{O}$ therefore $\mathbf{O}\in im(T)$ so condition (a) is satisfied. This also means that $im(T)$ is a non-empty subset of W.

<u>Checking condition (b)</u>:

Let $\mathbf{u}$ and $\mathbf{v}$ be vectors in $im(T)$ and $k,\ c$ be any scalars. We need to show $k\mathbf{u}+c\mathbf{v}$ is also in $im(T)$.

Since $\mathbf{u},\ \mathbf{v}\in im(T)$ therefore there must be vectors $\mathbf{u}',\ \mathbf{v}'\in V$ such that $T(\mathbf{u}')=\mathbf{u}$ and $T(\mathbf{v}')=\mathbf{v}$. We have

$$\begin{aligned} k\mathbf{u}+c\mathbf{v} &= kT(\mathbf{u}')+cT(\mathbf{v}') \\ &= T(k\mathbf{u}'+c\mathbf{v}') \qquad \left[\text{Because } T \text{ is linear}\right] \end{aligned}$$

We know $\mathbf{u}',\ \mathbf{v}'\in V$ and V is a vector space so therefore $k\mathbf{u}'+c\mathbf{v}'\in V$. This means that $k\mathbf{u}+c\mathbf{v}\in im(T)$.

Both the conditions of (3-5) are satisfied so we conclude that $im(T)$ is a subspace of W. ■

(c) *How do we find the dimensions of the kernel and image (range) of T?*

We can use the dimension theorem (5-5) which says:

$$\dim\big(range(T)\big)+\dim\big(\ker(T)\big)=n \qquad (*)$$

where n is the dimension of V if we have $T:\ V\to W$.

In our case we have $T:\ \mathbb{R}^3\to\mathbb{R}^3$ so $n=3$. We need to find the dimensions of either image of T or the kernel of T. Let us find the dimensions of kernel which is the solution set of $T(x,\quad y,\quad z)=\mathbf{O}$:

$$\begin{aligned} x + 2y - z &= 0 \qquad (1) \\ y + z &= 0 \qquad (2) \\ x + y - 2z &= 0 \qquad (3) \end{aligned}$$

From the middle equation (2) we have $y=-z$. Let $z=t$ where t is any real number. Then $y=-t$. Substituting these $y=-t$ and $z=t$ into the top equation (1) gives

$$x - 2t - t = 0 \quad \text{gives} \quad x=3t$$

Our solution set is the kernel of T which is given by

$$\ker(T)=\begin{pmatrix} x \\ y \\ z \end{pmatrix}=\begin{pmatrix} 3t \\ -t \\ t \end{pmatrix}=t\begin{pmatrix} 3 \\ -1 \\ 1 \end{pmatrix} \quad \text{where } t\in\mathbb{R}$$

Since $\ker(T)=span\left\{(3,\quad -1,\quad 1)^T\right\}$ therefore $\dim\big(\ker(T)\big)=1$.

Substituting $\dim\big(\ker(T)\big)=1$ and $n=3$ into the above equation (*) gives

$$\dim\big(range(T)\big)+1=3 \quad\Rightarrow\quad \dim\big(range(T)\big)=2$$

Hence we have $\dim(\ker(T))=1$ and $\dim(range(T))=2$. Remember range and image are identical terms so $\dim(image(T))=2$.

15. We are given that $\{T(\mathbf{v}_1),\ T(\mathbf{v}_2),\ \cdots,\ T(\mathbf{v}_n)\}$ is linearly independent and we need to prove that $S=\{\mathbf{v}_1,\ \mathbf{v}_2,\ \cdots,\ \mathbf{v}_n\}$ is linearly independent. *How?*

Required to prove that $k_1\mathbf{v}_1+k_2\mathbf{v}_2+\cdots+k_n\mathbf{v}_n=\mathbf{O} \quad \Rightarrow \quad k_1=k_2=\cdots=k_n=0$.

Proof.

Consider the linear combination

$$k_1\mathbf{v}_1+k_2\mathbf{v}_2+\cdots+k_n\mathbf{v}_n=\mathbf{O}$$

where $k_1,\ k_2,\ \cdots,\ k_n$ are scalars. Taking the transformation of this gives

$$T(k_1\mathbf{v}_1+k_2\mathbf{v}_2+\cdots+k_n\mathbf{v}_n)=T(\mathbf{O})$$

Since T is linear we have

$$k_1T(\mathbf{v}_1)+k_2T(\mathbf{v}_2)+\cdots+k_nT(\mathbf{v}_n)=T(\mathbf{O})=\mathbf{O}$$

We are given that $\{T(\mathbf{v}_1),\ T(\mathbf{v}_2),\ \cdots,\ T(\mathbf{v}_n)\}$ is linearly independent therefore

$$k_1=k_2=\cdots=k_n=0$$

We have $k_1\mathbf{v}_1+k_2\mathbf{v}_2+\cdots+k_n\mathbf{v}_n=\mathbf{O} \quad \Rightarrow \quad k_1=k_2=\cdots=k_n=0$ which means that $S=\{\mathbf{v}_1,\ \mathbf{v}_2,\ \cdots,\ \mathbf{v}_n\}$ is linearly independent.

■

We need to show that $S=\{\mathbf{v}_1,\ \mathbf{v}_2,\ \cdots,\ \mathbf{v}_n\}$ is linearly independent $\nRightarrow$ $\{T(\mathbf{v}_1),\ T(\mathbf{v}_2),\ \cdots,\ T(\mathbf{v}_n)\}$ is linearly independent.

For example consider the linear transformation $T:\mathbb{R}^2\to\mathbb{R}^2$ given by

$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right)=\begin{pmatrix} x-y \\ x-y \end{pmatrix}$$

Let $\mathbf{v}_1=\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $\mathbf{v}_2=\begin{pmatrix} 2 \\ 1 \end{pmatrix}$, then these are linearly **independent** but

$$T(\mathbf{v}_1)=T\left(\begin{pmatrix} 1 \\ 1 \end{pmatrix}\right)=\begin{pmatrix} 1-1 \\ 1-1 \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \end{pmatrix}=\mathbf{O}$$

Since $T(\mathbf{v}_1)=\mathbf{O}$ therefore $T(\mathbf{v}_1)$ and $T(\mathbf{v}_2)$ are linearly **dependent** because one of the vectors is the zero vector.

16. (a) A mapping $T:\ P_2\to P_3$ is a linear mapping if **both** the following conditions are satisfied:

$$T(\mathbf{p}+\mathbf{q})=T(\mathbf{p})+T(\mathbf{q}) \text{ and } T(k\mathbf{p})=kT(\mathbf{p})$$

where $\mathbf{p}$ and $\mathbf{q}$ are any vectors in P_2 and k is a scalar.

