Complete Solutions to Exercise 2.2

1. (a) We apply formula (2.5) which says $\cos(\theta)=\dfrac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{u}\|\|\mathbf{v}\|}$. We find each of the components $\mathbf{u}\cdot\mathbf{v}$, $\|\mathbf{u}\|$ and $\|\mathbf{v}\|$:

$$\mathbf{u}\cdot\mathbf{v}=\begin{pmatrix}1\\1\end{pmatrix}\cdot\begin{pmatrix}0\\1\end{pmatrix}=(1\times 0)+(1\times 1)=1$$

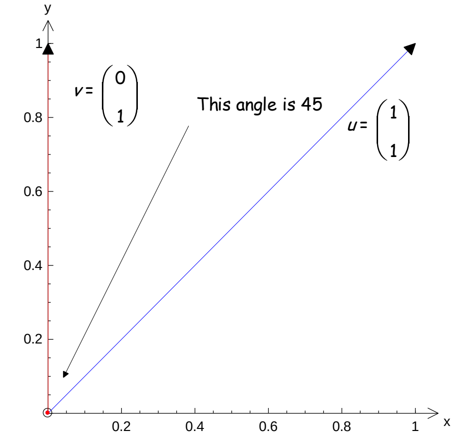
$$\|\mathbf{u}\|=\left\|\begin{pmatrix}1\\1\end{pmatrix}\right\|=\sqrt{1^2+1^2}=\sqrt{2}$$

$$\|\mathbf{v}\|=\left\|\begin{pmatrix}0\\1\end{pmatrix}\right\|=\sqrt{0^2+1^2}=1$$

Substituting these, $\mathbf{u}\cdot\mathbf{v}=1$, $\|\mathbf{u}\|=\sqrt{2}$ and $\|\mathbf{v}\|=1$ into $\cos(\theta)=\dfrac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{u}\|\|\mathbf{v}\|}$ gives

$$\cos(\theta)=\frac{1}{\sqrt{2}\times 1}=\frac{1}{\sqrt{2}} \quad \text{gives} \quad \theta=\cos^{-1}\left(\frac{1}{\sqrt{2}}\right)=45°$$

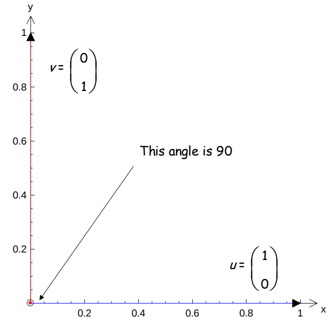
Hence the angle between the vectors is $\theta=45°$.



(b) We apply formula (2.5) $\cos(\theta)=\dfrac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{u}\|\|\mathbf{v}\|}$. We find each of the components $\mathbf{u}\cdot\mathbf{v}$, $\|\mathbf{u}\|$ and $\|\mathbf{v}\|$:

$$\mathbf{u}\cdot\mathbf{v}=\begin{pmatrix}1\\0\end{pmatrix}\cdot\begin{pmatrix}0\\1\end{pmatrix}=(1\times 0)+(0\times 1)=0$$

Since $\mathbf{u}\cdot\mathbf{v}=0$ so the two vectors are perpendicular. Hence the angle between the vectors is $\theta=90°$.



(c) We apply formula (2.5) $\cos(\theta)=\dfrac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{u}\|\|\mathbf{v}\|}$. Each of the components $\mathbf{u}\cdot\mathbf{v}$, $\|\mathbf{u}\|$ and $\|\mathbf{v}\|$ is equal to:

$$\mathbf{u}\cdot\mathbf{v}=\begin{pmatrix}-2\\3\end{pmatrix}\cdot\begin{pmatrix}\frac{1}{2}\\-\frac{1}{2}\end{pmatrix}=\left(-2\times\frac{1}{2}\right)+\left(3\times\left(-\frac{1}{2}\right)\right)=-\frac{5}{2}$$

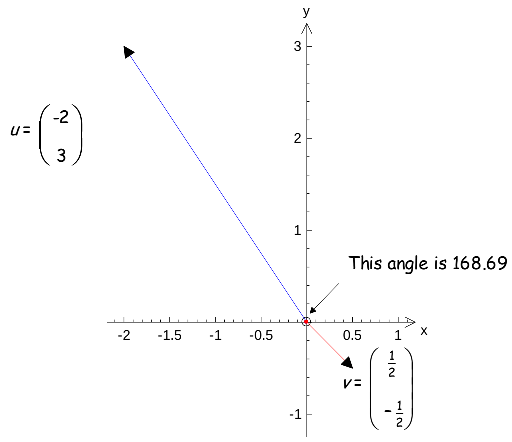
$$\|\mathbf{u}\|=\left\|\begin{pmatrix}-2\\3\end{pmatrix}\right\|=\sqrt{(-2)^2+3^2}=\sqrt{13}$$

$$\|\mathbf{v}\|=\left\|\begin{pmatrix}\frac{1}{2}\\-\frac{1}{2}\end{pmatrix}\right\|=\sqrt{\left(\frac{1}{2}\right)^2+\left(-\frac{1}{2}\right)^2}=\sqrt{\frac{1}{2}}=\frac{1}{\sqrt{2}}$$

Substituting these, $\mathbf{u}\cdot\mathbf{v}=-\frac{5}{2}$, $\|\mathbf{u}\|=\sqrt{13}$ and $\|\mathbf{v}\|=\frac{1}{\sqrt{2}}$ into $\cos(\theta)=\dfrac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{u}\|\|\mathbf{v}\|}$ gives

$$\cos(\theta)=\frac{-\frac{5}{2}}{\sqrt{13}\times\frac{1}{\sqrt{2}}}=-0.98 \quad \text{gives} \quad \theta=\cos^{-1}(-0.98)=168.69^\circ$$

Hence the angle between the vectors is $\theta=168.69^\circ$. Note that the two vectors are working in opposite direction.



