Complete Solutions to Miscellaneous Exercises 6

1. We have

$$\det(\mathbf{A}) = \det\begin{bmatrix} 0 & 1 & 5 \\ 3 & -6 & 9 \\ 2 & 6 & 1 \end{bmatrix}$$

$$= 0 - 1\det\begin{bmatrix} 3 & 9 \\ 2 & 1 \end{bmatrix} + 5\det\begin{bmatrix} 3 & -6 \\ 2 & 6 \end{bmatrix}$$

$$= -(3-18) + 5(18+12) = 15 + 150 = 165$$

Thus $\det(\mathbf{A}) = 165$ so we go for option ①

2. *How do we find the determinant of matrix* ***E****?*

Since the bottom row of **E** contains the most zeros therefore expand along this row:

$$\det(\mathbf{E}) = \begin{pmatrix} -1 & 7 & 8 \\ 0 & 5 & 6 \\ 0 & 0 & 4 \end{pmatrix} = 0 - 0 + 4\det\begin{pmatrix} -1 & 7 \\ 0 & 5 \end{pmatrix}$$

$$= 4(-5-0) = -20$$

Similarly we have

$$\det(\mathbf{F}) = \det\begin{pmatrix} 3 & 0 & 0 \\ -1 & -5 & 0 \\ 0 & 1 & -2 \end{pmatrix} = 0 - 0 - 2\det\begin{pmatrix} 3 & 0 \\ -1 & -5 \end{pmatrix}$$

$$= -2(-15-0) = 30$$

We have $\det(\mathbf{E}) = -20$ and $\det(\mathbf{F}) = 30$. *What is* $\det(\mathbf{EF})$ *equal to?*

$$\det(\mathbf{EF}) = \det(\mathbf{E})\det(\mathbf{F})$$

$$= -20 \times 30 = -600$$

What is $\det(\mathbf{E}+\mathbf{F})$ *equal to?*

$\det(\mathbf{E}+\mathbf{F}) \neq \det(\mathbf{E}) + \det(\mathbf{F})$ [Not Equal]. We need to add the two matrices **E** and **F** together first and then find the determinant of the result.

$$\mathbf{E}+\mathbf{F} = \begin{pmatrix} -1 & 7 & 8 \\ 0 & 5 & 6 \\ 0 & 0 & 4 \end{pmatrix} + \begin{pmatrix} 3 & 0 & 0 \\ -1 & -5 & 0 \\ 0 & 1 & -2 \end{pmatrix}$$

$$= \begin{pmatrix} 2 & 7 & 8 \\ -1 & 0 & 6 \\ 0 & 1 & 2 \end{pmatrix}$$

The determinant is given by

$$\begin{aligned}\det(\mathbf{E}+\mathbf{F}) &= \det\begin{pmatrix} 2 & 7 & 8 \\ -1 & 0 & 6 \\ 0 & 1 & 2 \end{pmatrix} \\ &= 0-\det\begin{pmatrix} 2 & 8 \\ -1 & 6 \end{pmatrix}+2\det\begin{pmatrix} 2 & 7 \\ -1 & 0 \end{pmatrix} \\ &= -(12+8)+2(7)=-6\end{aligned}$$

3. We carry out elementary row operations to simplify the given matrix. Labelling the rows:

$$\begin{array}{c} R_1 \\ R_2 \\ R_3 \\ R_4 \\ R_5 \end{array}\begin{bmatrix} 3 & 1 & 1 & 1 & 1 \\ 1 & 3 & 1 & 1 & 1 \\ 3 & 1 & 3 & 1 & 1 \\ 1 & 3 & 1 & 3 & 1 \\ 3 & 1 & 3 & 1 & 3 \end{bmatrix}$$

Executing the following row operations:

$$\begin{array}{r} R_1 \\ R_2^* = R_2 - R_1 \\ R_3^* = R_3 - R_1 \\ R_4^* = R_4 - R_1 \\ R_5^* = R_5 - R_1 \end{array}\begin{bmatrix} 3 & 1 & 1 & 1 & 1 \\ -2 & 2 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ -2 & 2 & 0 & 2 & 0 \\ 0 & 0 & 2 & 0 & 2 \end{bmatrix}$$

Carrying out the row operations $R_4^* - R_2^*$ and $R_5^* - R_3^*$:

$$\begin{array}{r} R_1 \\ R_2^* \\ R_3^* \\ R_4^{**} = R_4^* - R_2^* \\ R_5^{**} = R_5^* - R_3^* \end{array}\begin{bmatrix} 3 & 1 & 1 & 1 & 1 \\ -2 & 2 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 2 \end{bmatrix}$$