<u>Checking</u> $T(k\mathbf{p})=kT(\mathbf{p})$:

Let k be a scalar and $\mathbf{p}=a_0+a_1t+a_2t^2$ then applying

$$T(a_0+a_1t+a_2t^2):=3a_1+2a_2t+a_0t^2+(a_1+a_2)t^3$$

gives

$$
\begin{aligned}
T(k\mathbf{p}) &= T\left(a_0k + a_1kt + a_2kt^2\right) \\
&= 3a_1k + 2a_2kt + a_0kt^2 + (a_1k + a_2k)t^3 \\
&= k\underbrace{\left[3a_1 + 2a_2t + a_0t^2 + (a_1 + a_2)t^3\right]}_{=T(\mathbf{p})} \\
&= kT(\mathbf{p})
\end{aligned}
$$

(b) The matrix $\mathbf{A}$ is given by $\mathbf{A} = \left(\left[T(1)\right]_C \middle| \left[T(t)\right]_C \middle| \left[T(t^2)\right]_C\right)$. We need to find $T(1)$, $T(t)$ and $T(t^2)$ where $T\left(a_0 + a_1t + a_2t^2\right) := 3a_1 + 2a_2t + a_0t^2 + (a_1 + a_2)t^3$:

$$T(1) = T\left(1 + 0t + 0t^2\right) \underset{a_0=1,\ a_1=0\ \text{and}\ a_3=0}{=} 3(0) + 2(0)t + 1t^2 + (0+0)t^3 = t^2$$

$$T(t) = T\left(0 + 1t + 0t^2\right) \underset{a_0=0,\ a_1=1\ \text{and}\ a_3=0}{=} 3(1) + 2(0)t + 0t^2 + (1+0)t^3 = 3 + t^3$$

$$T(t^2) = T\left(0 + 0t + 1t^2\right) \underset{a_0=0,\ a_1=0\ \text{and}\ a_3=1}{=} 3(0) + 2(1)t + 0t^2 + (0+1)t^3 = 2t + t^3$$

We need to write each of these vectors $T(1)$, $T(t)$ and $T(t^2)$ as coordinates of the basis $C = \{1,\ t,\ t^2,\ t^3\}$:

$$
\begin{aligned}
T(1) &= t^2 = 0(1) + 0(t) + 1(t^2) + 0(t^3) \\
T(t) &= 3 + t^3 = 3(1) + 0(t) + 0(t^2) + 1(t^3) \\
T(t^2) &= 2t + t^3 = 0(1) + 2(t) + 0(t^2) + 1(t^3)
\end{aligned}
$$

What is $\mathbf{A} = \left(\left[T(1)\right]_C \middle| \left[T(t)\right]_C \middle| \left[T(t^2)\right]_C\right)$ *equal to?*

$$\mathbf{A} = \begin{pmatrix} 0 & 3 & 0 \\ 0 & 0 & 2 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix}$$

(c) By using the above matrix $\mathbf{A}$ we need to determine $T\left(2 + 5t - t^2\right)$. *What are the coordinates of* $2 + 5t - t^2$ *with respect to the basis* $B = \{1,\ t,\ t^2\}$*?*

$$\left[2 + 5t - t^2\right]_B = \begin{pmatrix} 2 \\ 5 \\ -1 \end{pmatrix}$$

We have $\mathbf{A}\left[2 + 5t - t^2\right]_B = \begin{pmatrix} 0 & 3 & 0 \\ 0 & 0 & 2 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix}\begin{pmatrix} 2 \\ 5 \\ -1 \end{pmatrix} = \begin{pmatrix} 15 \\ -2 \\ 2 \\ 4 \end{pmatrix}$. This means that

$$T\left(2 + 5t - t^2\right) = 15 - 2t + 2t^2 + 4t^3$$

<u>Check</u>: Applying $T\left(a_0 + a_1t + a_2t^2\right) := 3a_1 + 2a_2t + a_0t^2 + (a_1 + a_2)t^3$:

$$T\left(2+5t-t^2\right) \underset{a_0=2,\ a_1=5 \text{ and } a_3=-1}{=} 3(5)+2(-1)t+2t^2+(5-1)t^3 = 15-2t+2t^2+4t^3$$

(d) Yes we can do the same with $T(p) := t^3 + p(t)$ but we will need to replace the matrix **A** above with a new matrix.

17. (a) Let **u** and **v** be vectors in $\mathbb{R}^n$ and m be any scalar. Then $T:\mathbb{R}^n \to \mathbb{R}^k$ is a linear transformation if

$$T(\mathbf{u}+\mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v}) \text{ and } T(m\mathbf{u}) = mT(\mathbf{u}) \ (m \text{ is a scalar})$$

(b) (i) Let $\mathbf{u} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} d \\ e \\ f \end{bmatrix}$ then applying the given transformation

$T([x, \ y, \ z]) = [x+2y, \ 2y-3z, \ z+x, \ x]$ we have

$$T(\mathbf{u}) = T\left(\begin{bmatrix} a \\ b \\ c \end{bmatrix}\right) = \begin{bmatrix} a+2b \\ 2b-3c \\ c+a \\ a \end{bmatrix} \text{ and } T(\mathbf{v}) = T\left(\begin{bmatrix} d \\ e \\ f \end{bmatrix}\right) = \begin{bmatrix} d+2e \\ 2e-3f \\ f+d \\ d \end{bmatrix}$$

Evaluating the other way we have

$$T(\mathbf{u}+\mathbf{v}) = T\left(\begin{bmatrix} a \\ b \\ c \end{bmatrix} + \begin{bmatrix} d \\ e \\ f \end{bmatrix}\right)$$

$$= T\left(\begin{bmatrix} a+d \\ b+e \\ c+f \end{bmatrix}\right) = \begin{bmatrix} (a+d)+2(b+e) \\ 2(b+e)-3(c+f) \\ (c+f)+(a+d) \\ a+d \end{bmatrix} \quad \left[\text{Because } T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{pmatrix} x+2y \\ 2y-3z \\ z+x \\ x \end{pmatrix}\right]$$

$$= \begin{bmatrix} (a+2b)+(d+2e) \\ (2b-3c)+(2e-3f) \\ (c+a)+(f+d) \\ a+d \end{bmatrix}$$

$$= \underbrace{\begin{bmatrix} a+2b \\ 2b-3c \\ c+a \\ a \end{bmatrix}}_{=T(\mathbf{u})} + \underbrace{\begin{bmatrix} d+2e \\ 2e-3f \\ f+d \\ d \end{bmatrix}}_{=T(\mathbf{v})} = T(\mathbf{u}) + T(\mathbf{v})$$

Hence we have $T(\mathbf{u}+\mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$. Let us check the second condition, $T(m\mathbf{u}) = mT(\mathbf{u})$:

$$T(m\mathbf{u}) = T\left(m\begin{bmatrix} a \\ b \\ c \end{bmatrix}\right)$$

$$= T\left(\begin{bmatrix} ma \\ mb \\ mc \end{bmatrix}\right) = \begin{bmatrix} ma+2mb \\ 2mb-3mc \\ mc+ma \\ ma \end{bmatrix} = m\begin{bmatrix} a+2b \\ 2b-3c \\ c+a \\ a \end{bmatrix} = mT(\mathbf{u})$$

Thus the given transformation T is linear.