2. (a) We apply formula (2.5) which says $\cos(\theta)=\dfrac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{u}\|\|\mathbf{v}\|}$. We find each of the components $\mathbf{u}\cdot\mathbf{v}$, $\|\mathbf{u}\|$ and $\|\mathbf{v}\|$:

$$\mathbf{u}\cdot\mathbf{v}=\begin{pmatrix}-1\\1\\3\end{pmatrix}\cdot\begin{pmatrix}3\\-1\\5\end{pmatrix}=(-1\times3)+(1\times(-1))+(3\times5)=11$$

$$\|\mathbf{u}\|=\left\|\begin{pmatrix}-1\\1\\3\end{pmatrix}\right\|=\sqrt{(-1)^2+1^2+3^2}=\sqrt{11}$$

$$\|\mathbf{v}\| = \left\| \begin{pmatrix} 3 \\ -1 \\ 5 \end{pmatrix} \right\| = \sqrt{3^2 + (-1)^2 + 5^2} = \sqrt{35}$$

Substituting these, $\mathbf{u}\cdot\mathbf{v} = 11$, $\|\mathbf{u}\| = \sqrt{11}$ and $\|\mathbf{v}\| = \sqrt{35}$ into $\cos(\theta) = \dfrac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{u}\|\|\mathbf{v}\|}$ gives

$$\cos(\theta) = \frac{11}{\sqrt{11}\times\sqrt{35}} = \sqrt{\frac{11}{35}} \quad \text{gives } \theta = \cos^{-1}\left(\sqrt{\frac{11}{35}}\right) = 55.90^\circ$$

Hence the angle between the vectors is $\theta = 55.90^\circ$.

(b) We apply formula (2.5) $\cos(\theta) = \dfrac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{u}\|\|\mathbf{v}\|}$. We find each of the components $\mathbf{u}\cdot\mathbf{v}$, $\|\mathbf{u}\|$ and $\|\mathbf{v}\|$:

$$\mathbf{u}\cdot\mathbf{v} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 0 \\ 15 \end{pmatrix} = (1\times 0) + (0\times 0) + (0\times 15) = 0$$

Since $\mathbf{u} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ and $\mathbf{v} = \begin{pmatrix} 0 \\ 0 \\ 15 \end{pmatrix}$ are **not** the zero vectors therefore the norms of these **cannot equal zero**. Hence we have $\cos(\theta) = \dfrac{0}{\|\mathbf{u}\|\|\mathbf{v}\|} = 0$ because $\mathbf{u}\cdot\mathbf{v} = 0$. Therefore $\theta = \cos^{-1}(0) = 90^\circ$.

(c) We apply formula (2.5) $\cos(\theta) = \dfrac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{u}\|\|\mathbf{v}\|}$. We find each of the components $\mathbf{u}\cdot\mathbf{v}$, $\|\mathbf{u}\|$ and $\|\mathbf{v}\|$:

$$\mathbf{u}\cdot\mathbf{v} = \begin{pmatrix} -1 \\ 2 \\ 3 \end{pmatrix} \cdot \begin{pmatrix} \sqrt{2} \\ \dfrac{1}{\sqrt{2}} \\ -1 \end{pmatrix} = \left(-1\times\sqrt{2}\right) + \left(2\times\left(\frac{1}{\sqrt{2}}\right)\right) + \left(3\times(-1)\right) = -3,$$

$$\|\mathbf{u}\| = \left\| \begin{pmatrix} -1 \\ 2 \\ 3 \end{pmatrix} \right\| = \sqrt{(-1)^2 + 2^2 + 3^2} = \sqrt{14}$$

$$\|\mathbf{v}\| = \left\| \begin{pmatrix} \sqrt{2} \\ \dfrac{1}{\sqrt{2}} \\ -1 \end{pmatrix} \right\| = \sqrt{\left(\sqrt{2}\right)^2 + \left(\frac{1}{\sqrt{2}}\right)^2 + (-1)^2} = \sqrt{2 + \frac{1}{2} + 1} = \sqrt{\frac{7}{2}}$$

Substituting these, $\mathbf{u}\cdot\mathbf{v} = -3$, $\|\mathbf{u}\| = \sqrt{14}$ and $\|\mathbf{v}\| = \sqrt{\dfrac{7}{2}}$ into $\cos(\theta) = \dfrac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{u}\|\|\mathbf{v}\|}$:

$$\cos(\theta)=\frac{-3}{\sqrt{14}\times\sqrt{7/2}}=-\frac{3}{\sqrt{7^2}}=-\frac{3}{7} \quad \text{gives } \theta=\cos^{-1}\left(-\frac{3}{7}\right)=115.38°$$

Hence the angle between the vectors is $\theta=115.38°$.

3. (a) We need to determine the angle between the given vectors $\mathbf{u}=\begin{pmatrix}2\\3\\-8\\1\end{pmatrix}$, $\mathbf{v}=\begin{pmatrix}-1\\2\\-5\\-3\end{pmatrix}$ by using the formula $\cos(\theta)=\dfrac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{u}\|\|\mathbf{v}\|}$.

$$\mathbf{u}\cdot\mathbf{v}=\begin{pmatrix}2\\3\\-8\\1\end{pmatrix}\cdot\begin{pmatrix}-1\\2\\-5\\-3\end{pmatrix}=(2\times(-1))+(3\times2)+(-8\times(-5))+(1\times(-3))=41$$

$$\|\mathbf{u}\|=\left\|\begin{pmatrix}2\\3\\-8\\1\end{pmatrix}\right\|=\sqrt{2^2+3^2+(-8)^2+1^2}=\sqrt{78}$$

$$\|\mathbf{v}\|=\left\|\begin{pmatrix}-1\\2\\-5\\-3\end{pmatrix}\right\|=\sqrt{(-1)^2+2^2+(-5)^2+(-3)^2}=\sqrt{39}$$

Substituting these, $\mathbf{u}\cdot\mathbf{v}=41$, $\|\mathbf{u}\|=\sqrt{78}$ and $\|\mathbf{v}\|=\sqrt{39}$ into $\cos(\theta)=\dfrac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{u}\|\|\mathbf{v}\|}$ gives

$$\cos(\theta)=\frac{41}{\sqrt{78}\times\sqrt{39}}=0.743 \quad \text{gives } \theta=\cos^{-1}(0.743)=41.98°$$

Hence the angle between the vectors is $\theta=41.98°$.

(b) We need to find the angle between the vectors $\mathbf{u}=\begin{pmatrix}-2\\-3\\-1\\-1\end{pmatrix}$, $\mathbf{v}=\begin{pmatrix}1\\2\\3\\4\end{pmatrix}$:

$$\mathbf{u}\cdot\mathbf{v}=\begin{pmatrix}-2\\-3\\-1\\-1\end{pmatrix}\cdot\begin{pmatrix}1\\2\\3\\4\end{pmatrix}=(-2\times1)+(-3\times2)+(-1\times3)+(-1\times4)=-15$$

$$\|\mathbf{u}\|=\left\|\begin{pmatrix}-2\\-3\\-1\\-1\end{pmatrix}\right\|=\sqrt{(-2)^2+(-3)^2+(-1)^2+(-1)^2}=\sqrt{15}$$

$$\|\mathbf{v}\| = \left\|\begin{pmatrix}1\\2\\3\\4\end{pmatrix}\right\| = \sqrt{1^2+2^2+3^2+4^2} = \sqrt{30}$$

Substituting these, $\mathbf{u}\cdot\mathbf{v} = -15$, $\|\mathbf{u}\| = \sqrt{15}$ and $\|\mathbf{v}\| = \sqrt{30}$ into $\cos(\theta) = \dfrac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{u}\|\|\mathbf{v}\|}$ gives

$$\cos(\theta) = -\frac{15}{\sqrt{15}\times\sqrt{30}} = -\frac{1}{\sqrt{2}} \text{ gives } \theta = \cos^{-1}\left(-\frac{1}{\sqrt{2}}\right) = 135°$$

Hence the angle between the given vectors is $\theta = 135°$.