Carrying out the row operation $3R_2^*$:

$$\begin{array}{r} R_1 \\ R_2^{**} = 3R_2^* \\ R_3^* \\ R_4^{**} \\ R_5^{**} \end{array}\begin{bmatrix} 3 & 1 & 1 & 1 & 1 \\ -6 & 6 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 2 \end{bmatrix}$$

Executing the row operation $R_2^{**} - 2R_1$:

$$\begin{array}{r} R_1 \\ R_2^{***} = R_2^{**} + 2R_1 \\ R_3^* \\ R_4^{**} \\ R_5^{**} \end{array}\begin{bmatrix} 3 & 1 & 1 & 1 & 1 \\ 0 & 8 & 2 & 2 & 2 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 2 \end{bmatrix}$$

This matrix is now a (upper) triangular matrix. *What is the determinant of this matrix?*

It is the multiple of all entries along the leading diagonal:

$$\det\begin{bmatrix}3 & 1 & 1 & 1 & 1\\ 0 & 8 & 2 & 2 & 2\\ 0 & 0 & 2 & 0 & 0\\ 0 & 0 & 0 & 2 & 0\\ 0 & 0 & 0 & 0 & 2\end{bmatrix} = 3\times 8\times 2\times 2\times 2 = 192$$

What is the determinant of the given matrix?

$\frac{192}{3} = 64$ because we have multiplied a row (shown in red above) by 3 therefore the 192 needs to be divided by 3 in order to get the determinant of the given matrix.
Hence determinant is 64.

4. We have

(a) $\det(\mathbf{AB}) = \det(\mathbf{A})\det(\mathbf{B})$

(b) $\det(\mathbf{A}^{-1}) = \frac{1}{\det(\mathbf{A})}$ provided $\det(\mathbf{A}) \neq 0$

(c) $\det(\mathbf{A}+\mathbf{B}) = \text{No Formula}$

(d) $\det(3\mathbf{A}) = 3^n \det(\mathbf{A})$

(e) $\det(\mathbf{A}^T) = \det(\mathbf{A})$

5. To find the determinant of 4 by 4 matrix it is less arithmetic effort to use row operations. Labelling the rows of the given matrix we have

$$\begin{array}{c} R_1 \\ R_2 \\ R_3 \\ R_4 \end{array} \begin{pmatrix} 1 & 2 & 0 & 0\\ 0 & 1 & 2 & 0\\ 0 & 0 & 1 & 2\\ -3 & 4 & -5 & 6\end{pmatrix}$$

Carrying out the row operation $R_4 + 3R_1$:

$$\begin{array}{c} R_1 \\ R_2 \\ R_3 \\ R_4{*} = R_4 + 3R_1 \end{array} \begin{pmatrix} 1 & 2 & 0 & 0\\ 0 & 1 & 2 & 0\\ 0 & 0 & 1 & 2\\ 0 & 10 & -5 & 6\end{pmatrix}$$

Implementing the row operation $R_4{*} - 10R_2$:

$$\begin{array}{c} R_1 \\ R_2 \\ R_3 \\ R_4{**} = R_4{*} - 10R_2 \end{array} \begin{pmatrix} 1 & 2 & 0 & 0\\ 0 & 1 & 2 & 0\\ 0 & 0 & 1 & 2\\ 0 & 0 & -25 & 6\end{pmatrix}$$

Executing the row operation $R_4{**} + 25R_3$:

$$\begin{array}{r} R_1 \\ R_2 \\ R_3 \\ R_4{}^{\dagger} = R_4{}^{**} + 25R_3 \end{array} \begin{pmatrix} 1 & 2 & 0 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 56 \end{pmatrix}$$

We have an upper triangular matrix so the determinant of this matrix is the product of all the entries along the leading diagonal, which is $1\times1\times1\times56 = 56$.
Hence $\det(\mathbf{B}) = 56$.