(ii) The standard matrix **S** of the given linear transformation

$$T([x, \quad y, \quad z]) = [x+2y, \quad 2y-3z, \quad z+x, \quad x]$$

is determined by reading off the coefficients of x, y and z:

$$\begin{array}{cc} & \begin{array}{ccc} x & y & z \end{array} \\ \mathbf{S} = & \begin{bmatrix} 1 & 2 & 0 \\ 0 & 2 & -3 \\ 1 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \end{array} \qquad \left[\text{Because } T\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x+2y \\ 2y-3z \\ z+x \\ x \end{pmatrix}\right]$$

(c) We need to prove that $T(\mathbf{v})$ ($\mathbf{v} \in \mathbb{R}^n$) is uniquely determined by the vectors $T(\mathbf{b}_1),\ T(\mathbf{b}_2),\ \cdots,\ T(\mathbf{b}_n)$ where $\beta = \{\mathbf{b}_1, \quad \mathbf{b}_2, \quad \cdots, \quad \mathbf{b}_n\}$ is a basis of $\mathbb{R}^n$.

Proof.

Let $\mathbf{v} \in \mathbb{R}^n$ be an arbitrary vector. Since $\beta = \{\mathbf{b}_1, \quad \mathbf{b}_2, \quad \cdots, \quad \mathbf{b}_n\}$ is a basis of $\mathbb{R}^n$ therefore we can write the vector **v uniquely** as

$$\mathbf{v} = k_1\mathbf{b}_1 + k_2\mathbf{b}_2 + \cdots + k_n\mathbf{b}_n$$

where the k's are scalars. Taking the transformation of this we have

$$T(\mathbf{v}) = T(k_1\mathbf{b}_1 + k_2\mathbf{b}_2 + \cdots + k_n\mathbf{b}_n)$$

$$= k_1T(\mathbf{b}_1) + k_2T(\mathbf{b}_2) + \cdots + k_nT(\mathbf{b}_n) \qquad [\text{Because } T \text{ is linear}]$$

Thus $T(\mathbf{v})$ can be expressed uniquely by the vectors $T(\mathbf{b}_1),\ T(\mathbf{b}_2),\ \cdots,\ T(\mathbf{b}_n)$.

■

18. (a) Need to show that $\beta = \{1+x,\ 1-x,\ e^x,\ (1+x)e^x\}$ is linearly independent.

Let $k_1,\ k_2,\ k_3$ and k_4 be scalars such that

$$k_1(1+x) + k_2(1-x) + k_3e^x + k_4(1+x)e^x = 0$$

Expanding this out gives

$$(k_1+k_2) + (k_1-k_2)x + (k_3+k_4)e^x + k_4xe^x = 0$$

Equating coefficients of xe^x gives $k_4 = 0$.

Equating coefficients of e^x gives $k_3 + k_4 = 0 \quad \Rightarrow \quad k_3 = 0 \quad [\text{Because } k_4 = 0]$.

Equating coefficients of x gives $k_1 - k_2 = 0 \quad \Rightarrow \quad k_1 = k_2$.

Equating constants gives $k_1 + k_2 = 0 \quad \Rightarrow \quad k_1 = -k_2$.

From the last two lines we have $k_1 = k_2 = 0$. Hence all our scalars $k_1 = k_2 = k_3 = k_4 = 0$.

What does this mean?

Means that the vectors in $\beta=\{1+x,\ 1-x,\ e^x,\ (1+x)e^x\}$ are linearly independent.
Since these vectors span the given subspace V therefore they are a basis for V.
(b) Using the expanded version from part (a) which is:

$$(k_1+k_2)+(k_1-k_2)x+(k_3+k_4)e^x+k_4xe^x=1-xe^x \qquad (☼)$$

Equating coefficients of xe^x in (☼) gives $k_4=-1$.
Equating coefficients e^x gives $k_3+k_4=0$. Since $k_4=-1$ therefore $k_3=1$.
Equating coefficients of x and constants of (☼) gives the simultaneous equations

$$\left.\begin{aligned}k_1-k_2&=0\\k_1+k_2&=1\end{aligned}\right\}\Rightarrow\ k_1=k_2=\frac{1}{2}$$

The coordinates of the given vector $\mathbf{u}=1-xe^x$ with respect to the basis β is

$$[\mathbf{u}]^\beta=\begin{bmatrix}k_1\\k_2\\k_3\\k_4\end{bmatrix}=\begin{bmatrix}1/2\\1/2\\1\\-1\end{bmatrix}$$

(c) We need to differentiate each of the terms in $\beta=\{1+x,\ 1-x,\ e^x,\ (1+x)e^x\}$:

$$\frac{d}{dx}(1+x)=1,\ \frac{d}{dx}(1-x)=-1,\ \frac{d}{dx}(e^x)=e^x,\ \frac{d}{dx}(1+x)e^x=2e^x+xe^x$$

Writing each of these $1,\ -1,\ e^x,\ \ 2e^x+xe^x$ in terms of the basis vectors β we have:

$$(k_1+k_2)+(k_1-k_2)x+(k_3+k_4)e^x+k_4xe^x=1 \text{ implies } k_1=k_2=\frac{1}{2},\quad k_3=k_4=0$$

$$(k_1+k_2)+(k_1-k_2)x+(k_3+k_4)e^x+k_4xe^x=-1 \text{ implies } k_1=k_2=-\frac{1}{2},\quad k_3=k_4=0$$

$$(k_1+k_2)+(k_1-k_2)x+(k_3+k_4)e^x+k_4xe^x=e^x \text{ implies } k_1=k_2=k_4=0,\quad k_3=1$$

$$(k_1+k_2)+(k_1-k_2)x+(k_3+k_4)e^x+k_4xe^x=2e^x+xe^x \text{ implies } k_1=k_2=0,\quad k_3=k_4=1$$

We have

$$[1]^\beta=\begin{bmatrix}1/2\\1/2\\0\\0\end{bmatrix},\ [-1]^\beta=\begin{bmatrix}-1/2\\-1/2\\0\\0\end{bmatrix},\ \left[e^x\right]^\beta=\begin{bmatrix}0\\0\\1\\0\end{bmatrix},\ \left[2e^x+xe^x\right]^\beta=\begin{bmatrix}0\\0\\1\\1\end{bmatrix}$$

The matrix representation of this differentiation operator is given by

$$[D]_\beta^\beta=\left[[1]^\beta\ \middle|\ [-1]^\beta\ \middle|\ \left[e^x\right]^\beta\ \middle|\ \left[2e^x+xe^x\right]^\beta\right]$$

The matrix is $[D]_\beta^\beta=\begin{bmatrix}1/2&-1/2&0&0\\1/2&-1/2&0&0\\0&0&1&1\\0&0&0&1\end{bmatrix}$.

(d) We first differentiate the given $\mathbf{u}=1-xe^x$:

$$\frac{d}{dx}\left[1-xe^x\right]=-\left[e^x+xe^x\right]=-e^x-xe^x$$

$[D\mathbf{u}]^{\beta}$ is the coordinates of $D\mathbf{u}$ with respect to the basis β. This means we need to find the scalars $k_1,\ k_2,\ k_3$ and k_4 such that

$$(k_1+k_2)+(k_1-k_2)x+(k_3+k_4)e^x+k_4xe^x=-e^x-xe^x$$

which gives $k_1=k_2=0,\ k_4=-1$ and $k_3=0$. Thus $[D\mathbf{u}]^{\beta}=\begin{bmatrix}0\\0\\0\\-1\end{bmatrix}$.