(c) We need to find the angle between the given vectors $\mathbf{u} = \begin{pmatrix}\pi\\\sqrt{2}\\0\\1\end{pmatrix}$, $\mathbf{v} = \begin{pmatrix}1/\pi\\\sqrt{2}\\-1\\1\end{pmatrix}$:

$$\mathbf{u}\cdot\mathbf{v} = \begin{pmatrix}\pi\\\sqrt{2}\\0\\1\end{pmatrix}\cdot\begin{pmatrix}1/\pi\\\sqrt{2}\\-1\\1\end{pmatrix} = \left(\pi\times\frac{1}{\pi}\right) + \left(\sqrt{2}\times\sqrt{2}\right) + \left(0\times(-1)\right) + (1\times1) = 4$$

$$\|\mathbf{u}\| = \left\|\begin{pmatrix}\pi\\\sqrt{2}\\0\\1\end{pmatrix}\right\| = \sqrt{\pi^2+\left(\sqrt{2}\right)^2+0^2+1^2} = 3.587$$

$$\|\mathbf{v}\| = \left\|\begin{pmatrix}1/\pi\\\sqrt{2}\\-1\\1\end{pmatrix}\right\| = \sqrt{\left(\frac{1}{\pi}\right)^2+\left(\sqrt{2}\right)^2+(-1)^2+1^2} = \sqrt{4.101} = 2.025$$

Substituting these, $\mathbf{u}\cdot\mathbf{v} = 4$, $\|\mathbf{u}\| = 3.587$ and $\|\mathbf{v}\| = 2.025$ into $\cos(\theta) = \dfrac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{u}\|\|\mathbf{v}\|}$:

$$\cos(\theta) = \frac{4}{3.587\times2.025} = 0.551 \text{ gives } \theta = \cos^{-1}(0.551) = 56.56°$$

Hence the angle between the vectors is $\theta = 56.56°$.

4. Vectors are orthogonal if and only if $\mathbf{u}\cdot\mathbf{v} = 0$.

(a) We need to find k so that dot product of the vectors $\mathbf{u} = \begin{pmatrix}-1\\5\\k\end{pmatrix}$, $\mathbf{v} = \begin{pmatrix}-3\\2\\7\end{pmatrix}$ is 0.

$$\mathbf{u}\cdot\mathbf{v}=\begin{pmatrix}-1\\5\\k\end{pmatrix}\cdot\begin{pmatrix}-3\\2\\7\end{pmatrix}=\left(-1\times(-3)\right)+\left(5\times 2\right)+\left(k\times 7\right)$$

$$=3+10+7k=7k+13=0 \quad \text{which gives } k=-\frac{13}{7}$$

We have $k=-\frac{13}{7}$.

(b) Similarly we have

$$\mathbf{u}\cdot\mathbf{v}=\begin{pmatrix}2\\-1\\3\end{pmatrix}\cdot\begin{pmatrix}3\\1\\k\end{pmatrix}=\left(2\times 3\right)+\left(-1\times 1\right)+\left(3\times k\right)$$

$$=6-1+3k=3k+5=0 \quad \text{which gives } k=-\frac{5}{3}$$

Hence $k=-\frac{5}{3}$.

(c) We need to find k so that $\mathbf{u}\cdot\mathbf{v}=0$:

$$\mathbf{u}\cdot\mathbf{v}=\begin{pmatrix}0\\-k\\\sqrt{2}\end{pmatrix}\cdot\begin{pmatrix}-7\\5\\k\end{pmatrix}=\left(0\times(-7)\right)+\left(-k\times 5\right)+\left(\sqrt{2}\times k\right)$$

$$=0-5k+\sqrt{2}k=\left(\sqrt{2}-5\right)k=0 \quad \text{which gives } k=0$$

Hence $k=0$.

5. We use formula (2.7) $\hat{\mathbf{u}}=\frac{1}{\|\mathbf{u}\|}\mathbf{u}$:

(a) We have $\mathbf{u}=\begin{pmatrix}2\\3\end{pmatrix}$ therefore $\|\mathbf{u}\|=\left\|\begin{pmatrix}2\\3\end{pmatrix}\right\|=\sqrt{2^2+3^2}=\sqrt{13}$ and

$$\hat{\mathbf{u}}=\frac{1}{\|\mathbf{u}\|}\mathbf{u}=\frac{1}{\sqrt{13}}\begin{pmatrix}2\\3\end{pmatrix}$$

(b) Similarly we can expand this to $\mathbb{R}^3$:

$$\|\mathbf{u}\|=\left\|\begin{pmatrix}1\\2\\3\end{pmatrix}\right\|=\sqrt{1^2+2^2+3^2}=\sqrt{14}$$

The unit vector is given by

$$\hat{\mathbf{u}}=\frac{1}{\|\mathbf{u}\|}\mathbf{u}=\frac{1}{\sqrt{14}}\begin{pmatrix}1\\2\\3\end{pmatrix}$$

We have $\hat{\mathbf{u}}=\frac{1}{\sqrt{14}}\begin{pmatrix}1\\2\\3\end{pmatrix}$.

(c) We can rewrite the given vector as $\mathbf{u}=\begin{pmatrix}\frac{1}{2}\\ -\frac{1}{2}\\ \frac{1}{4}\end{pmatrix}=\frac{1}{4}\begin{pmatrix}2\\ -2\\ 1\end{pmatrix}$ and use the earlier Proposition (2-7) property (b) $\|k\mathbf{u}\|=|k|\|\mathbf{u}\|$. (This reduces our arithmetic).

$$\|\mathbf{u}\|=\left\|\frac{1}{4}\begin{pmatrix}2\\ -2\\ 1\end{pmatrix}\right\|=\frac{1}{4}\left\|\begin{pmatrix}2\\ -2\\ 1\end{pmatrix}\right\|=\frac{1}{4}\sqrt{\left(2^2+(-2)^2+1^2\right)}=\frac{1}{4}\sqrt{9}=\frac{3}{4}$$

$$\hat{\mathbf{u}}=\frac{1}{\|\mathbf{u}\|}\mathbf{u}=\frac{1}{3/4}\begin{pmatrix}\frac{1}{2}\\ -\frac{1}{2}\\ \frac{1}{4}\end{pmatrix}=\frac{4}{3}\frac{1}{4}\begin{pmatrix}2\\ -2\\ 1\end{pmatrix}=\frac{1}{3}\begin{pmatrix}2\\ -2\\ 1\end{pmatrix}\qquad\left[\text{Cancelling 4's}\right]$$

Hence $\hat{\mathbf{u}}=\frac{1}{3}\begin{pmatrix}2\\ -2\\ 1\end{pmatrix}$.

