Complete Solutions to Exercises 4.1

1. (a) We have

$$\langle \mathbf{f},\ \mathbf{g}\rangle = \int_0^1 f(x)g(x)\,dx = \int_0^1 xx^2\ \mathrm{d}x \qquad \left[\text{Substituting } f(x)=x \text{ and } g(x)=x^2\right]$$

$$= \int_0^1 x^3\ \mathrm{d}x = \left[\frac{x^4}{4}\right]_0^1 = \frac{1}{4}$$

(b) Since $\langle \mathbf{g},\ \mathbf{f}\rangle = \int_0^1 g(x)\,f(x)\,dx$ is an inner product therefore

$$\langle \mathbf{g},\ \mathbf{f}\rangle = \langle \mathbf{f},\ \mathbf{g}\rangle \underset{\text{By Part (a)}}{=} \frac{1}{4}$$

(c) Again since $\langle 3\mathbf{f},\ \mathbf{g}\rangle$ is an inner product therefore by (4-1) Part (c) we can take the scalar, 3, out:

$$\langle 3\mathbf{f},\ \mathbf{g}\rangle = 3\underbrace{\langle \mathbf{f},\ \mathbf{g}\rangle}_{=\frac{1}{4}\text{ By Part (a)}} = 3\frac{1}{4} = \frac{3}{4}$$

(d) *What is* $\langle \mathbf{f},\ \mathbf{f}\rangle$ *equal to?*

$$\langle \mathbf{f},\ \mathbf{f}\rangle = \int_0^1 f(x)f(x)\,dx = \int_0^1 xx\ \mathrm{d}x \qquad \left[\text{Substituting } f(x)=x\right]$$

$$= \int_0^1 x^2\ \mathrm{d}x = \left[\frac{x^3}{3}\right]_0^1 = \frac{1}{3}$$

(e) By (4.1) we have $\|\mathbf{f}\| = \sqrt{\langle \mathbf{f},\ \mathbf{f}\rangle}$ therefore

$$\|\mathbf{f}\| = \sqrt{\langle \mathbf{f},\ \mathbf{f}\rangle} \underset{\text{By Part (d)}}{=} \sqrt{\frac{1}{3}} = \frac{1}{\sqrt{3}}$$

(f) We have

$$\langle \mathbf{g},\ \mathbf{g}\rangle = \int_0^1 g(x)g(x)\,dx = \int_0^1 x^2x^2\ \mathrm{d}x \qquad \left[\text{Substituting } g(x)=x^2\right]$$

$$= \int_0^1 x^4\ \mathrm{d}x = \left[\frac{x^5}{5}\right]_0^1 = \frac{1}{5}$$

(g) Again as in part (e) we have $\|\mathbf{g}\| = \sqrt{\langle \mathbf{g},\ \mathbf{g}\rangle}$ therefore

$$\|\mathbf{g}\| = \sqrt{\langle \mathbf{g},\ \mathbf{g}\rangle} \underset{\text{By Part (f)}}{=} \sqrt{\frac{1}{5}} = \frac{1}{\sqrt{5}}$$

(4-1) part (c) $\langle k\mathbf{u},\ \mathbf{v}\rangle = k\langle \mathbf{u},\ \mathbf{v}\rangle$

2. (a) We have

$$\langle \mathbf{f},\ \mathbf{g}\rangle = \int_{-1}^{1} f(x)g(x)\,dx = \int_{-1}^{1} xx^3\ \mathrm{d}x \qquad \left[\text{Substituting } f(x) = x \text{ and } g(x) = x^3\right]$$

$$= \int_{-1}^{1} x^4\ \mathrm{d}x = \left[\frac{x^5}{5}\right]_{-1}^{1} = \frac{1}{5} - \left(-\frac{1}{5}\right) = \frac{2}{5}$$

(b) We have

$$\langle \mathbf{g},\ \mathbf{f}\rangle = \langle \mathbf{f},\ \mathbf{g}\rangle \underset{\text{By Part (a)}}{=} \frac{2}{5}$$

(c) Again $\langle 3\mathbf{f},\ \mathbf{g}\rangle$ is an inner product therefore by (4-1) Part (c) we have

$$\langle 3\mathbf{f},\ \mathbf{g}\rangle = 3\ \underbrace{\langle \mathbf{f},\ \mathbf{g}\rangle}_{=\frac{2}{5}\text{ By Part (a)}} = 3\frac{2}{5} = \frac{6}{5}$$

(d) *What is* $\langle \mathbf{f},\ \mathbf{f}\rangle$ *equal to?*

$$\langle \mathbf{f},\ \mathbf{f}\rangle = \int_{-1}^{1} f(x)f(x)\,dx = \int_{-1}^{1} xx\ \mathrm{d}x \qquad \left[\text{Substituting } f(x) = x\right]$$

$$= \int_{-1}^{1} x^2\ \mathrm{d}x = \left[\frac{x^3}{3}\right]_{-1}^{1} = \frac{1}{3} - \left(-\frac{1}{3}\right) = \frac{2}{3}$$

(e) By (4.1) we have $\|\mathbf{f}\| = \sqrt{\langle \mathbf{f},\ \mathbf{f}\rangle}$ therefore

$$\|\mathbf{f}\| = \sqrt{\langle \mathbf{f},\ \mathbf{f}\rangle} \underset{\text{By Part (d)}}{=} \sqrt{\frac{2}{3}}$$

(f) We have

$$\langle \mathbf{g},\ \mathbf{g}\rangle = \int_{-1}^{1} g(x)g(x)\,dx = \int_{-1}^{1} x^3x^3\ \mathrm{d}x$$

$$= \int_{-1}^{1} x^6\ \mathrm{d}x = \left[\frac{x^7}{7}\right]_{-1}^{1} = \frac{1}{7} - \left(-\frac{1}{7}\right) = \frac{2}{7}$$

(g) Again as in part (e) we have $\|\mathbf{g}\| = \sqrt{\langle \mathbf{g},\ \mathbf{g}\rangle}$ therefore

$$\|\mathbf{g}\| = \sqrt{\langle \mathbf{g},\ \mathbf{g}\rangle} \underset{\text{By Part (f)}}{=} \sqrt{\frac{2}{7}}$$

3. We are given $\langle \mathbf{p},\ \mathbf{q}\rangle = c_0d_0 + c_1d_1 + c_2d_2$ where

$$\mathbf{p} = 2 - 3x + 5x^2 \quad \text{and} \quad \mathbf{q} = 7 + 5x - 4x^2$$

(a) We have

$$\langle \mathbf{p},\ \mathbf{q}\rangle = \left\langle 2 - 3x + 5x^2,\ 7 + 5x - 4x^2\right\rangle$$

$$= (2\times 7) + (-3\times 5) + \big(5\times(-4)\big) = -21$$

(4-1) part (c) $\langle k\mathbf{u},\ \mathbf{v}\rangle = k\langle \mathbf{u},\ \mathbf{v}\rangle$