Working out $[D]_{\beta}^{\beta}[\mathbf{u}]^{\beta}=\begin{bmatrix}1/2 & -1/2 & 0 & 0\\1/2 & -1/2 & 0 & 0\\0 & 0 & 1 & 1\\0 & 0 & 0 & 1\end{bmatrix}\underbrace{\begin{bmatrix}1/2\\1/2\\1\\-1\end{bmatrix}}_{\text{From part (b)}}=\begin{bmatrix}0\\0\\0\\-1\end{bmatrix}$.

Hence we have $[D\mathbf{u}]^{\beta}=[D]_{\beta}^{\beta}[\mathbf{u}]^{\beta}$.

19. (a) i. Let $\mathbf{u}=\begin{pmatrix}a\\b\end{pmatrix}$ and $\mathbf{v}=\begin{pmatrix}c\\d\end{pmatrix}$ be our vectors in $\mathbb{R}^2$. *How do we show whether the given map f is linear or not?*

Need to show that both $f(\mathbf{u}+\mathbf{v})=f(\mathbf{u})+f(\mathbf{v})$ and $f(k\mathbf{u})=kf(\mathbf{u})$, where k is a scalar, are satisfied. This is definition (5-2).

We are given $f:\mathbb{R}^2\to P_5:\ (a,\quad b)\to(a+b)x^5$.

<u>Checking</u> $f(\mathbf{u}+\mathbf{v})=f(\mathbf{u})+f(\mathbf{v})$:

$$\begin{aligned}f(\mathbf{u}+\mathbf{v})&=f\left(\begin{bmatrix}a\\b\end{bmatrix}+\begin{bmatrix}c\\d\end{bmatrix}\right)\\&=f\left(\begin{bmatrix}a+c\\b+d\end{bmatrix}\right)\underset{\substack{\text{Applying the}\\\text{given map}}}{=}\left[(a+c)+(b+d)\right]x^5\\&=(a+b)x^5+(c+d)x^5\\&=f(\mathbf{u})+f(\mathbf{v})\end{aligned}$$

<u>Checking</u> $f(k\mathbf{u})=kf(\mathbf{u})$:

$$\begin{aligned}f(k\mathbf{u})&=f\left(k\begin{bmatrix}a\\b\end{bmatrix}\right)\\&=f\left(\begin{bmatrix}ka\\kb\end{bmatrix}\right)\underset{\substack{\text{Applying the}\\\text{given map}}}{=}(ka+kb)x^5\\&=k(a+b)x^5=kf(\mathbf{u})\end{aligned}$$

Thus since f satisfies both $f(\mathbf{u}+\mathbf{v})=f(\mathbf{u})+f(\mathbf{v})$ and $f(k\mathbf{u})=kf(\mathbf{u})$ therefore f is a linear map.

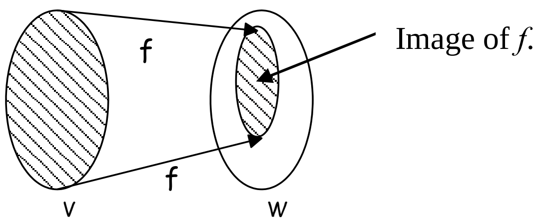
ii. We need to check whether $f:\ M(2,\ 2)\to \mathbb{R}^2 : \begin{pmatrix} a & b \\ c & d \end{pmatrix} \to (ad,\quad bc)$ is linear or not. Let $\mathbf{A}=\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ then $f(\mathbf{A})=\begin{bmatrix} ad \\ bc \end{bmatrix}$.

<u>Checking</u> $f(k\mathbf{A})=kf(\mathbf{A})$. It is easier to check this condition first.

$$\begin{aligned} f(k\mathbf{A}) &= f\left(k\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) \\ &= f\left(\begin{pmatrix} ka & kb \\ kc & kd \end{pmatrix}\right) \\ &\underset{\substack{\text{Applying the} \\ \text{given map}}}{=} \begin{bmatrix} kakd \\ kbkc \end{bmatrix} = k^2\begin{bmatrix} ad \\ bc \end{bmatrix} = k^2 f(\mathbf{A}) \neq kf(\mathbf{A}) \quad [\text{Not Equal}] \end{aligned}$$

Hence the given map is **not** linear because $f(k\mathbf{A})\neq kf(\mathbf{A})$ $[\text{Not Equal}]$.

(b) The **image** (range) of a map $f:\ V\to W$ is the set $\{f(\mathbf{v}) \mid \mathbf{v}\in V\}$, this can be drawn as:



The kernel are the elements in the vector space V such that $f(\mathbf{v})=\mathbf{O}$, that is all the elements in V of $f:\ V\to W$ which get mapped to the zero vector.
The **dimension** of the image (range) of f is called the **rank** of f.
The **dimension** of the kernel of f is called the **nullity** of f.
Let n be the dimension of the given vector space V. The Rank-Nullity theorem states:

$$rank(f)+nullity(f)=n$$

(c) Since $nullity(f)=1$ therefore f is **not** injective (**not** one-to-one) because by:

> Proposition (5-8). Let $T:\ V\to W$ be a linear transformation. T is one to one (injective) $\Leftrightarrow$ $nullity(T)=0$.

For f to be injective (one-to-one) we need $nullity(f)=0$ but we have $nullity(f)=1$ which means f is **not** one-to-one.
By substituting $nullity(f)=1$ into $rank(f)+nullity(f)=n$ we have

$$rank(f)=n-1$$

Since $f:\ P_2\to\mathbb{R}^3$ therefore $n=3$ (dimension of P_2) and $rank(f)=3-1=2$.
Dimension of $\mathbb{R}^3$ is also 3. In our case we have $rank(f)=2$ but $\dim(\mathbb{R}^3)=3$
This means that f is **not surjective (not** onto) because by:

> Proposition (5-10). Let $T:\ V\to W$ be a linear transformation. Then T is onto (surjective) $\Leftrightarrow$ $rank(T)=\dim(W)$.

(d) To prove that the given map is injective we need to show that $\ker(f)=\{\mathbf{O}\}$. *What is* $\ker(f)$ *equal to?*

It is those elements $(x,\ y,\ z,\ t)$ in $\mathbb{R}^4$ such that

$$f(x,\ y,\ z,\ t)=(x+y,\ 0,\ z+t)=(0,\ 0,\ 0)$$

Writing this in conventional manner we have

$$f\begin{pmatrix}x\\y\\z\\t\end{pmatrix}=\begin{pmatrix}x+y\\0\\z+t\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix}$$

From the first and last rows we have

$$x+y=0 \quad \text{gives} \quad x=-y$$
$$z+t=0 \quad \text{gives} \quad z=-t$$

Let $y=a$ and $t=b$ where a and b are any real numbers then $x=-y=-a$ and $z=-t=-b$. Hence elements in $\mathbb{R}^4$ which are mapped to the zero vectors are $(-a,\ a,\ -b,\ b)$ where a and b are any real numbers. Thus

$$\ker(f)=\begin{pmatrix}-a\\a\\-b\\b\end{pmatrix}=a\begin{pmatrix}-1\\1\\0\\0\end{pmatrix}+b\begin{pmatrix}0\\0\\-1\\1\end{pmatrix}\neq\mathbf{O}$$

(does **not** equal zero) therefore f is **not injective**.