(d) We need to normalize the given vector $\mathbf{u}=\begin{pmatrix}\sqrt{2}\\ 2\\ -\sqrt{2}\\ \sqrt{2}\end{pmatrix}$ which can be written as

$$\mathbf{u}=\begin{pmatrix}\sqrt{2}\\ 2\\ -\sqrt{2}\\ \sqrt{2}\end{pmatrix}=\sqrt{2}\begin{pmatrix}1\\ \sqrt{2}\\ -1\\ 1\end{pmatrix}$$

Again we can use Proposition (2-7) property (b) $\|k\mathbf{u}\|=|k|\|\mathbf{u}\|$:

$$\|\mathbf{u}\|=\left\|\sqrt{2}\begin{pmatrix}1\\ \sqrt{2}\\ -1\\ 1\end{pmatrix}\right\|=\sqrt{2}\left\|\begin{pmatrix}1\\ \sqrt{2}\\ -1\\ 1\end{pmatrix}\right\|=\sqrt{2}\sqrt{\left(1^2+\left(\sqrt{2}\right)^2+(-1)^2+1^2\right)}=\sqrt{2}\sqrt{5}$$

$$\hat{\mathbf{u}}=\frac{1}{\|\mathbf{u}\|}\mathbf{u}=\frac{1}{\sqrt{5}\sqrt{2}}\begin{pmatrix}\sqrt{2}\\ 2\\ -\sqrt{2}\\ \sqrt{2}\end{pmatrix}=\frac{1}{\sqrt{5}\sqrt{2}}\sqrt{2}\begin{pmatrix}1\\ \sqrt{2}\\ -1\\ 1\end{pmatrix}=\frac{1}{\sqrt{5}}\begin{pmatrix}1\\ \sqrt{2}\\ -1\\ 1\end{pmatrix}\qquad\left[\text{Cancelling }\sqrt{2}\right]$$

The unit vector is $\hat{\mathbf{u}} = \frac{1}{\sqrt{5}}\begin{pmatrix} 1 \\ \sqrt{2} \\ -1 \\ 1 \end{pmatrix}$.

(e) We need to normalize the vector $\mathbf{u} = \begin{pmatrix} -\pi/5 \\ \pi \\ -\pi \\ \pi/10 \\ 0 \end{pmatrix}$ which is equal to

$$\mathbf{u} = \begin{pmatrix} -\pi/5 \\ \pi \\ -\pi \\ \pi/10 \\ 0 \end{pmatrix} = \frac{\pi}{10}\begin{pmatrix} -2 \\ 10 \\ -10 \\ 1 \\ 0 \end{pmatrix}$$

The norm (length) of this vector is

$$\|\mathbf{u}\| = \left\| \frac{\pi}{10}\begin{pmatrix} -2 \\ 10 \\ -10 \\ 1 \\ 0 \end{pmatrix} \right\| = \frac{\pi}{10}\left\| \begin{pmatrix} -2 \\ 10 \\ -10 \\ 1 \\ 0 \end{pmatrix} \right\|$$

$$= \frac{\pi}{10}\left(\sqrt{(-2)^2 + 10^2 + (-10)^2 + 1^2 + 0^2}\right) = \frac{\pi}{10}\sqrt{205}$$

The unit vector is given by

$$\hat{\mathbf{u}} = \frac{1}{\|\mathbf{u}\|}\mathbf{u} = \frac{1}{\frac{\pi}{10}\sqrt{205}}\frac{\pi}{10}\begin{pmatrix} -2 \\ 10 \\ -10 \\ 1 \\ 0 \end{pmatrix} = \frac{1}{\sqrt{205}}\begin{pmatrix} -2 \\ 10 \\ -10 \\ 1 \\ 0 \end{pmatrix} \qquad \left[\text{Cancelling } \frac{\pi}{10}\text{'}s\right]$$

Hence $\hat{\mathbf{u}} = \frac{1}{\sqrt{205}}\begin{pmatrix} -2 \\ 10 \\ -10 \\ 1 \\ 0 \end{pmatrix}$.

6. For unit vector means that the norm of the given vector is 1, that is

$$\|\hat{\mathbf{u}}\| = \left\| \begin{pmatrix} 1/\sqrt{2} \\ 1/2 \\ k \end{pmatrix} \right\| = \sqrt{\left(\frac{1}{\sqrt{2}}\right)^2 + \left(\frac{1}{2}\right)^2 + k^2} = 1$$

How can we find the value of k from $\sqrt{\left(\frac{1}{\sqrt{2}}\right)^2+\left(\frac{1}{2}\right)^2+k^2}=1$*?*

Square both sides and simplify:

$$\left(\frac{1}{\sqrt{2}}\right)^2+\left(\frac{1}{2}\right)^2+k^2=\frac{1}{2}+\frac{1}{4}+k^2$$

$$=k^2+\frac{3}{4}=1 \quad \text{gives} \quad k^2=\frac{1}{4}$$

Therefore $k=\pm\sqrt{\frac{1}{4}}=\pm\frac{1}{2}$. Hence $k=\frac{1}{2}$ or $k=-\frac{1}{2}$.

7. (a) We need to prove the norm of $\mathbf{u}=\begin{pmatrix}\cos(\theta)\\ \sin(\theta)\end{pmatrix}$ is 1:

$$\|\mathbf{u}\|=\left\|\begin{pmatrix}\cos(\theta)\\ \sin(\theta)\end{pmatrix}\right\|=\sqrt{[\cos(\theta)]^2+[\sin(\theta)]^2}$$

$$=\sqrt{\cos^2(\theta)+\sin^2(\theta)}$$

$$=\sqrt{1}=1 \qquad \begin{bmatrix}\text{Follows by Fundamental identity}\\ \cos^2(\theta)+\sin^2(\theta)\equiv 1\end{bmatrix}$$

(b) When $\theta=\frac{\pi}{4}$ we have

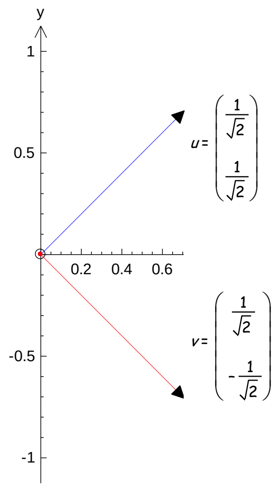
$$\mathbf{u}=\begin{pmatrix}\cos(\theta)\\ \sin(\theta)\end{pmatrix}=\begin{pmatrix}\cos\left(\frac{\pi}{4}\right)\\ \sin\left(\frac{\pi}{4}\right)\end{pmatrix}=\begin{pmatrix}\frac{1}{\sqrt{2}}\\ \frac{1}{\sqrt{2}}\end{pmatrix}$$

See diagram below for plot of the vector **u**.