(b) Since $\langle \mathbf{p},\ \mathbf{q}\rangle = \langle \mathbf{q},\ \mathbf{p}\rangle$ we have

$$\langle \mathbf{p},\ \mathbf{q}\rangle = \langle \mathbf{q},\ \mathbf{p}\rangle = -21$$

(c) By theorem (4-1) Part (b) we have $\langle \mathbf{p},\ -3\mathbf{q}\rangle = -3\langle \mathbf{p},\ \mathbf{q}\rangle$ therefore

$$\langle \mathbf{p},\ -3\mathbf{q}\rangle = -3 \underbrace{\langle \mathbf{p},\ \mathbf{q}\rangle}_{=-21 \text{ By Part (a)}} = -3\times(-21) = 63$$

(d) We have

$$\begin{aligned}\langle \mathbf{p},\ \mathbf{p}\rangle &= \langle 2-3x+5x^2,\ 2-3x+5x^2\rangle \\ &= (2\times 2)+(-3\times -3)+(5\times 5) = 38\end{aligned}$$

(e) We have $\|\mathbf{p}\| = \sqrt{\langle \mathbf{p},\ \mathbf{p}\rangle} = \sqrt{38}$.

(f) We have

$$\begin{aligned}\langle \mathbf{q},\ \mathbf{q}\rangle &= \langle 7+5x-4x^2,\ 7+5x-4x^2\rangle \\ &= (7\times 7)+(5\times 5)+(-4\times(-4)) = 90\end{aligned}$$

(g) As before we have $\|\mathbf{q}\| = \sqrt{\langle \mathbf{q},\ \mathbf{q}\rangle} = \sqrt{90}$.

4. By checking *all* 4 axioms of Definition (4-1).

Check (a): Need to check that $\langle \mathbf{u},\ \mathbf{w}\rangle = \langle \mathbf{w},\ \mathbf{u}\rangle$.

$$\begin{aligned}\langle \mathbf{u},\ \mathbf{w}\rangle = \mathbf{u}^T\mathbf{A}\mathbf{w} &= \begin{pmatrix} u_1 & u_2\end{pmatrix}\begin{pmatrix}1 & 2\\ 2 & 5\end{pmatrix}\begin{pmatrix}w_1\\ w_2\end{pmatrix} \\ &= \begin{pmatrix} u_1 & u_2\end{pmatrix}\begin{pmatrix}w_1+2w_2\\ 2w_1+5w_2\end{pmatrix} && \left[\text{Matrix Multiplication}\right] \\ &= u_1(w_1+2w_2)+u_2(2w_1+5w_2) \\ &= u_1w_1+2u_1w_2+2u_2w_1+5u_2w_2 && \left[\text{Expanding}\right]\end{aligned}$$

Going the other way we have

$$\begin{aligned}\langle \mathbf{w},\ \mathbf{u}\rangle = \mathbf{w}^T\mathbf{A}\mathbf{u} &= \begin{pmatrix} w_1 & w_2\end{pmatrix}\begin{pmatrix}1 & 2\\ 2 & 5\end{pmatrix}\begin{pmatrix}u_1\\ u_2\end{pmatrix} \\ &= \begin{pmatrix} w_1 & w_2\end{pmatrix}\begin{pmatrix}u_1+2u_2\\ 2u_1+5u_2\end{pmatrix} && \left[\text{Matrix Multiplication}\right] \\ &= w_1(u_1+2u_2)+w_2(2u_1+5u_2) \\ &= u_1w_1+2u_2w_1+2w_2u_1+5w_2u_2 && \left[\text{Expanding}\right]\end{aligned}$$

Comparing these two, $\langle \mathbf{u},\ \mathbf{w}\rangle = \mathbf{u}^T\mathbf{A}\mathbf{w} = u_1w_1+2u_1w_2+2u_2w_1+5u_2w_2$ and $\langle \mathbf{w},\ \mathbf{u}\rangle = \mathbf{w}^T\mathbf{A}\mathbf{u} = u_1w_1+2u_2w_1+2w_2u_1+5w_2u_2$, we have

$$\langle \mathbf{u},\ \mathbf{w}\rangle = \langle \mathbf{w},\ \mathbf{u}\rangle$$

Hence we have shown part (a) of definition (4-1).

Check (b): Need to check that $\langle \mathbf{u}+\mathbf{v},\ \mathbf{w}\rangle = \langle \mathbf{u},\ \mathbf{w}\rangle + \langle \mathbf{v},\ \mathbf{w}\rangle$. Let $\mathbf{v} = \begin{pmatrix}v_1\\ v_2\end{pmatrix}$ then we have

$$\begin{aligned}\langle \mathbf{u}+\mathbf{v},\ \mathbf{w}\rangle &= (\mathbf{u}+\mathbf{v})^T\,\mathbf{Aw}\\ &= \begin{pmatrix} u_1+v_1 \\ u_2+v_2\end{pmatrix}^T \begin{pmatrix} 1 & 2 \\ 2 & 5\end{pmatrix}\begin{pmatrix} w_1 \\ w_2\end{pmatrix}\\ &= \begin{pmatrix} u_1+v_1 & u_2+v_2\end{pmatrix}\begin{pmatrix} 1 & 2 \\ 2 & 5\end{pmatrix}\begin{pmatrix} w_1 \\ w_2\end{pmatrix}\\ &= \begin{pmatrix} u_1+v_1 & u_2+v_2\end{pmatrix}\begin{pmatrix} w_1+2w_2 \\ 2w_1+5w_2\end{pmatrix}\\ &= (u_1+v_1)(w_1+2w_2)+(u_2+v_2)(2w_1+5w_2)\\ &= u_1w_1+2u_1w_2+v_1w_1+2v_1w_2+2u_2w_1+5u_2w_2+2v_2w_1+5v_2w_2\end{aligned}$$

Similarly by using the results of part (i) above we have

$$\begin{aligned}\langle \mathbf{u},\ \mathbf{w}\rangle+\langle \mathbf{v},\ \mathbf{w}\rangle &= \underbrace{u_1w_1+2u_1w_2+2u_2w_1+5u_2w_2}_{=\langle \mathbf{u},\ \mathbf{w}\rangle}+\underbrace{v_1w_1+2v_1w_2+2v_2w_1+5v_2w_2}_{=\langle \mathbf{v},\ \mathbf{w}\rangle}\\ &= \langle \mathbf{u}+\mathbf{v},\ \mathbf{w}\rangle \qquad [\text{By Above}]\end{aligned}$$

Part (b) of definition (4-1) is satisfied.