A basis for $\ker(f)$ is $\left\{\begin{pmatrix}-1\\1\\0\\0\end{pmatrix},\begin{pmatrix}0\\0\\-1\\1\end{pmatrix}\right\}$. Since we have two linearly independent vectors therefore the dimension of the kernel is 2 which means $nullity(f)=2$.

What is the dimension of $\mathbb{R}^4$*?*

4. Using the dimension theorem with $n=4$ and $nullity(f)=2$ we have

$$Rank(f)+Nullity(f)=n$$
$$Rank(f)+2=4 \quad\Rightarrow\quad Rank(f)=2$$

Since we are given that $f:\mathbb{R}^4\to\mathbb{R}^3$ and $\dim(\mathbb{R}^3)=3$ but we have $Rank(f)=2$ therefore f is **not** onto (**not** surjective).

This means that the dimension of image (range) is 2. A basis for the image can be evaluated by:

$$f\begin{pmatrix}0\\1\\0\\0\end{pmatrix}=\begin{pmatrix}1\\0\\0\end{pmatrix},\quad f\begin{pmatrix}0\\0\\0\\1\end{pmatrix}=\begin{pmatrix}0\\0\\1\end{pmatrix}$$

A basis for the image is $\left\{\begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\0\\1\end{pmatrix}\right\}$.

20. *Do you remember what* $T_1 \circ T_2$ *means?*

Let **u** be a vector in the domain, $\mathbb{R}^2$, of T_2 then $(T_1 \circ T_2)(\mathbf{u}) = T_1(T_2(\mathbf{u}))$.

Let $\mathbf{u} = \begin{pmatrix}x\\y\end{pmatrix} \in \mathbb{R}^2$ then

$$\begin{aligned}(T_1 \circ T_2)(\mathbf{u}) &= T_1\left(T_2\left(\begin{bmatrix}x\\y\end{bmatrix}\right)\right)\\ &= T_1\left(\begin{bmatrix}x\\x+2y\end{bmatrix}\right) && \left[\text{Because } T_2\left(\begin{bmatrix}x\\y\end{bmatrix}\right) = \begin{bmatrix}x\\x+2y\end{bmatrix}\right]\\ &= \begin{bmatrix}2x+x+2y\\-x+x+2y\end{bmatrix} && \left[\text{Because } T_1\left(\begin{bmatrix}x\\y\end{bmatrix}\right) = \begin{bmatrix}2x+y\\-x+y\end{bmatrix}\right]\\ &= \begin{bmatrix}3x+2y\\2y\end{bmatrix}\end{aligned}$$

What is the standard matrix **S** *for this transformation?*

$$\mathbf{S} = \begin{bmatrix}3 & 2\\0 & 2\end{bmatrix}$$

Selection C is correct.

21. The standard matrix **B** for the given linear transformation

$$S((x_1,\ x_2,\ x_3)) = (3x_1+5x_2-x_3,\ \ 4x_2+3x_3,\ \ x_1-x_2+4x_3)$$

is given by reading off the coefficients of x_1, x_2 and x_3:

$$\mathbf{B} = \begin{array}{c}\begin{array}{ccc}x_1 & x_2 & x_3\end{array}\\ \begin{pmatrix}3 & 5 & -1\\0 & 4 & 3\\1 & -1 & 4\end{pmatrix}\end{array} \qquad \left[\text{Because } S\left(\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}\right) = \begin{bmatrix}3x_1+5x_2-x_3\\4x_2+3x_3\\x_1-x_2+4x_3\end{bmatrix}\right]$$

Therefore the matrix **C** is given by

$$\begin{aligned}\mathbf{Cx} &= (S \circ T)(\mathbf{x})\\ &= S(T(\mathbf{x}))\\ &= \mathbf{B}(\mathbf{Ax})\\ &= (\mathbf{BA})\mathbf{x} = \left(\begin{pmatrix}3 & 5 & -1\\0 & 4 & 3\\1 & -1 & 4\end{pmatrix}\begin{pmatrix}1 & 0 & 5\\2 & -1 & 0\\4 & 1 & 0\end{pmatrix}\right)\begin{pmatrix}\mathbf{x}_1\\\mathbf{x}_2\\\mathbf{x}_3\end{pmatrix} = \begin{pmatrix}9 & -6 & 15\\20 & -1 & 0\\15 & 5 & 5\end{pmatrix}\begin{pmatrix}\mathbf{x}_1\\\mathbf{x}_2\\\mathbf{x}_3\end{pmatrix}\end{aligned}$$

Hence matrix $\mathbf{C} = \mathbf{BA} = \begin{pmatrix} 9 & -6 & 15 \\ 20 & -1 & 0 \\ 15 & 5 & 5 \end{pmatrix}$.

22. (a) Let **x** and **y** be vectors in $\mathbb{R}^4$. To prove that ϕ is a linear mapping (transformation) we need to show **both** the following conditions:

$$\phi(\mathbf{x}+\mathbf{y}) = \phi(\mathbf{x}) + \phi(\mathbf{y}) \text{ and } \phi(k\mathbf{x}) = k\phi(\mathbf{x}) \quad (k \text{ is scalar})$$

We have

$$\begin{aligned} \phi(\mathbf{x}+\mathbf{y}) &= \mathbf{A}(\mathbf{x}+\mathbf{y}) \\ &= \mathbf{Ax} + \mathbf{Ay} \\ &= \phi(\mathbf{x}) + \phi(\mathbf{y}) \end{aligned}$$

Let k be scalar then $\phi(k\mathbf{x}) = \mathbf{A}(k\mathbf{x}) = k\mathbf{Ax} = k\phi(\mathbf{x})$.

Since we have $\phi(\mathbf{x}+\mathbf{y}) = \phi(\mathbf{x}) + \phi(\mathbf{y})$ and $\phi(k\mathbf{x}) = k\phi(\mathbf{x})$ therefore ϕ is a linear map (transformation).