(c) Similarly substituting $\theta=\frac{\pi}{4}$ into **v** we have

$$\mathbf{v}=\begin{pmatrix}\cos(\theta)\\ -\sin(\theta)\end{pmatrix}=\begin{pmatrix}\cos\left(\frac{\pi}{4}\right)\\ -\sin\left(\frac{\pi}{4}\right)\end{pmatrix}=\begin{pmatrix}\frac{1}{\sqrt{2}}\\ -\frac{1}{\sqrt{2}}\end{pmatrix}$$

In $\mathbb{R}^2$ we have



(d) From the diagram it looks as if the vectors **u** and **v** are orthogonal but we can check this by using the formula $\cos(\theta) = \dfrac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{u}\|\|\mathbf{v}\|}$.

Evaluating the dot product, $\mathbf{u}\cdot\mathbf{v}$, $\|\mathbf{u}\|$ and $\|\mathbf{v}\|$ for $\mathbf{u} = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}$ and $\mathbf{v} = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{pmatrix}$

we have

$$\mathbf{u}\cdot\mathbf{v} = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix} \cdot \begin{pmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{pmatrix} = \left(\frac{1}{\sqrt{2}} \times \frac{1}{\sqrt{2}}\right) + \left(\frac{1}{\sqrt{2}} \times \left(-\frac{1}{\sqrt{2}}\right)\right) = \frac{1}{2} - \frac{1}{2} = 0$$

The given vectors **u** and **v** are **not** zero vectors therefore $\|\mathbf{u}\|$ and $\|\mathbf{v}\|$ **cannot equal zero**. Therefore vectors **u** and **v** are orthogonal because the dot product is zero, $\mathbf{u}\cdot\mathbf{v} = 0$.

8. We need to prove that $\mathbf{u} = \begin{pmatrix} a \\ b \end{pmatrix}$ and $\mathbf{v} = \begin{pmatrix} -b \\ a \end{pmatrix}$ are orthogonal which means we only need to show that $\mathbf{u}\cdot\mathbf{v} = 0$.

$$\mathbf{u}\cdot\mathbf{v} = \begin{pmatrix} a \\ b \end{pmatrix} \cdot \begin{pmatrix} -b \\ a \end{pmatrix} = -ab + ab = 0$$

Since $a \neq 0$ or $b \neq 0$ therefore the vectors **u** and **v** **cannot** be the zero vectors so therefore $\cos(\theta) = \dfrac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{u}\|\|\mathbf{v}\|} = \dfrac{0}{\|\mathbf{u}\|\|\mathbf{v}\|} = 0$. Taking inverse cos gives $\theta = 90°$ therefore the given vectors are orthogonal.

9. We are given that $\mathbf{u} = \begin{pmatrix} \cos(A) \\ \sin(A) \end{pmatrix}$, $\mathbf{v} = \begin{pmatrix} \cos(B) \\ \sin(B) \end{pmatrix}$ and need to show that

$$\mathbf{u}\cdot\mathbf{v} = \cos(A - B)$$

$$\mathbf{u}\cdot\mathbf{v}=\begin{pmatrix}\cos(A)\\ \sin(A)\end{pmatrix}\cdot\begin{pmatrix}\cos(B)\\ \sin(B)\end{pmatrix}$$
$$=\cos(A)\cos(B)+\sin(A)\sin(B)$$
$$=\cos(A-B)\qquad\left[\begin{array}{l}\text{By Trigonometric identity}\\ \cos(A)\cos(B)+\sin(A)\sin(B)=\cos(A-B)\end{array}\right]$$

10. *How do we prove that the angle between a nonzero vector* $\mathbf{u}$ *and* $-\mathbf{u}$ *is* π *?*

Use the formula for the cos of the angle between two vectors $\cos(\theta)=\dfrac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{u}\|\|\mathbf{v}\|}$ where $\mathbf{u}=\mathbf{u}$ and $\mathbf{v}=-\mathbf{u}$.

Proof. Let $\mathbf{u}$ be a nonzero vector in $\mathbb{R}^n$. Then $-\mathbf{u}=(-1)\mathbf{u}$.

$$\mathbf{u}\cdot(-\mathbf{u})=\mathbf{u}\cdot(-1\mathbf{u})$$
$$=(-1\mathbf{u})\cdot\mathbf{u}$$
$$=(-1)(\mathbf{u}\cdot\mathbf{u})=(-1)\|\mathbf{u}\|^2\qquad\left[\text{By (3-7)}\ \ \|\mathbf{u}\|=\sqrt{\mathbf{u}\cdot\mathbf{u}}\right]$$

What is the norm $\|-\mathbf{u}\|$ *equal to?*

Using Proposition (2-7) property (b) $\|k\mathbf{u}\|=|k|\|\mathbf{u}\|$ we have

$$\|-\mathbf{u}\|=\|(-1)\mathbf{u}\|$$
$$=|-1|\|\mathbf{u}\|=1\|\mathbf{u}\|=\|\mathbf{u}\|\qquad\left[\text{Because }|-1|=1\right]$$

Substituting $\mathbf{u}\cdot(-\mathbf{u})=(-1)\|\mathbf{u}\|^2$ and $\|-\mathbf{u}\|=\|\mathbf{u}\|$ into $\cos(\theta)=\dfrac{\mathbf{u}\cdot(-\mathbf{u})}{\|\mathbf{u}\|\|\mathbf{u}\|}$ we have

$$\cos(\theta)=\frac{(-1)\|\mathbf{u}\|^2}{\|\mathbf{u}\|\|\mathbf{u}\|}=(-1)\frac{\|\mathbf{u}\|^2}{\|\mathbf{u}\|^2}=-1\qquad\left[\text{Cancelling Out }\|\mathbf{u}\|^2\right]$$

Taking inverse cos gives $\theta=\cos^{-1}(-1)=180°$ or π radians.