6. For the given linear system

$$\begin{array}{ccccccc} 2x_1 & + & x_2 & + & x_3 & = & 8 \\ 3x_1 & - & 2x_2 & - & x_3 & = & 1 \\ 4x_1 & - & 7x_2 & + & 3x_3 & = & 10 \end{array}$$

We can express this in matrix form as $\mathbf{Ax} = \mathbf{b}$ where

$$\mathbf{A} = \begin{pmatrix} 2 & 1 & 1 \\ 3 & -2 & -1 \\ 4 & -7 & 3 \end{pmatrix}, \ \mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \text{ and } \mathbf{b} = \begin{pmatrix} 8 \\ 1 \\ 10 \end{pmatrix}$$

We can only apply Cramer's rule subject to $\det(\mathbf{A})$ is **not** zero. Checking the determinant of the above matrix **A**:

$$\begin{aligned} \det \mathbf{A} &= \begin{pmatrix} 2 & 1 & 1 \\ 3 & -2 & -1 \\ 4 & -7 & 3 \end{pmatrix} \\ &= 2\det\begin{pmatrix} -2 & -1 \\ -7 & 3 \end{pmatrix} - 1\det\begin{pmatrix} 3 & -1 \\ 4 & 3 \end{pmatrix} + 1\det\begin{pmatrix} 3 & -2 \\ 4 & -7 \end{pmatrix} \\ &= 2(-6-7) - (9+4) + (-21+8) = -52 \end{aligned}$$

Since $\det(\mathbf{A}) = -52 \neq 0$ therefore we can apply Cramer's rule. Applying this rule we have

$$x_2 = \frac{\det(\mathbf{A}_2(\mathbf{b}))}{\det(\mathbf{A})}$$

We know det(A) but what is $\det(\mathbf{A}_2(\mathbf{b}))$ *equal to?*

It is the matrix **A** but with the second column filled in the column vector **b**. Thus we have

$$x_2 = \frac{\det(\mathbf{A}_2(\mathbf{b}))}{\det(\mathbf{A})} = \frac{\det\begin{pmatrix} 2 & 8 & 1 \\ 3 & 1 & -1 \\ 4 & 10 & 3 \end{pmatrix}}{\det\begin{pmatrix} 2 & 1 & 1 \\ 3 & -2 & -1 \\ 4 & -7 & 3 \end{pmatrix}}$$

7. *How do we find* x_2*?*

By applying Cramer's rule. We have

$$x_2 = \frac{\det(\mathbf{A}_2(\mathbf{b}))}{\det(\mathbf{A})} = \frac{\det(\mathbf{A}_2(\mathbf{b}))}{-420} \qquad \begin{bmatrix}\text{Because we are given} \\ \det(\mathbf{A}) = -420\end{bmatrix}$$

So we need to determine $\det(\mathbf{A}_2(\mathbf{b}))$. *What is* $\det(\mathbf{A}_2(\mathbf{b}))$ *equal to?*

This is the given matrix **A** but the second column in **A** is replaced by the column $\mathbf{b} = \begin{bmatrix}1 \\ 0 \\ 1\end{bmatrix}$. Thus we have $\mathbf{A}_2(\mathbf{b}) = \begin{bmatrix}2 & 1 & -1 \\ 1 & 0 & 1 \\ 2 & 1 & 1\end{bmatrix}$. Thus

$$\det(\mathbf{A}_2(\mathbf{b})) = \det\begin{bmatrix}2 & 1 & -1 \\ 1 & 0 & 1 \\ 2 & 1 & 1\end{bmatrix}$$

$$= -1\det\begin{bmatrix}1 & 1 \\ 2 & 1\end{bmatrix} - 1\det\begin{bmatrix}2 & -1 \\ 1 & 1\end{bmatrix} \qquad \begin{bmatrix}\text{Expanding the} \\ \text{middle column}\end{bmatrix}$$

$$= -[1-2] - [2+1] = -2$$

Substituting $\det(\mathbf{A}_2(\mathbf{b})) = -2$ into the above $x_2 = \dfrac{\det(\mathbf{A}_2(\mathbf{b}))}{-420}$ gives

$$x_2 = \frac{\det(\mathbf{A}_2(\mathbf{b}))}{-420} = \frac{-2}{-420} = \frac{1}{210}$$

Thus $x_2 = \dfrac{1}{210}$.