We are given that $\mathbf{e}_1 = (1, \quad 0, \quad 0, \quad 0)^T$ therefore

$$\begin{aligned} \phi(\mathbf{e}_1) &= \mathbf{Ae}_1 \\ &= \begin{pmatrix} 1 & 0 & -1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \\ -1 & 5 & 2 & 2 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \\ -1 \end{pmatrix} \end{aligned}$$

(b) Since we need to find the determinant of a 4×4 matrix therefore it should easier to first carry out some simple row operations. Labelling the rows of matrix **A** we have

$$\begin{array}{c} R_1 \\ R_2 \\ R_3 \\ R_4 \end{array} \begin{pmatrix} 1 & 0 & -1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \\ -1 & 5 & 2 & 2 \end{pmatrix}$$

Carrying out the row operations $R_2 - R_1$, $R_3 - R_1$ and $R_4 + R_1$ we have

$$\begin{array}{c} R_1 \\ R_2{*} = R_2 - R_1 \\ R_3{*} = R_3 - R_1 \\ R_4{*} = R_4 + R_1 \end{array} \begin{pmatrix} 1 & 0 & -1 & 1 \\ 0 & 1 & 2 & 0 \\ 0 & 2 & 4 & 3 \\ 0 & 5 & 1 & 3 \end{pmatrix} \qquad (\dagger)$$

The determinant of this last matrix can be found by expanding along the first column:

$$\det\begin{pmatrix}1&0&-1&1\\0&1&2&0\\0&2&4&3\\0&5&1&3\end{pmatrix}=1\det\begin{pmatrix}1&2&0\\2&4&3\\5&1&3\end{pmatrix}$$

$$\underset{\substack{\text{Expanding along}\\\text{the first row}}}{=}\det\begin{pmatrix}4&3\\1&3\end{pmatrix}-2\det\begin{pmatrix}2&3\\5&3\end{pmatrix}$$

$$=(12-3)-2(6-15)=9-2(-9)=27$$

All the row operations carried out in (†) does **not** change the determinant therefore $\det(\mathbf{A})=27$.

(c) For a basis for $\ker(\phi)$ we need to place matrix **A** into a reduced row echelon matrix **R** and then solve the homogeneous system $\mathbf{Rx}=\mathbf{O}$. From (†) in part (b) we have

$$\begin{array}{c}R_1\\R_2*\\R_3*\\R_4*\end{array}\begin{pmatrix}1&0&-1&1\\0&1&2&0\\0&2&4&3\\0&5&1&3\end{pmatrix}$$

Carrying out the row operation R_3*-2R_2* yields

$$\begin{array}{c}R_1\\R_2*\\R_3'=R_3*-2R_2*\\R_4*\end{array}\begin{pmatrix}1&0&-1&1\\0&1&2&0\\0&0&0&3\\0&5&1&3\end{pmatrix}$$

Executing the row operations R_4*-R_3' and $R_3'/3$ gives

$$\begin{array}{c}R_1\\R_2*\\R_3''=R_3'/3\\R_4'=R_4*-R_3'\end{array}\begin{pmatrix}1&0&-1&1\\0&1&2&0\\0&0&0&1\\0&5&1&0\end{pmatrix}$$

Carrying out the row operation $R_4'-5R_2*$ gives

$$\begin{array}{c}R_1\\R_2*\\R_3''\\R_4''=R_4'-5R_2*\end{array}\begin{pmatrix}1&0&-1&1\\0&1&2&0\\0&0&0&1\\0&0&-9&0\end{pmatrix}$$

Dividing the bottom row by -9 yields

$$\begin{array}{c}R_1\\R_2*\\R_3''\\R_4^{\dagger}=R_4'-5R_2*\end{array}\begin{pmatrix}1&0&-1&1\\0&1&2&0\\0&0&0&1\\0&0&1&0\end{pmatrix}$$

Carrying out the row operations R_1-R_3'', R_1+R_4'' and R_2*-2R_4'' gives

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

Interchanging the bottom two rows gives the reduced row echelon form matrix **R**:

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \mathbf{R}$$

Solving $\mathbf{Rx} = \mathbf{O}$:

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \text{ gives } x_1 = x_2 = x_3 = x_4 = 0$$

This means that $\ker(\phi) = \mathbf{O}$ and so there is **no basis** for $\ker(\phi)$.

Since $\ker(\phi) = \mathbf{O}$ therefore $nullity(\phi) = 0$ and we are given that $\phi: \mathbb{R}^4 \to \mathbb{R}^4$ therefore $im(\varphi) = \mathbb{R}^4$ and a basis for $\mathbb{R}^4$ is the standard basis:

$$\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\}$$

The given linear map ϕ is invertible because matrix **A** which represents ϕ is invertible since $\det(\mathbf{A}) = 27 \neq 0$ [Not equal to zero].

23. (a) (i) Let **u** and **v** be vectors in V then T is a linear transformation if

$$T(\mathbf{u}+\mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v}) \text{ and } T(k\mathbf{u}) = kT(\mathbf{u}) \text{ where } k \text{ is a scalar.}$$

(ii) The matrix **A** representing T with respect to the basis $B = \{\overrightarrow{u_1}, \cdots, \overrightarrow{u_n}\}$ is

$$\mathbf{A} = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{pmatrix}$$

where $T(\overrightarrow{u_1}) = \begin{pmatrix} a_{11} \\ \vdots \\ a_{m1} \end{pmatrix}, \ \cdots, T(\overrightarrow{u_n}) = \begin{pmatrix} a_{1n} \\ \vdots \\ a_{mn} \end{pmatrix}$.

(b) We are given that $B = \{\sin x, \ \cos x\}$ which are the basis vectors. We need to find $T(\sin x)$, $T(\cos x)$ and write these in terms of $\begin{bmatrix} a\sin x \\ b\cos x \end{bmatrix}$.

We have

$$\begin{aligned} T(\sin x) &= (\sin x)' + (\sin x)'' && \left[\text{Because} \quad T(f(x)) = f'(x) + f''(x)\right] \\ &= \cos x + (\cos x)' \\ &= \cos x - \sin x \end{aligned}$$

Similarly we have

$$\begin{aligned} T(\cos x) &= (\cos x)' + (\cos x)'' && \left[\text{Because} \quad T(f(x)) = f'(x) + f''(x)\right] \\ &= -\sin x + (-\sin x)' \\ &= -\sin x - \cos x \end{aligned}$$

Collecting the above we have

$$T(\sin x) = \cos x - \sin x = (-1)\sin x + (1)\cos x \quad \Rightarrow \quad \left[T(\sin x)\right]_B = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

$$T(\cos x) = -\sin x - \cos x = (-1)\sin x + (-1)\cos x \quad \Rightarrow \quad \left[T(\cos x)\right]_B = \begin{bmatrix} -1 \\ -1 \end{bmatrix}$$

What is the matrix representation of T ?

Let $[T]_B$ be the matrix representing the given linear transformation T with respect to the basis $B = \{\sin x,\ \cos x\}$:

$$\begin{aligned} [T]_B &= \left(\left[T(\sin x)\right]_B \;\middle|\; \left[T(\cos x)\right]_B\right) \\ &= \begin{pmatrix} -1 & -1 \\ 1 & -1 \end{pmatrix} \end{aligned}$$

(c) By Proposition (5-21) we have T is invertible $\Leftrightarrow$ the matrix $[T]_B$ is invertible.

How do we determine whether $[T]_B = \begin{pmatrix} -1 & -1 \\ 1 & -1 \end{pmatrix}$ *is invertible?*

Check that the determinant is **not** equal to zero:

$$\det\begin{pmatrix} -1 & -1 \\ 1 & -1 \end{pmatrix} = 1 + 1 = 2 \neq 0$$

Thus the given linear transformation is invertible and taking the inverse gives

$$[T]_B^{-1} = \frac{1}{2}\begin{pmatrix} -1 & 1 \\ -1 & -1 \end{pmatrix}$$

(d) To find $f(x)$ such that $f''(x) + f'(x) = 2\sin x + 3\cos x$ we need to use $[T]_B^{-1}$ found in part (c) and the vector $\begin{pmatrix} 2 \\ 3 \end{pmatrix}$ because we are given $2\sin x + 3\cos x$:

$$[T]_B^{-1}\begin{pmatrix} 2 \\ 3 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} -1 & 1 \\ -1 & -1 \end{pmatrix}\begin{pmatrix} 2 \\ 3 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 \\ -5 \end{pmatrix}$$

Hence $f(x) = \frac{1}{2}\sin x - \frac{5}{2}\cos x$.