<u>Check (c)</u>: Need to check that $\langle k\mathbf{u},\ \mathbf{w}\rangle = k\langle \mathbf{u},\ \mathbf{w}\rangle$:

$$\begin{aligned}\langle k\mathbf{u},\ \mathbf{w}\rangle = (k\mathbf{u})^T\,\mathbf{Aw} &= k\left(\mathbf{u}^T\right)\mathbf{Aw} \qquad [\text{Because } k \text{ is a scalar therefore } k^T = k]\\ &= k\mathbf{u}^T\mathbf{Aw}\\ &= k\langle \mathbf{u},\ \mathbf{w}\rangle \qquad [\text{We are given } \mathbf{u}^T\mathbf{Aw} = \langle \mathbf{u},\ \mathbf{w}\rangle]\end{aligned}$$

Hence part (c) is satisfied.

<u>Check (d)</u>:

Need to show $\langle \mathbf{u},\ \mathbf{u}\rangle \ge 0$ and $\langle \mathbf{u},\ \mathbf{u}\rangle = 0 \iff \mathbf{u} = \mathbf{O}$:

$$\begin{aligned}\langle \mathbf{u},\ \mathbf{u}\rangle = \mathbf{u}^T\mathbf{Au} &= \begin{pmatrix} u_1 & u_2\end{pmatrix}\begin{pmatrix} 1 & 2 \\ 2 & 5\end{pmatrix}\begin{pmatrix} u_1 \\ u_2\end{pmatrix}\\ &= \begin{pmatrix} u_1 & u_2\end{pmatrix}\begin{pmatrix} u_1+2u_2 \\ 2u_1+5u_2\end{pmatrix}\\ &= u_1(u_1+2u_2)+u_2(2u_1+5u_2)\\ &= (u_1)^2+\underbrace{2u_1u_2+2u_2u_1}_{=4u_1u_2}+5(u_2)^2 = (u_1)^2+4u_1u_2+5(u_2)^2\end{aligned}$$

We can rewrite the last line as

$$\begin{aligned}\langle \mathbf{u},\ \mathbf{u}\rangle &= (u_1)^2+4u_1u_2+5(u_2)^2\\ &= \underbrace{(u_1)^2+4u_1u_2+4(u_2)^2}_{=(u_1+2u_2)^2}+(u_2)^2\\ &= (u_1+2u_2)^2+(u_2)^2 \ge 0 \qquad [\text{Because we have square numbers}]\end{aligned}$$

Also

$$\langle \mathbf{u},\ \mathbf{u}\rangle = (u_1+2u_2)^2+(u_2)^2 = 0 \quad \iff \quad u_1 = u_2 = 0$$

$u_1 = u_2 = 0$ means that $\mathbf{u} = \mathbf{O}$. Hence part (d) is fulfilled therefore *all* 4 axioms are satisfied, so we conclude that $\langle \mathbf{u},\ \mathbf{w}\rangle = \mathbf{u}^T\mathbf{Aw}$ is an inner product for $\mathbb{R}^2$.

5. We are given $\langle \mathbf{A},\ \mathbf{B}\rangle = tr\left(\mathbf{B}^T\mathbf{A}\right)$ where

$$\mathbf{A} = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix},\ \mathbf{B} = \begin{pmatrix} 5 & 6 \\ 7 & 8 \end{pmatrix} \text{ and } \mathbf{C} = \begin{pmatrix} -1 & 1 \\ 2 & 5 \end{pmatrix}$$

(a) We need to find $\langle \mathbf{A},\ \mathbf{B}\rangle$:

$$\begin{aligned} \langle \mathbf{A},\ \mathbf{B}\rangle = tr\left(\mathbf{B}^T\mathbf{A}\right) &= tr\left[\begin{pmatrix} 5 & 6 \\ 7 & 8 \end{pmatrix}^T \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}\right] \\ &= tr\left[\begin{pmatrix} 5 & 7 \\ 6 & 8 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}\right] && [\text{Transposed } \mathbf{B}] \\ &= tr\begin{pmatrix} 26 & 38 \\ 30 & 44 \end{pmatrix} && [\text{Matrix Multiplication}] \\ &= 26 + 44 = 70 \end{aligned}$$

(b) $\langle 5\mathbf{A},\ \mathbf{B}\rangle = 5\langle \mathbf{A},\ \mathbf{B}\rangle \underset{\text{By Part (a)}}{=} 5\times 70 = 350$

(c) We can write $-\mathbf{A} = (-1)\mathbf{A}$ and $-\mathbf{B} = (-1)\mathbf{B}$ and then use (4-1) part (c) to take out the scalars:

$$\langle -\mathbf{A},\ -\mathbf{B}\rangle = \langle (-1)\mathbf{A},\ (-1)\mathbf{B}\rangle = (-1)(-1)\langle \mathbf{A},\ \mathbf{B}\rangle \underset{\text{By Part (a)}}{=} \langle \mathbf{A},\ \mathbf{B}\rangle = 70$$

(d) We have $\|\mathbf{A}\| = \sqrt{\langle \mathbf{A},\ \mathbf{A}\rangle}$ but we first find $\|\mathbf{A}\|^2 = \langle \mathbf{A},\ \mathbf{A}\rangle$ and then take the square root:

$$\begin{aligned} \langle \mathbf{A},\ \mathbf{A}\rangle = tr\left(\mathbf{A}^T\mathbf{A}\right) &= tr\left[\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}^T \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}\right] \\ &= tr\left[\begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}\right] && [\text{Transposed } \mathbf{A}] \\ &= tr\begin{pmatrix} 10 & 14 \\ 14 & 20 \end{pmatrix} && [\text{Matrix Multiplication}] \\ &= 10 + 20 = 30 \end{aligned}$$

Hence $\|\mathbf{A}\| = \sqrt{\langle \mathbf{A},\ \mathbf{A}\rangle} = \sqrt{30}$

(e) Similarly we have

$$\begin{aligned} \|\mathbf{B}\| = \langle \mathbf{B},\ \mathbf{B}\rangle = tr\left(\mathbf{B}^T\mathbf{B}\right) &= tr\left[\begin{pmatrix} 5 & 6 \\ 7 & 8 \end{pmatrix}^T \begin{pmatrix} 5 & 6 \\ 7 & 8 \end{pmatrix}\right] \\ &= tr\left[\begin{pmatrix} 5 & 7 \\ 6 & 8 \end{pmatrix}\begin{pmatrix} 5 & 6 \\ 7 & 8 \end{pmatrix}\right] && [\text{Transposed } \mathbf{B}] \\ &= tr\begin{pmatrix} 74 & 86 \\ 86 & 100 \end{pmatrix} && [\text{Matrix Multiplication}] \\ &= 74 + 100 = 174 \end{aligned}$$

Hence $\|\mathbf{B}\| = \sqrt{\langle \mathbf{B},\ \mathbf{B}\rangle} = \sqrt{174}$.