8. We are given the linear system

$$\begin{array}{ccccccc} & & 5x_2 & + & x_3 & = & -8 \\ x_1 & - & 2x_2 & - & x_3 & = & 2 \\ 6x_1 & + & x_2 & - & 3x_3 & = & -8 \end{array}$$

and we need to solve this system by applying Cramer's rule. *Before we use Cramer's rule what do we need to check?*

The determinant of the coefficients of the unknowns is **not** zero. We can write the above linear system in matrix form as $\mathbf{Ax} = \mathbf{b}$ where

$$\mathbf{A} = \begin{pmatrix}0 & 5 & 1 \\ 1 & -2 & -1 \\ 6 & 1 & -3\end{pmatrix}, \ \mathbf{x} = \begin{pmatrix}x_1 \\ x_2 \\ x_3\end{pmatrix}, \ \mathbf{b} = \begin{pmatrix}-8 \\ 2 \\ -8\end{pmatrix}$$

The determinant of the matrix **A** is

$$\det \mathbf{A} = \begin{vmatrix}0 & 5 & 1 \\ 1 & -2 & -1 \\ 6 & 1 & -3\end{vmatrix} = -1\det\begin{pmatrix}5 & 1 \\ 1 & -3\end{pmatrix} + 6\det\begin{pmatrix}5 & 1 \\ -2 & -1\end{pmatrix}$$

$$= -(-15-1) + 6(-5+2) = -2$$

We have $\det(\mathbf{A}) = -2$. *How do we determine the unknowns* x_1, x_2 *and* x_3 *using Cramer's rule?*

The formula for Cramer's rule (6-16) is given by

$$x_1 = \frac{\det(\mathbf{A}_1(\mathbf{b}))}{\det(\mathbf{A})}, \quad x_2 = \frac{\det(\mathbf{A}_2(\mathbf{b}))}{\det(\mathbf{A})}, \quad x_3 = \frac{\det(\mathbf{A}_3(\mathbf{b}))}{\det(\mathbf{A})}$$

What is $\mathbf{A}_1(\mathbf{b})$, $\mathbf{A}_2(\mathbf{b})$ *and* $\mathbf{A}_3(\mathbf{b})$ *equal to?*

$\mathbf{A}_1(\mathbf{b})$ is the matrix **A** with the first column replaced by the column vector **b** stated above. Similarly we have $\mathbf{A}_2(\mathbf{b})$ and $\mathbf{A}_3(\mathbf{b})$. Thus

$$\mathbf{A}_1(\mathbf{b}) = \begin{pmatrix} -8 & 5 & 1 \\ 2 & -2 & -1 \\ -8 & 1 & -3 \end{pmatrix}, \ \mathbf{A}_2(\mathbf{b}) = \begin{pmatrix} 0 & -8 & 1 \\ 1 & 2 & -1 \\ 6 & -8 & -3 \end{pmatrix} \text{ and } \mathbf{A}_3(\mathbf{b}) = \begin{pmatrix} 0 & 5 & -8 \\ 1 & -2 & 2 \\ 6 & 1 & -8 \end{pmatrix}$$

We need to find the determinant of each of these.

$$\det\left[\mathbf{A}_1(\mathbf{b})\right] = \det\begin{pmatrix} -8 & 5 & 1 \\ 2 & -2 & -1 \\ -8 & 1 & -3 \end{pmatrix} = 0$$

$$\det\left[\mathbf{A}_2(\mathbf{b})\right] = \det\begin{pmatrix} 0 & -8 & 1 \\ 1 & 2 & -1 \\ 6 & -8 & -3 \end{pmatrix} = 4$$

$$\det\left[\mathbf{A}_3(\mathbf{b})\right] = \det\begin{pmatrix} 0 & 5 & -8 \\ 1 & -2 & 2 \\ 6 & 1 & -8 \end{pmatrix} = -4$$

Substituting these $\det\left[\mathbf{A}_1(\mathbf{b})\right] = 0$, $\det\left[\mathbf{A}_2(\mathbf{b})\right] = 4$, $\det\left[\mathbf{A}_3(\mathbf{b})\right] = -4$ and $\det(\mathbf{A}) = -2$ into

$$x_1 = \frac{\det(\mathbf{A}_1(\mathbf{b}))}{\det(\mathbf{A})}, \quad x_2 = \frac{\det(\mathbf{A}_2(\mathbf{b}))}{\det(\mathbf{A})}, \quad x_3 = \frac{\det(\mathbf{A}_3(\mathbf{b}))}{\det(\mathbf{A})}$$

gives

$$x_1 = \frac{0}{-2} = 0, \quad x_2 = \frac{4}{-2} = -2, \quad x_3 = \frac{-4}{-2} = 2$$

Our solution to the system is $x_1 = 0$, $x_2 = -2$ and $x_3 = 2$. You may check this result by substituting these into the given linear system.