■

11. Remember $\mathbf{u}=\begin{pmatrix}1 & 1 & 1\end{pmatrix}^T=\begin{pmatrix}1\\1\\1\end{pmatrix}$. Let $\mathbf{v}=\begin{pmatrix}x\\y\\z\end{pmatrix}$ then

$$\begin{pmatrix}1\\1\\1\end{pmatrix}\cdot\begin{pmatrix}x\\y\\z\end{pmatrix}=x+y+z=0\quad\Rightarrow\quad x=-y-z$$

Let $y=z=1$ then $x=-y-z=-1-1=-2$. The vector $\begin{pmatrix}1\\1\\-2\end{pmatrix}$ is orthogonal to $\mathbf{u}=\begin{pmatrix}1\\1\\1\end{pmatrix}$.

There are an infinite number of solutions to the equation $x=-y-z$ because we have 1 equation and 3 unknowns so there are $3-1=2$ free variables. Let $y=s$ and $z=t$ where *s* and *t* are any real numbers then $x=-y-z=-s-t$. Hence

$$\mathbf{v} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -s-t \\ s \\ t \end{pmatrix} \text{ is orthogonal to the vector } \mathbf{u} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

for any real numbers *s* and *t*.

12\. (a) We need to find the shortest distance between $\begin{pmatrix} 1 & 1 \end{pmatrix}^T$ and $y = x+1$. The equation $y = x+1$ can be written as $x - y + 1 = 0$ and in dot product form we have

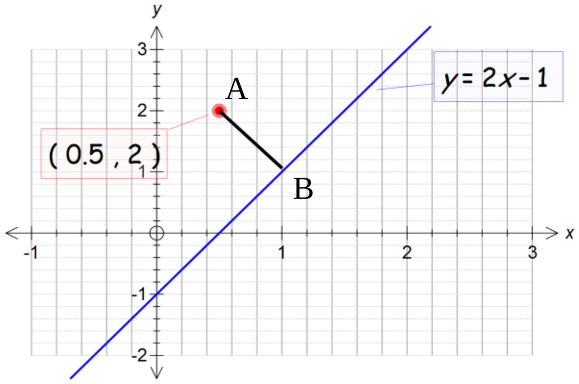
$$\begin{pmatrix} 1 \\ -1 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix} + 1 = 0 \quad \text{[This is of the form } \mathbf{v} \cdot \mathbf{x} + c = 0\text{]}$$

Applying the formula given in the text $\dfrac{|\mathbf{u} \cdot \mathbf{v} + c|}{\|\mathbf{v}\|}$ with $\mathbf{u} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, $\mathbf{v} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ and $c = 1$:

$$\frac{|\mathbf{u} \cdot \mathbf{v} + c|}{\|\mathbf{v}\|} = \frac{\left|\begin{pmatrix} 1 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ -1 \end{pmatrix} + 1\right|}{\left\|\begin{pmatrix} 1 \\ -1 \end{pmatrix}\right\|} = \frac{1-1+1}{\sqrt{(1)^2 + (-1)^2}} = \frac{1}{\sqrt{2}} = 0.71 \text{ (2 dp)}$$

The shortest distance between the line $y = x+1$ and the vector $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ is 0.71.

(b) We need to find the distance of the line AB shown below:



We can write the straight line $y = 2x-1$ as $2x - y - 1 = 0$ and with the dot product:

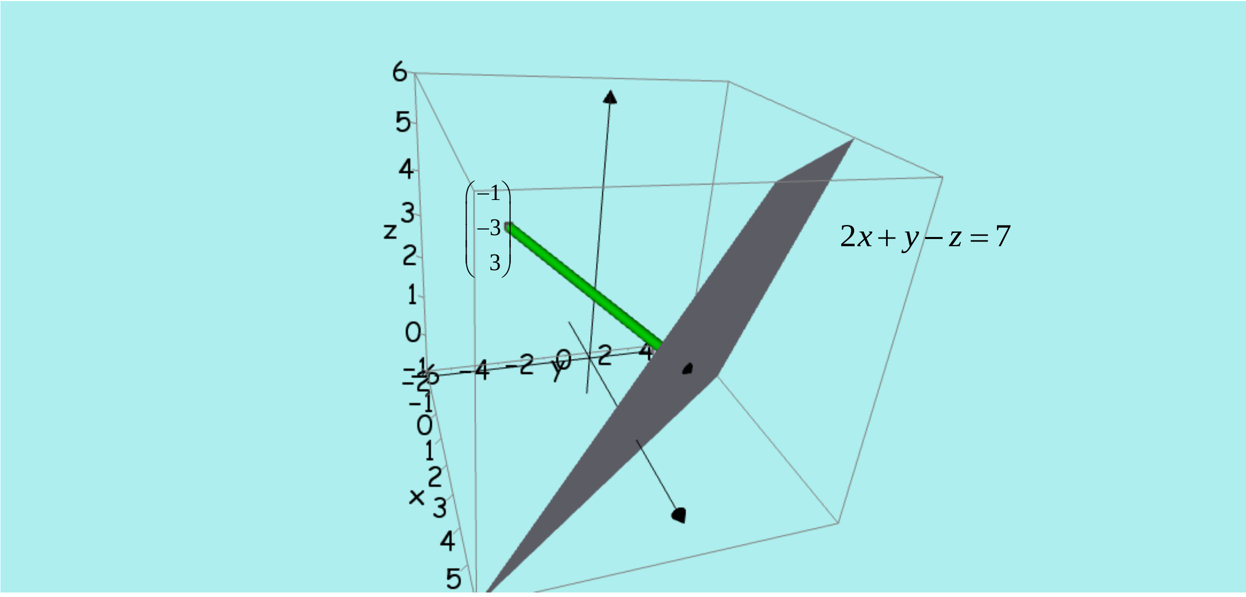
$$\begin{pmatrix} 2 \\ -1 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix} - 1 = 0 \quad \text{[This is of the form } \mathbf{v} \cdot \mathbf{x} + c = 0\text{]}$$

Using the above formula with $\mathbf{u} = \begin{pmatrix} 0.5 \\ 2 \end{pmatrix}$, $\mathbf{v} = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$ and $c = -1$:

$$\frac{|\mathbf{u} \cdot \mathbf{v} + c|}{\|\mathbf{v}\|} = \frac{|(0.5 \times 2) + (2 \times (-1)) - 1|}{\sqrt{2^2 + (-1)^2}} = \frac{|1-2-1|}{\sqrt{5}} = \frac{2}{\sqrt{5}} = 0.89 \text{ (2dp)}$$

Length $\text{AB} = 0.89$.

(c) Similarly we have to find the shortest distance between $\begin{pmatrix} -1 & -3 & 3 \end{pmatrix}^T$ and the plane $2x + y - z = 7$. We need to find the distance of the line shown below.