24. *How do we show that* $\langle \mathbf{u},\ \mathbf{v}\rangle_V := \langle T(\mathbf{u}),\ T(\mathbf{v})\rangle$ *is an inner product?*

We need to use the following from Chapter 4:

> Definition (4-1).
> An inner product on a real vector space V is an operation which assigns to each pair of vectors, **u** and **v**, a **unique** real number $\langle \mathbf{u},\ \mathbf{v}\rangle$ which satisfies the following axioms for **all** vectors **u**, **v** and **w** in V and **all** scalars k.
> (a) $\langle \mathbf{u},\ \mathbf{v}\rangle = \langle \mathbf{v},\ \mathbf{u}\rangle$ [Commutative Law]
> (b) $\langle \mathbf{u}+\mathbf{v},\ \mathbf{w}\rangle = \langle \mathbf{u},\ \mathbf{w}\rangle + \langle \mathbf{v},\ \mathbf{w}\rangle$ [Distributive Law]
> (c) $\langle k\mathbf{u}, \mathbf{v}\rangle = k\langle \mathbf{u},\ \mathbf{v}\rangle$
> (d) $\langle \mathbf{u},\ \mathbf{u}\rangle \ge 0$ and we have $\langle \mathbf{u},\ \mathbf{u}\rangle = 0$ if and only if $\mathbf{u} = \mathbf{O}$

Proof.

Let $\mathbf{u},\ \mathbf{v},\ \mathbf{w} \in V$ and we are given that $T:\ V \to \mathbb{R}^n$. The standard inner product on $\mathbb{R}^n$ is the dot or scalar product denoted by $\cdot$. We have

$$\langle \mathbf{u},\ \mathbf{v}\rangle = \langle T(\mathbf{u}),\ T(\mathbf{v})\rangle = T(\mathbf{u})\cdot T(\mathbf{v})$$

<u>Checking (a)</u>:
We have

$$\begin{aligned}\langle \mathbf{u},\ \mathbf{v}\rangle &= \langle T(\mathbf{u}),\ T(\mathbf{v})\rangle \\ &= T(\mathbf{u})\cdot T(\mathbf{v}) \underset{\text{Because }\cdot\text{ is an Inner Product}}{=} T(\mathbf{v})\cdot T(\mathbf{u}) = \langle \mathbf{v},\ \mathbf{u}\rangle\end{aligned}$$

Thus part (a) of Definition (4-1) is satisfied.
<u>Checking (b)</u>:
We have

$$\begin{aligned}\langle \mathbf{u}+\mathbf{v},\ \mathbf{w}\rangle &= \langle T(\mathbf{u}+\mathbf{v}),\ T(\mathbf{w})\rangle \\ &= \left[T(\mathbf{u}+\mathbf{v})\right]\cdot T(\mathbf{w}) \\ &= \underbrace{\left[T(\mathbf{u})+T(\mathbf{v})\right]}_{\text{Because } T \text{ is linear}}\cdot T(\mathbf{w}) \\ &= \left[T(\mathbf{u})\cdot T(\mathbf{w})\right] + \left[T(\mathbf{v})\cdot T(\mathbf{w})\right] \qquad \left[\text{Because }\cdot\text{ is an I.P.}\right] \\ &= \langle \mathbf{u},\ \mathbf{w}\rangle + \langle \mathbf{v},\ \mathbf{w}\rangle\end{aligned}$$

Hence part (b) is satisfied.
<u>Checking (c)</u>:
Let k be a scalar. We have

$$\begin{aligned}\langle k\mathbf{u}, \mathbf{v}\rangle &= \langle T(k\mathbf{u}), T(\mathbf{v})\rangle \\ &= \langle kT(\mathbf{u}), T(\mathbf{v})\rangle \qquad \left[\text{Because } T \text{ is linear}\right] \\ &= \left[kT(\mathbf{u})\right]\cdot T(\mathbf{v}) = k\left[T(\mathbf{u})\cdot T(\mathbf{v})\right] = k\langle \mathbf{u},\ \mathbf{v}\rangle\end{aligned}$$

Part (c) is satisfied.
<u>Checking (d)</u>:
We have

$$\begin{aligned}\langle \mathbf{u},\ \mathbf{u}\rangle &= \langle T(\mathbf{u}),\ T(\mathbf{u})\rangle \\ &= T(\mathbf{u})\cdot T(\mathbf{u}) \ge 0 \qquad \left[\text{Because }\cdot\text{ is an inner product}\right]\end{aligned}$$

Also

$$0 = \langle \mathbf{u},\ \mathbf{u} \rangle = \langle T(\mathbf{u}),\ T(\mathbf{u}) \rangle = T(\mathbf{u}) \cdot T(\mathbf{u}) \quad \Rightarrow \quad T(\mathbf{u}) = \mathbf{O}$$

Since we are given that T is one-to-one therefore $\ker(T) = \mathbf{O}$ because Proposition (5-7) says:

T is one-to-one $\Leftrightarrow$ $\ker(T) = \mathbf{O}$

This means that $T(\mathbf{u}) = \mathbf{O}$ implies $\mathbf{u} = \mathbf{O}$. Part (d) is satisfied.

Since **all** four parts of definition (4-1) is satisfied therefore $\langle \mathbf{u},\ \mathbf{v} \rangle_V := \langle T(\mathbf{u}),\ T(\mathbf{v}) \rangle$ is an inner product.

■

25. We need to prove that if $\mathbf{v}_1$ and $\mathbf{v}_2$ are linear independent vectors in V where $T:\ V \to W$ is a linearly one-to-one transformation then $T(\mathbf{v}_1)$ and $T(\mathbf{v}_2)$ are linearly independent.

Proof.

Consider the linear combination

$$c_1 T(\mathbf{v}_1) + c_2 T(\mathbf{v}_2) = \mathbf{O}$$

where c_1 and c_2 are scalars. Since T is linear we have

$$T(c_1\mathbf{v}_1 + c_2\mathbf{v}_2) = \mathbf{O}$$

We are also given that T is one-to-one which means that $\ker(T) = \mathbf{O}$ because Proposition (5-7) says that:

$T:\ V \to W$ is one-to-one $\Leftrightarrow$ $\ker(T) = \mathbf{O}$

Therefore from $T(c_1\mathbf{v}_1 + c_2\mathbf{v}_2) = \mathbf{O}$ we have

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 = \mathbf{O}$$

We are given that $\mathbf{v}_1$ and $\mathbf{v}_2$ are linearly independent which implies that $c_1 = c_2 = 0$.

Thus we have $c_1 T(\mathbf{v}_1) + c_2 T(\mathbf{v}_2) = \mathbf{O} \Rightarrow c_1 = c_2 = 0 \Rightarrow T(\mathbf{v}_1)$ and $T(\mathbf{v}_2)$ are linearly independent. This is our required result.