9. To find a determinant of a 4 by 4 matrix it is less arduous to use elementary row operations. We have

$$\begin{array}{c} R_1 \\ R_2 \\ R_3 \\ R_4 \end{array} \begin{pmatrix} 2 & 1 & 3 & 1 \\ 0 & 8 & -2 & 5 \\ -6 & -3 & -5 & -5 \\ 2 & 5 & 3 & -4 \end{pmatrix}$$

Carry out the row operations $R_3 + 3R_1$ and $R_4 - R_1$:

$$\begin{array}{c} R_1 \\ R_2 \\ R_3{*} = R_3 + 3R_1 \\ R_4{*} = R_4 - R_1 \end{array} \begin{pmatrix} 2 & 1 & 3 & 1 \\ 0 & 8 & -2 & 5 \\ 0 & 0 & 4 & -2 \\ 0 & 4 & 0 & -5 \end{pmatrix}$$

Carrying out the row operation $R_2 + R_4*$:

$$\begin{array}{c} R_1 \\ R_2* = R_2 + R_4* \\ R_3* \\ R_4* \end{array} \begin{pmatrix} 2 & 1 & 3 & 1 \\ 0 & 12 & -2 & 0 \\ 0 & 0 & 4 & -2 \\ 0 & 4 & 0 & -5 \end{pmatrix}$$

This last matrix has the same determinant as the matrix given to us above because the only row operation that has been used is adding a multiple of one row to another. Remember this operation does **not** change the determinant. Thus we have

$$\det\begin{pmatrix} 2 & 1 & 3 & 1 \\ 0 & 12 & -2 & 0 \\ 0 & 0 & 4 & -2 \\ 0 & 4 & 0 & -5 \end{pmatrix} = 2\det\begin{pmatrix} 12 & -2 & 0 \\ 0 & 4 & -2 \\ 4 & 0 & -5 \end{pmatrix} \qquad \begin{bmatrix} \text{Expanding along} \\ \text{the first column} \end{bmatrix}$$

$$= 2\left[4\det\begin{pmatrix} 12 & 0 \\ 4 & -5 \end{pmatrix} + 2\det\begin{pmatrix} 12 & -2 \\ 4 & 0 \end{pmatrix}\right]$$

$$= 2\left[4(-60) + 2(0+8)\right] = -448$$

Hence the determinant of the given matrix is -448.

10. We apply elementary row operations to establish some zeros into the matrix so that the evaluation of the determinant is made straightforward.

$$\begin{array}{c} R_1 \\ R_2 \\ R_3 \\ R_4 \end{array} \begin{pmatrix} 10 & -2 & 3 & 16 \\ 1 & 1 & 1 & 1 \\ 9 & -3 & 2 & 15 \\ 110 & 23 & 12 & -15 \end{pmatrix}$$

Interchanging the top 2 rows gives

$$\begin{array}{c} R_1* = R_2 \\ R_2* = R_1 \\ R_3 \\ R_4 \end{array} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 10 & -2 & 3 & 16 \\ 9 & -3 & 2 & 15 \\ 110 & 23 & 12 & -15 \end{pmatrix}$$

Carrying out the row operations R_2*-10R_1*, $R_3 - 9R_1*$ and $R_4 - 110R_1*$:

$$\begin{array}{c} R_1* \\ R_2** = R_2* - 10R_1* \\ R_3* = R_3 - 9R_1* \\ R_4* = R_4 - 110R_1* \end{array} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & -12 & -7 & 6 \\ 0 & -12 & -7 & 6 \\ 0 & -87 & -98 & -125 \end{pmatrix}$$

Since the middle two rows, R_2** and R_3* are identical therefore the determinant of the matrix is 0. Hence the determinant of the original matrix is also 0.

11. The determinant of the given matrix is

$$\det(\mathbf{A}) = \det\begin{pmatrix} 0 & 1 & 3 \\ 1 & 2 & 0 \\ 4 & 5 & 1 \end{pmatrix}$$
$$= 0 - 1\det\begin{pmatrix} 1 & 0 \\ 4 & 1 \end{pmatrix} + 3\det\begin{pmatrix} 1 & 2 \\ 4 & 5 \end{pmatrix} = -(1) + 3(5-8) = -10$$

Hence $\det(\mathbf{A}) = -10$.