(f) We need to find $\langle \mathbf{A},\ \mathbf{C}\rangle$:

$$\langle \mathbf{A},\ \mathbf{C}\rangle = tr\left(\mathbf{C}^T\mathbf{A}\right) = tr\left[\begin{pmatrix}-1 & 1\\ 2 & 5\end{pmatrix}^T\begin{pmatrix}1 & 2\\ 3 & 4\end{pmatrix}\right]$$

$$= tr\left[\begin{pmatrix}-1 & 2\\ 1 & 5\end{pmatrix}\begin{pmatrix}1 & 2\\ 3 & 4\end{pmatrix}\right] \qquad [\text{Transposed } \mathbf{C}]$$

$$= tr\begin{pmatrix}5 & 6\\ 16 & 22\end{pmatrix} \qquad [\text{Matrix Multiplication}]$$

$$= 5 + 22 = 27$$

(g) We need to determine $\langle \mathbf{B},\ \mathbf{C}\rangle$:

$$\langle \mathbf{B},\ \mathbf{C}\rangle = tr\left(\mathbf{C}^T\mathbf{B}\right) = tr\left[\begin{pmatrix}-1 & 1\\ 2 & 5\end{pmatrix}^T\begin{pmatrix}5 & 6\\ 7 & 8\end{pmatrix}\right]$$

$$= tr\left[\begin{pmatrix}-1 & 2\\ 1 & 5\end{pmatrix}\begin{pmatrix}5 & 6\\ 7 & 8\end{pmatrix}\right] \qquad [\text{Transposed } \mathbf{C}]$$

$$= tr\begin{pmatrix}9 & 10\\ 40 & 46\end{pmatrix} \qquad [\text{Matrix Multiplication}]$$

$$= 9 + 46 = 55$$

(h) We use (4-1) part (b) which says $\langle \mathbf{u}+\mathbf{v},\ \mathbf{w}\rangle = \langle \mathbf{u},\ \mathbf{w}\rangle + \langle \mathbf{v},\ \mathbf{w}\rangle$

$$\langle \mathbf{A}+\mathbf{B},\ \mathbf{C}\rangle = \langle \mathbf{A},\ \mathbf{C}\rangle + \langle \mathbf{B},\ \mathbf{C}\rangle = \underbrace{27}_{\text{By Part (f)}} + \underbrace{55}_{\text{By Part (g)}} = 82$$

(i) Similarly we have for $\langle \mathbf{A},\ \mathbf{C}+\mathbf{B}\rangle$:

$$\langle \mathbf{A},\ \mathbf{C}+\mathbf{B}\rangle = \langle \mathbf{A},\ \mathbf{C}\rangle + \langle \mathbf{A},\ \mathbf{B}\rangle = \underbrace{27}_{\text{By Part (f)}} + \underbrace{70}_{\text{By Part (a)}} = 97$$

6. (a) We need to prove that $\langle \mathbf{u},\ \mathbf{v}-\mathbf{w}\rangle = \langle \mathbf{u},\ \mathbf{v}\rangle - \langle \mathbf{u},\ \mathbf{w}\rangle$.

Proof. We can write $-\mathbf{w} = (-1)\mathbf{w}$ which means we have

$$\langle \mathbf{u},\ \mathbf{v}-\mathbf{w}\rangle = \langle \mathbf{u},\ \mathbf{v}+(-1)\mathbf{w}\rangle$$

$$= \langle \mathbf{u},\ \mathbf{v}\rangle + \langle \mathbf{u},\ (-1)\mathbf{w}\rangle \qquad [\text{By theorem (4-1) Part (c)}]$$

$$= \langle \mathbf{u},\ \mathbf{v}\rangle + (-1)\langle \mathbf{u},\ \mathbf{w}\rangle \qquad [\text{By theorem (4-1) Part (b)}]$$

$$= \langle \mathbf{u},\ \mathbf{v}\rangle - \langle \mathbf{u},\ \mathbf{w}\rangle$$

■

(b) Required to prove $\langle \mathbf{u}-\mathbf{v},\ \mathbf{w}\rangle = \langle \mathbf{u},\ \mathbf{w}\rangle - \langle \mathbf{v},\ \mathbf{w}\rangle$:

Proof. Similarly we have

$$\langle \mathbf{u}-\mathbf{v},\ \mathbf{w}\rangle = \langle \mathbf{u}+(-1)\mathbf{v},\ \mathbf{w}\rangle$$

$$= \langle \mathbf{u},\ \mathbf{w}\rangle + \langle (-1)\mathbf{v},\ \mathbf{w}\rangle \qquad [\text{By definiton (4-1) Part (b)}]$$

$$= \langle \mathbf{u},\ \mathbf{w}\rangle + (-1)\langle \mathbf{v},\ \mathbf{w}\rangle \qquad [\text{By definition (4-1) Part (c)}]$$

$$= \langle \mathbf{u},\ \mathbf{w}\rangle - \langle \mathbf{v},\ \mathbf{w}\rangle$$

■

(c) We need to prove $\langle k_1\mathbf{u},\ k_2\mathbf{v}\rangle = k_1k_2\langle \mathbf{u},\ \mathbf{v}\rangle$:

Proof. We have

$$\begin{aligned}\langle k_1\mathbf{u},\ k_2\mathbf{v}\rangle &= k_1\langle \mathbf{u},\ k_2\mathbf{v}\rangle\\ &= k_1k_2\langle \mathbf{u},\ \mathbf{v}\rangle\end{aligned}$$

■

(d) Required to prove

$$\langle k_1\mathbf{u}+k_2\mathbf{v},\ k_3\mathbf{w}+k_4\mathbf{x}\rangle = k_1k_3\langle \mathbf{u},\ \mathbf{w}\rangle + k_1k_4\langle \mathbf{u},\ \mathbf{x}\rangle + k_2k_3\langle \mathbf{v},\ \mathbf{w}\rangle + k_2k_4\langle \mathbf{v},\ \mathbf{x}\rangle$$

Proof.