The equation of the plane $2x+y-z=7$ can be written as $2x+y-z-7=0$ and in dot product form we have

$$\begin{pmatrix}2\\1\\-1\end{pmatrix}\cdot\begin{pmatrix}x\\y\\z\end{pmatrix}-7=0 \quad [\text{This is of the form } \mathbf{v}\cdot\mathbf{x}+c=0]$$

Applying the formula given in the text $\frac{|\mathbf{u}\cdot\mathbf{v}+c|}{\|\mathbf{v}\|}$ with $\mathbf{u}=\begin{pmatrix}-1\\-3\\3\end{pmatrix}$, $\mathbf{v}=\begin{pmatrix}2\\1\\-1\end{pmatrix}$ and $c=-7$:

$$\frac{|\mathbf{u}\cdot\mathbf{v}+c|}{\|\mathbf{v}\|}=\frac{\left|\begin{pmatrix}-1\\-3\\3\end{pmatrix}\cdot\begin{pmatrix}2\\1\\-1\end{pmatrix}-7\right|}{\left\|\begin{pmatrix}2\\1\\-1\end{pmatrix}\right\|}=\frac{\left|(-1\times 2)+(-3\times 1)+(3\times(-1))-7\right|}{\sqrt{(2)^2+1^2+(-1)^2}}=\frac{15}{\sqrt{6}}=6.12 \text{ (2 dp)}$$

The shortest distance between the plane $2x+y-z=7$ and the vector $\begin{pmatrix}-1\\-3\\3\end{pmatrix}$ is 6.12.

(d) Need to find the distance between $\begin{pmatrix}1 & 2 & 3 & 4\end{pmatrix}^T$ and $x+2y+z+w=10$.

The equation of the plane $x+2y+z+w=10$ can be written as $x+2y+z+w-10=0$ and in dot product form we have

$$\begin{pmatrix}1\\2\\1\\1\end{pmatrix}\cdot\begin{pmatrix}x\\y\\z\\w\end{pmatrix}-10=0 \quad [\text{This is of the form } \mathbf{v}\cdot\mathbf{x}+c=0]$$

Applying the formula given in the text $\frac{|\mathbf{u}\cdot\mathbf{v}+c|}{\|\mathbf{v}\|}$ with $\mathbf{u}=\begin{pmatrix}1\\2\\3\\4\end{pmatrix}$, $\mathbf{v}=\begin{pmatrix}1\\2\\1\\1\end{pmatrix}$ and $c=-10$:

$$\frac{|\mathbf{u}\cdot\mathbf{v}+c|}{\|\mathbf{v}\|}=\frac{\left|\begin{pmatrix}1\\2\\3\\4\end{pmatrix}\cdot\begin{pmatrix}1\\2\\1\\1\end{pmatrix}-10\right|}{\left\|\begin{pmatrix}1\\2\\1\\1\end{pmatrix}\right\|}=\frac{|(1\times1)+(2\times2)+(3\times1)+(4\times1)-10|}{\sqrt{1^2+2^2+1^2+1^2}}=\frac{2}{\sqrt{7}}=0.76 \text{ (2 dp)}$$

The shortest distance between the plane $x+2y+z+w=10$ and the vector $\begin{pmatrix}1 & 2 & 3 & 4\end{pmatrix}^T$ is 0.76.

13. Need to prove the Cauchy Schwarz Inequality by looking at

$$(k\mathbf{u}+\mathbf{v})\cdot(k\mathbf{u}+\mathbf{v})$$

We give the whole proof rather than the parts stated in the question.

Proof. Let **u** and **v** be nonzero vectors in $\mathbb{R}^n$. Consider the vector $k\mathbf{u}+\mathbf{v}$ where *k* is a scalar. By Proposition (2-4) property (d) we have

$$(k\mathbf{u}+\mathbf{v})\cdot(k\mathbf{u}+\mathbf{v})\geq 0 \qquad \left[\text{Because by (2-4) property (d)} \quad \mathbf{w}\cdot\mathbf{w}\geq 0\right]$$

Expanding the Left Hand Side we have

$$\begin{aligned}(k\mathbf{u}+\mathbf{v})\cdot(k\mathbf{u}+\mathbf{v}) &= k\mathbf{u}\cdot k\mathbf{u}+k\mathbf{u}\cdot\mathbf{v}+\mathbf{v}\cdot k\mathbf{u}+(\mathbf{v}\cdot\mathbf{v})\\ &= k^2(\mathbf{u}\cdot\mathbf{u})+\underbrace{k(\mathbf{u}\cdot\mathbf{v})+k(\mathbf{u}\cdot\mathbf{v})}_{=2k\mathbf{u}\cdot\mathbf{v}}+(\mathbf{v}\cdot\mathbf{v})\\ &= k^2(\mathbf{u}\cdot\mathbf{u})+2k(\mathbf{u}\cdot\mathbf{v})+(\mathbf{v}\cdot\mathbf{v})\end{aligned}$$

We can write the last term as a quadratic, that is ak^2+bk+c. We have

$$k^2(\mathbf{u}\cdot\mathbf{u})+2k(\mathbf{u}\cdot\mathbf{v})+(\mathbf{v}\cdot\mathbf{v})=ak^2+bk+c$$

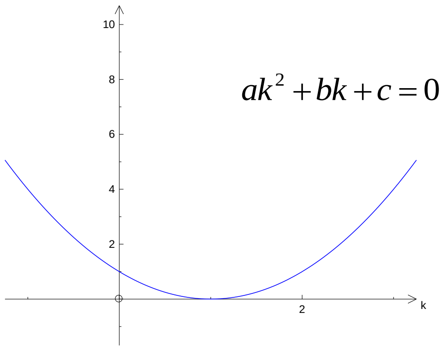
Equating coefficients gives

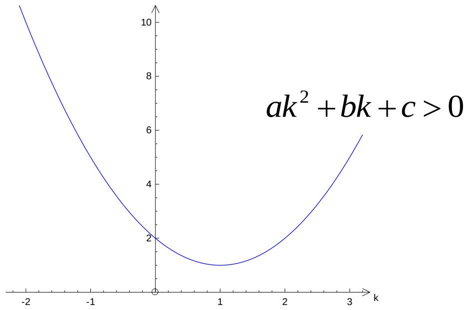
$$a=\mathbf{u}\cdot\mathbf{u},\ b=2(\mathbf{u}\cdot\mathbf{v}) \text{ and } c=\mathbf{v}\cdot\mathbf{v}$$