(b) *How do we find the inverse matrix using elementary row operations?*

By converting $(\mathbf{A} \mid \mathbf{I})$ to $(\mathbf{I} \mid \mathbf{B})$. If this is possible then the matrix **B** is the inverse matrix of **A**. We have

$$\begin{array}{c} R_1 \\ R_2 \\ R_3 \end{array} \left(\begin{array}{ccc|ccc} 0 & 1 & 3 & 1 & 0 & 0 \\ 1 & 2 & 0 & 0 & 1 & 0 \\ 4 & 5 & 1 & 0 & 0 & 1 \end{array}\right)$$

Interchanging the middle and top row gives

$$\begin{array}{c} R_1* = R_2 \\ R_2* = R_1 \\ R_3 \end{array} \left(\begin{array}{ccc|ccc} 1 & 2 & 0 & 0 & 1 & 0 \\ 0 & 1 & 3 & 1 & 0 & 0 \\ 4 & 5 & 1 & 0 & 0 & 1 \end{array}\right)$$

Performing the row operation $R_3 - 4R_1$ gives:

$$\begin{array}{c} R_1* \\ R_2* \\ R_3* = R_3 - 4R_1 \end{array} \left(\begin{array}{ccc|ccc} 1 & 2 & 0 & 0 & 1 & 0 \\ 0 & 1 & 3 & 1 & 0 & 0 \\ 0 & -3 & 1 & 0 & -4 & 1 \end{array}\right)$$

Executing the row operation $R_3 * + 3R_2 *$:

$$\begin{array}{c} R_1* \\ R_2* \\ R_3** = R_3* + 3R_2* \end{array} \left(\begin{array}{ccc|ccc} 1 & 2 & 0 & 0 & 1 & 0 \\ 0 & 1 & 3 & 1 & 0 & 0 \\ 0 & 0 & 10 & 3 & -4 & 1 \end{array}\right) \qquad (\text{☼})$$

Executing the row operation $R_1 * - 2R_2 *$ yields:

$$\begin{array}{c} R_1** = R_1* - 2R_2* \\ R_2* \\ R_3** \end{array} \left(\begin{array}{ccc|ccc} 1 & 0 & -6 & -2 & 1 & 0 \\ 0 & 1 & 3 & 1 & 0 & 0 \\ 0 & 0 & 10 & 3 & -4 & 1 \end{array}\right)$$

Dividing the bottom row by 10 gives

$$\begin{array}{c} R_1** \\ R_2* \\ R_3^\dagger = R_3**/10 \end{array} \left(\begin{array}{ccc|ccc} 1 & 0 & -6 & -2 & 1 & 0 \\ 0 & 1 & 3 & 1 & 0 & 0 \\ 0 & 0 & 1 & 3/10 & -4/10 & 1/10 \end{array}\right)$$

Carrying out the row operation $R_2 * - 3R_3^\dagger$:

$$\begin{array}{c} R_1** \\ R_2** = R_2* - 3R_3^\dagger \\ R_3^\dagger = R_3**/10 \end{array} \left(\begin{array}{ccc|ccc} 1 & 0 & -6 & -2 & 1 & 0 \\ 0 & 1 & 0 & 1/10 & 12/10 & -3/10 \\ 0 & 0 & 1 & 3/10 & -4/10 & 1/10 \end{array}\right)$$

Executing the elementary row operation $R_1 ** + 6R_3^\dagger$:

$$\begin{array}{c} R_1^\dagger = R_1{**}+6R_3^\dagger \\ R_2{**} \\ R_3^\dagger \end{array} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -2/10 & -14/10 & 6/10 \\ 0 & 1 & 0 & 1/10 & 12/10 & -3/10 \\ 0 & 0 & 1 & 3/10 & -4/10 & 1/10 \end{array}\right)$$

What is the inverse matrix $\mathbf{A}^{-1}$ *equal to?*
It is the matrix on the Right Hand Side of the vertical bar above:

$$\mathbf{A}^{-1} = \begin{pmatrix} -2/10 & -14/10 & 6/10 \\ 1/10 & 12/10 & -3/10 \\ 3/10 & -4/10 & 1/10 \end{pmatrix} = \frac{1}{10}\begin{pmatrix} -2 & -14 & 6 \\ 1 & 12 & -3 \\ 3 & -4 & 1 \end{pmatrix}$$

(c) In the above derivation we see at the point we reach the augmented matrix in (☼) we have an upper triangular matrix, that is we have

$$\begin{pmatrix} 1 & 2 & 0 \\ 0 & 1 & 3 \\ 0 & 0 & 10 \end{pmatrix}$$

What is the determinant of this matrix?

$$1\times 1\times 10 = 10$$

So far we have only carried out 3 row operations- interchanging rows, $R_3 - 4R_1$ and $R_1{*} - 2R_2{*}$. Remember adding a multiple of one row to another does not change the value of the determinant. However interchanging rows changes the sign of the determinant of the matrix. Hence we have $\det(\mathbf{A}) = -10$.