We have

$$\begin{aligned}\langle k_1\mathbf{u}+k_2\mathbf{v},\ k_3\mathbf{w}+k_4\mathbf{x}\rangle &= \langle k_1\mathbf{u},\ k_3\mathbf{w}+k_4\mathbf{x}\rangle + \langle k_2\mathbf{v},\ k_3\mathbf{w}+k_4\mathbf{x}\rangle\\ &= \langle k_1\mathbf{u},\ k_3\mathbf{w}\rangle + \langle k_1\mathbf{u},\ k_4\mathbf{x}\rangle + \langle k_2\mathbf{v},\ k_3\mathbf{w}\rangle + \langle k_2\mathbf{v},\ k_4\mathbf{x}\rangle\\ &= k_1k_3\langle \mathbf{u},\ \mathbf{w}\rangle + k_1k_4\langle \mathbf{u},\ \mathbf{x}\rangle + k_2k_3\langle \mathbf{v},\ \mathbf{w}\rangle + k_2k_4\langle \mathbf{v},\ \mathbf{x}\rangle\end{aligned}$$

■

(e) Need to prove

$$\langle k_1\mathbf{u}_1+k_2\mathbf{u}_2+\cdots+k_n\mathbf{u}_n,\ \mathbf{w}\rangle = k_1\langle \mathbf{u}_1,\ \mathbf{w}\rangle + k_2\langle \mathbf{u}_2,\ \mathbf{w}\rangle+\cdots+k_n\langle \mathbf{u}_n,\ \mathbf{w}\rangle$$

Proof. We use mathematical induction.

Clearly the result is true for $n=1$, that is $\langle k_1\mathbf{u}_1,\ \mathbf{w}\rangle = k_1\langle \mathbf{u}_1,\ \mathbf{w}\rangle$

Assume the result is true for $n=m$, that is

$$\langle k_1\mathbf{u}_1+k_2\mathbf{u}_2+\cdots+k_m\mathbf{u}_m,\ \mathbf{w}\rangle = k_1\langle \mathbf{u}_1,\ \mathbf{w}\rangle + k_2\langle \mathbf{u}_2,\ \mathbf{w}\rangle+\cdots+k_m\langle \mathbf{u}_m,\ \mathbf{w}\rangle \qquad (*)$$

Required to prove the result for $n=m+1$, that is we need to prove

$$\langle k_1\mathbf{u}_1+\cdots+k_m\mathbf{u}_m+k_{m+1}\mathbf{u}_{m+1},\ \mathbf{w}\rangle = k_1\langle \mathbf{u}_1,\ \mathbf{w}\rangle+\cdots+k_m\langle \mathbf{u}_m,\ \mathbf{w}\rangle + k_{m+1}\langle \mathbf{u}_{m+1},\ \mathbf{w}\rangle$$

Examining the Left Hand Side we have

$$\begin{aligned}\langle k_1\mathbf{u}_1+\cdots+k_m\mathbf{u}_m+k_{m+1}\mathbf{u}_{m+1},\ \mathbf{w}\rangle &= \langle (k_1\mathbf{u}_1+\cdots+k_m\mathbf{u}_m)+k_{m+1}\mathbf{u}_{m+1},\ \mathbf{w}\rangle\\ &= \langle (k_1\mathbf{u}_1+\cdots+k_m\mathbf{u}_m),\ \mathbf{w}\rangle + \langle k_{m+1}\mathbf{u}_{m+1},\ \mathbf{w}\rangle\\ &= \underbrace{k_1\langle \mathbf{u}_1,\ \mathbf{w}\rangle+\cdots+k_m\langle \mathbf{u}_m,\ \mathbf{w}\rangle}_{\text{By }(*)} + k_{m+1}\langle \mathbf{u}_{m+1},\ \mathbf{w}\rangle\end{aligned}$$

Hence we have our result.

■

7. We are given that $f(x)=x+1,\ \ g(x)=x^2$ and $h(x)=x-1$:

(a) We have

$$\langle \mathbf{f},\ \mathbf{g}\rangle = \int_0^1 f(x)g(x)\,dx = \int_0^1 (x+1)x^2\,\mathrm{d}x \quad \left[\text{Substituting } f(x)=x+1 \text{ and } g(x)=x^2\right]$$

$$= \int_0^1 \left(x^3+x^2\right)\mathrm{d}x = \left[\frac{x^4}{4}+\frac{x^3}{3}\right]_0^1 = \frac{1}{4}+\frac{1}{3} = \frac{7}{12}$$

(b) Similarly we have

$$\langle \mathbf{g},\ \mathbf{h}\rangle = \int_0^1 g(x)h(x)\,dx = \int_0^1 x^2(x-1)\,\mathrm{d}x \qquad \left[\text{Subs } g(x)=x^2 \text{ and } h(x)=x-1\right]$$

$$= \int_0^1 \left(x^3 - x^2\right)\mathrm{d}x = \left[\frac{x^4}{4} - \frac{x^3}{3}\right]_0^1 = \frac{1}{4} - \frac{1}{3} = -\frac{1}{12}$$

(c) We have

$$\langle \mathbf{f},\ \mathbf{h}\rangle = \int_0^1 f(x)h(x)\,dx = \int_0^1 (x+1)(x-1)\,\mathrm{d}x \qquad \left[\begin{array}{l}\text{Substituting } f(x)=x+1\\ \text{and } \ h(x)=x-1\end{array}\right]$$

$$= \int_0^1 \left(x^2-1\right)\mathrm{d}x$$

$$= \left[\frac{x^3}{3} - x\right]_0^1 = \frac{1}{3} - 1 = -\frac{2}{3}$$

(d) We use definition (4-1) part (b) which says $\langle \mathbf{u}+\mathbf{v},\ \mathbf{w}\rangle = \langle \mathbf{u},\ \mathbf{w}\rangle + \langle \mathbf{v},\ \mathbf{w}\rangle$ to evaluate $\langle \mathbf{f}+\mathbf{g},\ \mathbf{h}\rangle$:

$$\langle \mathbf{f}+\mathbf{g},\ \mathbf{h}\rangle = \underbrace{\langle \mathbf{f},\ \mathbf{h}\rangle}_{=-\frac{2}{3} \text{ By Part (c)}} + \underbrace{\langle \mathbf{g},\ \mathbf{h}\rangle}_{=-\frac{1}{12} \text{ By Part (b)}} = -\frac{2}{3} - \frac{1}{12} = -\frac{3}{4}$$

(e) Similarly we have

$$\langle \mathbf{f},\ \mathbf{h}+\mathbf{g}\rangle = \underbrace{\langle \mathbf{f},\ \mathbf{h}\rangle}_{=-\frac{2}{3} \text{ By Part (c)}} + \underbrace{\langle \mathbf{f},\ \mathbf{g}\rangle}_{=\frac{7}{12} \text{ By Part (a)}} = -\frac{2}{3} + \frac{7}{12} = -\frac{1}{12}$$