■

26. The following is true and proof of this follows:

If V is a vector space and $T:\ V \to V$ is an injective linear transformation, then T is surjective.

Proof.

Since T is injective which is another term for one-to-one therefore $\ker(T) = \mathbf{O}$ because Proposition (5-7) says:

T is one-to-one $\Leftrightarrow$ $\ker(T) = \mathbf{O}$

This means that the dimension of $\ker(T)$ is zero. Using the Dimension Theorem (5-5) which states

$$\dim(\ker(T)) + \dim(range(T)) = n \qquad (n \text{ is the dimension of } V)$$

We have

$$0 + \dim(range(T)) = \dim(V)$$

Thus $\dim(range(T)) = \dim(V)$. By

> Proposition (5-10). Let $T: V \to W$ be a linear transformation. Then T is onto $\Leftrightarrow$ $rank(T) = \dim(W)$.

We conclude T is onto or surjective because $\dim(range(T)) = rank(T) = \dim(V)$.

■

27. (a) We need to show that $\operatorname{Im}(A) \subset Ker(A)$ given $A^2 = 0$ and $A: V \to V$.

Proof.

Suppose $\operatorname{Im}(A) \not\subset Ker(A)$ that is $\operatorname{Im}(A)$ is **not** subset of $\ker(A)$. This means that there is a vector $\mathbf{y} \in \operatorname{Im}(A)$ such that $\mathbf{y} \notin \ker(A)$. Since $\mathbf{y} \in \operatorname{Im}(A)$ therefore there is a vector $\mathbf{x} \in V$ such that $A(\mathbf{x}) = \mathbf{y}$. Consider $A^2(\mathbf{x})$:

$$\begin{aligned}
A^2(\mathbf{x}) &= (A \circ A)(\mathbf{x}) \\
&= A(A(\mathbf{x})) \\
&= A(\mathbf{y}) && [\text{Because } A(\mathbf{x}) = \mathbf{y}] \\
&\neq \mathbf{O} && [\text{Because } \mathbf{y} \notin \ker(A)]
\end{aligned}$$

Remember the definition of $\ker(A)$ are those elements $\mathbf{u}$ in V such that $A(\mathbf{u}) = \mathbf{O}$.

Since $\mathbf{y} \notin \ker(A)$ therefore $A(\mathbf{y}) \neq \mathbf{O}$.

We have $A^2(\mathbf{x}) \neq \mathbf{O}$ which means that $A^2 \neq 0$. This is a contradiction because we are given that $A^2 = 0$ therefore our supposition $\operatorname{Im}(A) \not\subset Ker(A)$ must be wrong so $\operatorname{Im}(A) \subset Ker(A)$ which is our required result.

■

(b) We need to show that the rank of A is at most 5.

The $rank(A) = \dim(\operatorname{Im}(A))$ and $nullity(A) = \dim(ker(A))$.

Proof.

From part (a) we have $\operatorname{Im}(A) \subset Ker(A)$ which means that

$$\dim(\operatorname{Im}(A)) \leq \dim(Ker(A)) \text{ or } rank(A) \leq nullity(A)$$

By the rank-nullity theorem which is (5-5) we have

$$rank(A) + nullity(A) = \dim(V) = 10$$

Suppose $rank(A) > 5$ then

$$\begin{aligned}
nullity(A) &= 10 - rank(A) \\
&\leq 4 \quad \text{because we are supposing } rank(A) > 5
\end{aligned}$$

This means that $rank(A) > nullity(A)$. This is impossible because in the above we had $rank(A) \leq nullity(A)$. Thus our supposition $rank(A) > 5$ must be wrong so $rank(A) \leq 5$ which is our required result.

■

28. Let **A**, **B** and **C** be in M_{nn}. Then T is linear if $T(\mathbf{A}+\mathbf{C})=T(\mathbf{A})+T(\mathbf{C})$ and $T(k\mathbf{A})=kT(\mathbf{A})$ where k is a scalar. Checking the first result:

$$\begin{aligned} T(\mathbf{A}+\mathbf{C}) &= (\mathbf{A}+\mathbf{C})\mathbf{B}+\mathbf{B}(\mathbf{A}+\mathbf{C}) \\ &= \mathbf{AB}+\mathbf{CB}+\mathbf{BA}+\mathbf{BC} \\ T(\mathbf{A})+T(\mathbf{C}) &= \mathbf{AB}+\mathbf{BA}+\mathbf{CB}+\mathbf{BC} \end{aligned}$$

We have $T(\mathbf{A}+\mathbf{C})=T(\mathbf{A})+T(\mathbf{C})$.

Checking $T(k\mathbf{A})=kT(\mathbf{A})$:

$$\begin{aligned} T(k\mathbf{A}) &= (k\mathbf{A})\mathbf{B}+\mathbf{B}(k\mathbf{A}) \\ &= k(\mathbf{AB})+k(\mathbf{BA}) \\ &= k(\mathbf{AB}+\mathbf{BA})=kT(\mathbf{A}) \end{aligned}$$

Since we have $T(\mathbf{A}+\mathbf{C})=T(\mathbf{A})+T(\mathbf{C})$ and $T(k\mathbf{A})=kT(\mathbf{A})$ therefore T is a linear transformation.

29. Let $\mathbf{u}=\begin{pmatrix} a \\ b \end{pmatrix}$ and $\mathbf{v}=\begin{pmatrix} c \\ d \end{pmatrix}$. To show that T is **not** linear we prove

$$T(\mathbf{u}+\mathbf{v}) \neq T(\mathbf{u})+T(\mathbf{v}) \qquad [\text{Not Equal}]$$

We have

$$\begin{aligned} T(\mathbf{u}+\mathbf{v}) &= T\left(\begin{bmatrix} a \\ b \end{bmatrix}+\begin{bmatrix} c \\ d \end{bmatrix}\right) \\ &= T\left(\begin{bmatrix} a+c \\ b+d \end{bmatrix}\right) \\ &= \begin{bmatrix} e^{a+c} \\ e^{b+d} \end{bmatrix} && \left[\text{Because } T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right)=\begin{bmatrix} e^x \\ e^y \end{bmatrix}\right] \\ &= \begin{bmatrix} e^a e^c \\ e^b e^d \end{bmatrix} && [\text{Using the rules of indices}] \end{aligned}$$

$$\begin{aligned} T(\mathbf{u})+T(\mathbf{v}) &= T\left(\begin{bmatrix} a \\ b \end{bmatrix}\right)+T\left(\begin{bmatrix} c \\ d \end{bmatrix}\right) \\ &= \begin{bmatrix} e^a \\ e^b \end{bmatrix}+\begin{bmatrix} e^c \\ e^d \end{bmatrix}=\begin{bmatrix} e^a+e^c \\ e^b+e^d \end{bmatrix} \neq \begin{bmatrix} e^a e^c \\ e^b e^d \end{bmatrix} \end{aligned}$$

Thus $T(\mathbf{u})+T(\mathbf{v}) \neq T(\mathbf{u})+T(\mathbf{v})$ which means that T is **not** a linear transformation.