12.(a) First we find the determinant of the given matrix:

$$\det(\mathbf{A}) = \det\begin{bmatrix} 2 & -1 & -4 \\ -1 & 1 & 2 \\ -1 & 1 & 3 \end{bmatrix}$$

$$= 2\det\begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix} + 1\det\begin{bmatrix} -1 & 2 \\ -1 & 3 \end{bmatrix} - 4\det\begin{bmatrix} -1 & 1 \\ -1 & 1 \end{bmatrix}$$

$$= 2(3-2) + (-3+2) - 4(-1+1) = 1$$

Since $\det(\mathbf{A}) = 1 \neq 0$ therefore the inverse of the matrix **A** exists.

Find the cofactors of each entry and adjoint which the matrix of cofactors transposed .
You can do this in your own time and establish that

$$\mathbf{A}^{-1} = \begin{pmatrix} 1 & -1 & 2 \\ 1 & 2 & 0 \\ 0 & -1 & 1 \end{pmatrix}$$

(b) *What is the inverse of* $\mathbf{A}^t$ *?*

From the text we have $\left(\mathbf{A}^t\right)^{-1} = \left(\mathbf{A}^{-1}\right)^t = \begin{pmatrix} 1 & -1 & 2 \\ 1 & 2 & 0 \\ 0 & -1 & 1 \end{pmatrix}^t = \begin{pmatrix} 1 & 1 & 0 \\ -1 & 2 & -1 \\ 2 & 0 & 1 \end{pmatrix}.$

What is the inverse of $3\mathbf{A}$*?*
We have

$$(3\mathbf{A})^{-1} = 3^{-1}\mathbf{A}^{-1} = \frac{1}{3}\mathbf{A}^{-1} = \frac{1}{3}\begin{pmatrix} 1 & -1 & 2 \\ 1 & 2 & 0 \\ 0 & -1 & 1 \end{pmatrix} \qquad \left[\text{Because } \mathbf{A}^{-1} = \begin{pmatrix} 1 & -1 & 2 \\ 1 & 2 & 0 \\ 0 & -1 & 1 \end{pmatrix}\right]$$

What is the inverse of $\mathbf{A}^2$ *?*
We have

$$\left(\mathbf{A}^2\right)^{-1} = (\mathbf{AA})^{-1} = \mathbf{A}^{-1}\mathbf{A}^{-1}$$

$$= \begin{pmatrix} 1 & -1 & 2 \\ 1 & 2 & 0 \\ 0 & -1 & 1 \end{pmatrix}\begin{pmatrix} 1 & -1 & 2 \\ 1 & 2 & 0 \\ 0 & -1 & 1 \end{pmatrix} = \begin{pmatrix} 0 & -5 & 4 \\ 3 & 3 & 2 \\ -1 & -3 & 1 \end{pmatrix}$$

13. Since we are given a triangular matrix therefore the determinant is product of the entries on the leading diagonal. Since we are multiplying the terms above the leading diagonal by 5 therefore the determinant remains unchanged. Hence determinant of the new matrix is also 3

14. The adjoint of the matrix is the matrix of cofactors transposed. Thus for the 2 by 2 matrix we have

$$adj\mathbf{A} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

$$\det \mathbf{A} = \det\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = 1 - 1 = 0$$

Since det**A** is zero therefore the inverse matrix does **not** exist.
For the 3 by 3 matrix we need to find the cofactors of each entry in order to determine adj**A**. Considering each entry from the top row of $\mathbf{A} = \begin{bmatrix} 3 & -4 & 1 \\ 1 & -1 & 3 \\ 2 & -2 & 5 \end{bmatrix}$ we have:

Cofactor of 3 is $\det\begin{bmatrix} -1 & 3 \\ -2 & 5 \end{bmatrix} = -5 + 6 = 1$.

Cofactor of -4 is $-\det\begin{bmatrix} 1 & 3 \\ 2 & 5 \end{bmatrix} = -(5-6) = 1$.

Cofactor of 1 (in the top row) is $\det\begin{bmatrix} 1 & -1 \\ 2 & -2 \end{bmatrix} = 0$.

Cofactor of 1 (in the middle row) is $-\det\begin{bmatrix} -4 & 1 \\ -2 & 5 \end{bmatrix} = -(-20+2) = 18$.

Cofactor of -1 is $\det\begin{bmatrix} 3 & 1 \\ 2 & 5 \end{bmatrix} = (15-2) = 13$.