(f) We can write

$$\langle \mathbf{f}+3\mathbf{g},\ \mathbf{h}\rangle = \langle \mathbf{f},\ \mathbf{h}\rangle + \langle 3\mathbf{g},\ \mathbf{h}\rangle$$

$$= \underbrace{\langle \mathbf{f},\ \mathbf{h}\rangle}_{=-\frac{2}{3} \text{ By Part (c)}} + 3\underbrace{\langle \mathbf{g},\ \mathbf{h}\rangle}_{=-\frac{1}{12} \text{ By Part (b)}} = -\frac{2}{3} - 3\frac{1}{12} = -\frac{11}{12}$$

(g) Similarly we have

$$\langle \mathbf{f}-3\mathbf{g},\ \mathbf{h}\rangle = \langle \mathbf{f}+(-3)\mathbf{g},\ \mathbf{h}\rangle$$

$$= \langle \mathbf{f},\ \mathbf{h}\rangle + \langle -3\mathbf{g},\ \mathbf{h}\rangle$$

$$= \underbrace{\langle \mathbf{f},\ \mathbf{h}\rangle}_{=-\frac{2}{3} \text{ By Part (c)}} - 3\underbrace{\langle \mathbf{g},\ \mathbf{h}\rangle}_{=-\frac{1}{12} \text{ By Part (b)}} = -\frac{2}{3} - 3\left(-\frac{1}{12}\right) = -\frac{5}{12}$$

(h) We also have

$$\langle \mathbf{f}-\mathbf{g},\ \mathbf{h}\rangle = \langle \mathbf{f},\ \mathbf{h}\rangle + \langle -\mathbf{g},\ \mathbf{h}\rangle$$

$$= \underbrace{\langle \mathbf{f},\ \mathbf{h}\rangle}_{=-\frac{2}{3} \text{ By Part (c)}} - \underbrace{\langle \mathbf{g},\ \mathbf{h}\rangle}_{=-\frac{1}{12} \text{ By Part (b)}} = -\frac{2}{3} - \left(-\frac{1}{12}\right) = -\frac{7}{12}$$

(i) Note that we can take out all the scalars:

$$
\begin{aligned}
\langle 2\mathbf{f}+5\mathbf{g},\ 6\mathbf{h}\rangle &= \langle 2\mathbf{f},\ 6\mathbf{h}\rangle + \langle 5\mathbf{g},\ 6\mathbf{h}\rangle \\
&= (2\times 6)\langle \mathbf{f},\ \mathbf{h}\rangle + (5\times 6)\langle \mathbf{g},\ \mathbf{h}\rangle \\
&= 12\underbrace{\langle \mathbf{f},\ \mathbf{h}\rangle}_{=-\frac{2}{3}\text{ By Part (c)}} + 30\underbrace{\langle \mathbf{g},\ \mathbf{h}\rangle}_{=-\frac{1}{12}\text{ By Part (b)}} \\
&= 12\left(-\frac{2}{3}\right)+30\left(-\frac{1}{12}\right) = -10\frac{1}{2}
\end{aligned}
$$

(j) Similarly we have

$$
\begin{aligned}
\langle -10\mathbf{f},\ 2\mathbf{h}+5\mathbf{g}\rangle &= \langle 2\mathbf{h}+5\mathbf{g},\ -10\mathbf{f}\rangle \\
&= \langle 2\mathbf{h},\ -10\mathbf{f}\rangle + \langle 5\mathbf{g},\ -10\mathbf{f}\rangle \\
&= \left(2\times(-10)\right)\langle \mathbf{h},\ \mathbf{f}\rangle + \left(5\times(-10)\right)\langle \mathbf{g},\ \mathbf{f}\rangle \\
&= -20\langle \mathbf{f},\ \mathbf{h}\rangle - 50\langle \mathbf{f},\ \mathbf{g}\rangle \\
&= -20\underbrace{\langle \mathbf{f},\ \mathbf{h}\rangle}_{=-\frac{2}{3}\text{ By Part (c)}} - 50\underbrace{\langle \mathbf{f},\ \mathbf{g}\rangle}_{=\frac{7}{12}\text{ By Part (a)}} \\
&= -20\left(-\frac{2}{3}\right) - 50\left(\frac{7}{12}\right) = -15\frac{5}{6}
\end{aligned}
$$

8. The following is **not** an inner product

$$\langle \mathbf{f},\ \mathbf{g}\rangle = \int_0^1 \left[f(x)-g(x)\right]dx$$

because (4-1) axiom (d) fails, that is we do not have

$$\langle \mathbf{u},\ \mathbf{u}\rangle = 0 \ \Leftrightarrow\ \mathbf{u}=\mathbf{O}$$

We have

$$\langle \mathbf{f},\ \mathbf{f}\rangle = \int_0^1 \left[f(x)-f(x)\right]dx = \int_0^1 0\,dx = 0$$

Clearly $\mathbf{f} = f(x)$ may **not be** zero. For example $f(x)=x$ or any other non-zero function will do.

9. The following is not an inner product on $\mathbb{R}^2$:

$$\langle \mathbf{u},\ \mathbf{v}\rangle = \mathbf{u}^T\mathbf{A}\mathbf{v} \text{ where } \mathbf{A}=\begin{pmatrix}1 & 2\\ 3 & 4\end{pmatrix}$$

Let $\mathbf{u}=\begin{pmatrix}1\\2\end{pmatrix}$ and $\mathbf{v}=\begin{pmatrix}3\\4\end{pmatrix}$ then

$$\langle \mathbf{u},\ \mathbf{v}\rangle = \mathbf{u}^T\mathbf{A}\mathbf{v} = \begin{pmatrix}1 & 2\end{pmatrix}\begin{pmatrix}1 & 2\\ 3 & 4\end{pmatrix}\begin{pmatrix}3\\4\end{pmatrix} = 61$$

But

$$\langle \mathbf{v},\ \mathbf{u}\rangle = \mathbf{v}^T\mathbf{A}\mathbf{u} = \begin{pmatrix}3 & 4\end{pmatrix}\begin{pmatrix}1 & 2\\ 3 & 4\end{pmatrix}\begin{pmatrix}1\\2\end{pmatrix} = 59$$

We have $\langle \mathbf{u},\ \mathbf{v}\rangle = 61 \neq \langle \mathbf{v},\ \mathbf{u}\rangle = 59$ which means it is not commutative therefore not an inner product.