We have already established that this is greater than or equal to zero

$$k^2(\mathbf{u}\cdot\mathbf{u})+2(\mathbf{u}\cdot\mathbf{v})k+(\mathbf{v}\cdot\mathbf{v})=ak^2+bk+c\geq 0$$

This means that quadratic only has equal roots or what we call complex roots. The graphs of each case of the quadratic is:





We only have these solutions when the discriminate $b^2 \le 4ac$. Substituting $a = \mathbf{u} \cdot \mathbf{u}$, $b = 2\mathbf{u} \cdot \mathbf{v}$ and $c = \mathbf{v} \cdot \mathbf{v}$ into $b^2 \le 4ac$ we have

$$(2\mathbf{u} \cdot \mathbf{v})^2 \le 4(\mathbf{u} \cdot \mathbf{u})(\mathbf{v} \cdot \mathbf{v})$$

$$4(\mathbf{u} \cdot \mathbf{v})^2 \le 4(\mathbf{u} \cdot \mathbf{u})(\mathbf{v} \cdot \mathbf{v})$$

$$(\mathbf{u} \cdot \mathbf{v})^2 \le (\mathbf{u} \cdot \mathbf{u})(\mathbf{v} \cdot \mathbf{v}) \qquad [\text{Dividing through by } 4]$$

$$(\mathbf{u} \cdot \mathbf{v})^2 \le \|\mathbf{u}\|^2 \|\mathbf{v}\|^2 \qquad \left[\text{Because } \mathbf{u} \cdot \mathbf{u} = \|\mathbf{u}\|^2 \text{ and } \mathbf{v} \cdot \mathbf{v} = \|\mathbf{v}\|^2\right]$$

$$\sqrt{(\mathbf{u} \cdot \mathbf{v})^2} \le \sqrt{\|\mathbf{u}\|^2 \|\mathbf{v}\|^2} \qquad [\text{Taking Square Roots}]$$

$$|\mathbf{u} \cdot \mathbf{v}| \le \|\mathbf{u}\| \|\mathbf{v}\| \qquad \left[\text{Because } \sqrt{x^2} = |x|\right]$$

Hence we have CS inequality $|\mathbf{u} \cdot \mathbf{v}| \le \|\mathbf{u}\| \|\mathbf{v}\|$ for our nonzero vectors **u** and **v**.
If one (or both) of vectors are zero then

$$\mathbf{u} \cdot \mathbf{v} = 0 \text{ and } \|\mathbf{u}\| = 0 \text{ or } \|\mathbf{v}\| = 0$$

which means that the Cauchy Schwarz Inequality holds $|\mathbf{u} \cdot \mathbf{v}| \le \|\mathbf{u}\| \|\mathbf{v}\|$ in this case as well.

■

14. We need to prove that $-1 \le \cos(\theta) \le 1$ by using Cauchy Schwarz Inequality.

Proof. By (2.5) $\cos(\theta) = \dfrac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{u}\|\|\mathbf{v}\|}$ we have

$$|\cos(\theta)| = \left|\frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{u}\|\|\mathbf{v}\|}\right| = \frac{|\mathbf{u} \cdot \mathbf{v}|}{\|\mathbf{u}\|\|\mathbf{v}\|} \qquad \left[\begin{array}{l}\text{Because } \|\mathbf{u}\| \ge 0 \text{ and } \|\mathbf{v}\| \ge 0 \\ \text{therefore } \big|\|\mathbf{u}\|\big| = \|\mathbf{u}\| \text{ and } \big|\|\mathbf{v}\|\big| = \|\mathbf{v}\|\end{array}\right]$$

By the Cauchy Schwarz Inequality, $|\mathbf{u} \cdot \mathbf{v}| \le \|\mathbf{u}\| \|\mathbf{v}\|$, we have

$$|\cos(\theta)| = \frac{|\mathbf{u} \cdot \mathbf{v}|}{\|\mathbf{u}\|\|\mathbf{v}\|} \le \frac{\|\mathbf{u}\|\|\mathbf{v}\|}{\|\mathbf{u}\|\|\mathbf{v}\|} = 1 \qquad \left[\text{Cancelling } \|\mathbf{u}\|\|\mathbf{v}\|\right]$$

We have $|\cos(\theta)| \le 1$ which means that $-1 \le \cos(\theta) \le 1$. Hence we have our result.

■

15. *Proof*. By (2.5) $\cos(\theta) = \dfrac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{u}\|\|\mathbf{v}\|}$. Suppose $\cos(\beta) = \dfrac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{u}\|\|\mathbf{v}\|}$ where $0 \le \beta \le \pi$

Subtracting the two gives $\cos(\theta) - \cos(\beta) = 0$. Using the trigonometric identity

$$\cos(A)-\cos(B)=-2\sin\left(\frac{A+B}{2}\right)\sin\left(\frac{A-B}{2}\right)$$

We have

$$\cos(\theta)-\cos(\beta)=-2\sin\left(\frac{\theta+\beta}{2}\right)\sin\left(\frac{\theta-\beta}{2}\right)=0$$

This gives $\sin\left(\frac{\theta+\beta}{2}\right)=0$ or $\sin\left(\frac{\theta-\beta}{2}\right)=0$. From trigonometry we know

$$\sin(x)=0 \text{ gives } x=n\pi$$

Since $0\le\theta\le\pi$ and $0\le\beta\le\pi$ therefore we can only have $n=0$ or $n=1$ because all the other *n* values will give angles outside the stated range.

Let $n=1$ then $\sin\left(\frac{\theta+\beta}{2}\right)=0$ gives

$$\frac{\theta+\beta}{2}=\pi \Rightarrow \theta+\beta=2\pi \Rightarrow \theta=2\pi-\beta$$

The only way that angles lie within $0\le\theta\le\pi$ and $0\le\beta\le\pi$ is when $\beta=\pi$ and then $\theta=2\pi-\pi=\pi$ which means $\theta=\beta$ and so we have our result.

We **cannot have** $\sin\left(\frac{\theta-\beta}{2}\right)=0$ because

$$\frac{\theta-\beta}{2}=\pi \Rightarrow \theta-\beta=2\pi \Rightarrow \theta=2\pi+\beta$$

which gives $\theta>\pi$.
Let $n=0$ then we have

$$\frac{\theta+\beta}{2}=0 \Rightarrow \theta+\beta= \Rightarrow \theta=-\beta$$

Cannot have this result because both angles are positive, $0\le\theta\le\pi$ and $0\le\beta\le\pi$.

If $\sin\left(\frac{\theta-\beta}{2}\right)=0$ then we have

$$\frac{\theta-\beta}{2}=0 \Rightarrow \theta-\beta=0 \Rightarrow \theta=\beta$$

We have our required result $\theta=\beta$ which means that the angle θ between the two vectors is unique.

■