10. We are given $\langle \mathbf{u},\ \mathbf{v} \rangle = u_1v_1 + u_2v_2 + u_3v_3 - u_4v_4$ which is not an inner product. Consider $\mathbf{u} = \begin{pmatrix} 1 & 2 & 2 & 3 \end{pmatrix}^T$ then

$$\langle \mathbf{u},\ \mathbf{u} \rangle = \left\langle \begin{pmatrix} 1 \\ 2 \\ 2 \\ 3 \end{pmatrix},\ \begin{pmatrix} 1 \\ 2 \\ 2 \\ 3 \end{pmatrix} \right\rangle = 1^2 + 2^2 + 2^2 - 3^2 = 0$$

We have $\langle \mathbf{u},\ \mathbf{u} \rangle = 0$ but $\mathbf{u}$ is not the zero vector; $\mathbf{u} \neq \mathbf{O}$. Fails axiom (d) of the inner product definition (4-1).

11. We are given

$$\langle \mathbf{u},\ \mathbf{v} \rangle = -1,\ \langle \mathbf{u},\ \mathbf{w} \rangle = 5,\ \langle \mathbf{v},\ \mathbf{w} \rangle = 3,\ \|\mathbf{v}\| = 2,\ \|\mathbf{u}\| = 4 \text{ and } \|\mathbf{w}\| = 7$$

(a) Using the properties of inner product we have

$$\begin{aligned} \langle \mathbf{u},\ \mathbf{v}+\mathbf{w} \rangle &= \langle \mathbf{u},\ \mathbf{w} \rangle + \langle \mathbf{u},\ \mathbf{v} \rangle \\ &= 5 - 1 = 4 \end{aligned}$$

(b) Similarly

$$\begin{aligned} \langle 2\mathbf{u}+3\mathbf{v},\ 5\mathbf{v}-2\mathbf{w} \rangle &= \langle 2\mathbf{u},\ 5\mathbf{v}-2\mathbf{w} \rangle + \langle 3\mathbf{v},\ 5\mathbf{v}-2\mathbf{w} \rangle \\ &= \langle 2\mathbf{u},\ 5\mathbf{v} \rangle + \langle 2\mathbf{u},\ -2\mathbf{w} \rangle + \langle 3\mathbf{v},\ 5\mathbf{v} \rangle + \langle 3\mathbf{v},\ -2\mathbf{w} \rangle \\ &= 10\langle \mathbf{u},\ \mathbf{v} \rangle + (-4)\langle \mathbf{u},\ \mathbf{w} \rangle + 15\langle \mathbf{v},\ \mathbf{v} \rangle - 6\langle \mathbf{v},\ \mathbf{w} \rangle \\ &= 10(-1) - 4(5) + 15\langle \mathbf{v},\ \mathbf{v} \rangle - 6(3) \qquad (*) \end{aligned}$$

What is $\langle \mathbf{v},\ \mathbf{v} \rangle$ equal to?

$$\langle \mathbf{v},\ \mathbf{v} \rangle = \|\mathbf{v}\|^2$$

We are given $\|\mathbf{v}\| = 2$ therefore $\langle \mathbf{v},\ \mathbf{v} \rangle = 2^2 = 4$. Substituting this, $\langle \mathbf{v},\ \mathbf{v} \rangle = 4$ into (*) gives

$$\langle 2\mathbf{u}+3\mathbf{v},\ 5\mathbf{v}-2\mathbf{w} \rangle = 10(-1) - 4(5) + 15(4) - 6(3) = 12$$

(c) We need to find $\|\mathbf{u}-\mathbf{v}\|$ but first we find $\|\mathbf{u}-\mathbf{v}\|^2$ and then take the square root.

$$\begin{aligned} \|\mathbf{u}-\mathbf{v}\|^2 &= \langle \mathbf{u}-\mathbf{v},\ \mathbf{u}-\mathbf{v} \rangle \\ &= \langle \mathbf{u},\ \mathbf{u}-\mathbf{v} \rangle + \langle -\mathbf{v},\ \mathbf{u}-\mathbf{v} \rangle \\ &= \langle \mathbf{u},\ \mathbf{u} \rangle + \langle \mathbf{u},\ -\mathbf{v} \rangle + \langle -\mathbf{v},\ \mathbf{u} \rangle + \langle -\mathbf{v},\ -\mathbf{v} \rangle \\ &= \|\mathbf{u}\|^2 - \langle \mathbf{u},\ \mathbf{v} \rangle - \langle \mathbf{v},\ \mathbf{u} \rangle + \|\mathbf{v}\|^2 \\ &= 4^2 - (-1) - (-1) + 2^2 = 22 \end{aligned}$$

Hence $\|\mathbf{u}-\mathbf{v}\| = \sqrt{22}$.

(d) We have

$$\begin{aligned}\|\mathbf{u}-\mathbf{v}-\mathbf{w}\|^2 &= \|\mathbf{u}-(\mathbf{v}+\mathbf{w})\|^2\\ &= \langle \mathbf{u}-(\mathbf{v}+\mathbf{w}),\ \mathbf{u}-(\mathbf{v}+\mathbf{w})\rangle\\ &= \langle \mathbf{u},\ \mathbf{u}-(\mathbf{v}+\mathbf{w})\rangle + \langle -(\mathbf{v}+\mathbf{w}),\ \mathbf{u}-(\mathbf{v}+\mathbf{w})\rangle\\ &= \langle \mathbf{u},\ \mathbf{u}\rangle + \langle \mathbf{u},\ -(\mathbf{v}+\mathbf{w})\rangle - \left[\langle \mathbf{v}+\mathbf{w},\ \mathbf{u}-(\mathbf{v}+\mathbf{w})\rangle\right]\\ &= \langle \mathbf{u},\ \mathbf{u}\rangle - \langle \mathbf{u},\ \mathbf{v}+\mathbf{w}\rangle - \left[\langle \mathbf{v},\ \mathbf{u}-(\mathbf{v}+\mathbf{w})\rangle + \langle \mathbf{w},\ \mathbf{u}-(\mathbf{v}+\mathbf{w})\rangle\right] \qquad (\dagger)\end{aligned}$$

The evaluation will get rather messy if we continue in this manner. We examine the square bracket term separately below. Let x be the first term of the square bracket and y be the second term of the bracket on the Right Hand Side, that is:

$$\underbrace{\langle \mathbf{v},\ \mathbf{u}-(\mathbf{v}+\mathbf{w})\rangle}_{=x} + \underbrace{\langle \mathbf{w},\ \mathbf{u}-(\mathbf{v}+\mathbf{w})\rangle}_{=y}$$

$$\begin{aligned}x = \langle \mathbf{v},\ \mathbf{u}-(\mathbf{v}+\mathbf{w})\rangle &= \langle \mathbf{v},\ \mathbf{u}\rangle + \langle \mathbf{v},\ -(\mathbf{v}+\mathbf{w})\rangle\\ &= \langle \mathbf{v},\ \mathbf{u}\rangle - \langle \mathbf{v},\ \mathbf{v}+\mathbf{w}\rangle\\ &= \langle \mathbf{v},\ \mathbf{u}\rangle - \langle \mathbf{v},\ \mathbf{v}\rangle - \langle \mathbf{v},\ \mathbf{w}\rangle\\ &= \langle \mathbf{v},\ \mathbf{u}\rangle - \|\mathbf{v}\|^2 - \langle \mathbf{v},\ \mathbf{w}\rangle \qquad \left[\text{Because } \langle \mathbf{v},\ \mathbf{v}\rangle = \|\mathbf{v}\|^2\right]\\ &= -1-2^2-3=-8 \qquad \left[\text{Using the given values}\right]\end{aligned}$$

Also

$$\begin{aligned}y = \langle \mathbf{w},\ \mathbf{u}-(\mathbf{v}+\mathbf{w})\rangle &= \langle \mathbf{w},\ \mathbf{u}\rangle + \langle \mathbf{w},\ -(\mathbf{v}+\mathbf{w})\rangle\\ &= \langle \mathbf{w},\ \mathbf{u}\rangle - \langle \mathbf{w},\ \mathbf{v}+\mathbf{w}\rangle\\ &= \langle \mathbf{w},\ \mathbf{u}\rangle - \langle \mathbf{w},\ \mathbf{v}\rangle - \langle \mathbf{w},\ \mathbf{w}\rangle\\ &= \langle \mathbf{w},\ \mathbf{u}\rangle - \langle \mathbf{w},\ \mathbf{v}\rangle - \|\mathbf{w}\|^2 \qquad \left[\text{Because } \langle \mathbf{w},\ \mathbf{w}\rangle = \|\mathbf{w}\|^2\right]\\ &= 5-3-7^2 \qquad \left[\text{Substituting the given values}\right]\\ &= -47\end{aligned}$$

Considering the Left Hand term of $(\dagger)$:

$$\begin{aligned}\langle \mathbf{u},\ \mathbf{u}\rangle - \langle \mathbf{u},\ \mathbf{v}+\mathbf{w}\rangle &= \|\mathbf{u}\|^2 - \langle \mathbf{u},\ \mathbf{v}\rangle - \langle \mathbf{u},\ \mathbf{w}\rangle\\ &= 4^2-(-1)-5=12\end{aligned}$$

Substituting all these values, $x=-8$, $y=-47$ and $\langle \mathbf{u},\ \mathbf{u}\rangle - \langle \mathbf{u},\ \mathbf{v}+\mathbf{w}\rangle = 12$ into ($\dagger$) gives

$$\begin{aligned}\|\mathbf{u}-\mathbf{v}-\mathbf{w}\|^2 &= \underbrace{\langle \mathbf{u},\ \mathbf{u}\rangle - \langle \mathbf{u},\ \mathbf{v}+\mathbf{w}\rangle}_{=12} - [x+y]\\ &= 12-[-8-47]=67\end{aligned}$$

We have $\|\mathbf{u}-\mathbf{v}-\mathbf{w}\|^2 = 67$ therefore $\|\mathbf{u}-\mathbf{v}-\mathbf{w}\| = \sqrt{67}$.

12. (a) We need to prove $d(\mathbf{u},\ \mathbf{v}) = d(\mathbf{v},\ \mathbf{u})$:

Proof. Since $d(\mathbf{u},\ \mathbf{v}) = \|\mathbf{u}-\mathbf{v}\|$ we examine the square of this and then take the square root.

$$
\begin{aligned}
\|\mathbf{u}-\mathbf{v}\|^2 &= \langle \mathbf{u}-\mathbf{v},\ \mathbf{u}-\mathbf{v}\rangle \\
&= \langle -\mathbf{v}+\mathbf{u},\ -\mathbf{v}+\mathbf{u}\rangle \\
&= \langle -(\mathbf{v}-\mathbf{u}),\ -(\mathbf{v}-\mathbf{u})\rangle \\
&= (-1)(-1)\langle (\mathbf{v}-\mathbf{u}),\ (\mathbf{v}-\mathbf{u})\rangle \\
&= \langle \mathbf{v}-\mathbf{u},\ \mathbf{v}-\mathbf{u}\rangle = \|\mathbf{v}-\mathbf{u}\|^2
\end{aligned}
$$

Taking the square root of this $\|\mathbf{u}-\mathbf{v}\|^2 = \|\mathbf{v}-\mathbf{u}\|^2$ gives our result:

$$d(\mathbf{u},\ \mathbf{v}) = \|\mathbf{u}-\mathbf{v}\| = \|\mathbf{v}-\mathbf{u}\| = d(\mathbf{v},\ \mathbf{u})$$

■

(b) Required to prove $d(\mathbf{u},\ \mathbf{v}) \ge 0$:

Proof. We have

$$
\begin{aligned}
d(\mathbf{u},\ \mathbf{v}) &= \|\mathbf{u}-\mathbf{v}\| \\
&= \sqrt{\langle \mathbf{u}-\mathbf{v},\ \mathbf{u}-\mathbf{v}\rangle} \ge 0 \qquad \left[\text{Because } \langle \mathbf{u}-\mathbf{v},\ \mathbf{u}-\mathbf{v}\rangle \ge 0\right]
\end{aligned}
$$

■

(c) Need to prove $d(\mathbf{u},\ \mathbf{v}) = 0$ if and only if $\mathbf{u} = \mathbf{v}$.

Proof. We apply Proposition (4-2) Property (b) which says $\|\mathbf{u}\| = 0 \iff \mathbf{u} = \mathbf{O}$.
We have

$$d(\mathbf{u},\ \mathbf{v}) = \|\mathbf{u}-\mathbf{v}\| = 0 \quad \Leftrightarrow \quad \mathbf{u}-\mathbf{v} = \mathbf{O} \quad \Leftrightarrow \quad \mathbf{u} = \mathbf{v}$$

■

(d) We need to prove $d(k\mathbf{u},\ k\mathbf{v}) = |k|\, d(\mathbf{u},\ \mathbf{v})$:

Proof. Using the definition of distance function we have

$$
\begin{aligned}
d(k\mathbf{u},\ k\mathbf{v}) &= \|k\mathbf{u}-k\mathbf{v}\| \\
&= \|k(\mathbf{u}-\mathbf{v})\| \\
&= |k|\,\|\mathbf{u}-\mathbf{v}\| \qquad \left[\begin{array}{l}\text{By Proposition (4-2) property (c)} \\ \text{which is } \|k\mathbf{u}\| = |k|\,\|\mathbf{u}\|\end{array}\right] \\
&= |k|\, d(\mathbf{u},\ \mathbf{v})
\end{aligned}
$$

■