Cofactor of 3 is $-\det\begin{bmatrix} 3 & -4 \\ 2 & -2 \end{bmatrix} = -(-6+8) = -2$.

Cofactor of 2 is $\det\begin{bmatrix} -4 & 1 \\ -1 & 3 \end{bmatrix} = (-12+1) = -11$.

Cofactor of -2 is $-\det\begin{bmatrix}3 & 1\\1 & 3\end{bmatrix}=-(9-1)=-8$.

Cofactor of 5 is $\det\begin{bmatrix}3 & -4\\1 & -1\end{bmatrix}=(-3+4)=1$.

Thus the cofactor matrix **C** is given by

$$\mathbf{C}=\begin{pmatrix}1 & 1 & 0\\18 & 13 & -2\\-11 & -8 & 1\end{pmatrix}$$

What is the adjoint matrix equal to?

It is the cofactor matrix transposed, that is

$$adj\mathbf{A}=\mathbf{C}^T=\begin{pmatrix}1 & 1 & 0\\18 & 13 & -2\\-11 & -8 & 1\end{pmatrix}^T=\begin{pmatrix}1 & 18 & -11\\1 & 13 & -8\\0 & -2 & 1\end{pmatrix}$$

The determinant of the matrix can be found by expanding along the top row, that is

$$\begin{aligned}\det\mathbf{A}&=3(\text{cofactor of }3)-4(\text{Cofactor of }-4)+1(\text{Cofactor of }1)\\&=3(1)-4(1)+1(0)=-1\end{aligned}$$

Thus det**A=**-1. The inverse of the given matrix is

$$\mathbf{A}^{-1}=\frac{1}{\det\mathbf{A}}adj\mathbf{A}=-1\begin{pmatrix}1 & 18 & -11\\1 & 13 & -8\\0 & -2 & 1\end{pmatrix}=\begin{pmatrix}-1 & -18 & 11\\-1 & -13 & 8\\0 & 2 & -1\end{pmatrix}$$

15. (a) We need to find the determinant of $\mathbf{A}=\begin{bmatrix}2 & 0 & 0 & 0\\1 & -7 & -5 & 0\\3 & 8 & 6 & 0\\0 & 7 & 5 & 4\end{bmatrix}$. *What do you notice about this matrix?*

It is nearly a lower triangular matrix. The only problem is the -5 in the second row. We carry out the row operations to see if we can establish a 0 in this place:

$$\begin{array}{c}R_1\\R_2\\R_3\\R_4\end{array}\begin{bmatrix}2 & 0 & 0 & 0\\1 & -7 & -5 & 0\\3 & 8 & 6 & 0\\0 & 7 & 5 & 4\end{bmatrix}$$

Multiply the second row by 6 so that we have

$$\begin{array}{c}R_1\\R_2^*=6R_2\\R_3\\R_4\end{array}\begin{bmatrix}2 & 0 & 0 & 0\\6 & -42 & -30 & 0\\3 & 8 & 6 & 0\\0 & 7 & 5 & 4\end{bmatrix}$$

Now carrying out the row operation $R_2^*+5R_3$ we have

$$\begin{array}{c} R_1 \\ R_2^* + 5R_3 \\ R_3 \\ R_4 \end{array} \begin{bmatrix} 2 & 0 & 0 & 0 \\ 21 & -2 & 0 & 0 \\ 3 & 8 & 6 & 0 \\ 0 & 7 & 5 & 4 \end{bmatrix}$$

We now have a (lower) triangular matrix. *What is the determinant of this matrix?* The product of all the entries along the leading diagonal.

$$2\times(-2)\times 6\times 4 = -96$$

What is the determinant of the initial given matrix?

The first row operation of multiplying by 6 means we have to divide this -96 by 6 to find the determinant of the given matrix, that is

$$\det(\mathbf{A}) = \frac{-96}{6} = -16$$

(b) We know that $\det\left(\mathbf{A}^T\right) = \det(\mathbf{A}) = -16$.

(c) *What is the determinant of the inverse* $\mathbf{A}^{-1}$*?*

$$\det\left(\mathbf{A}^{-1}\right) = \frac{1}{\det(\mathbf{A})} = -\frac{1}{16}$$

16. We use row operations to simplify the given 4 by 4 matrix:

$$\begin{array}{c} R_1 \\ R_2 \\ R_3 \\ R_4 \end{array} \begin{pmatrix} 1 & 0 & 2 & 0 \\ 0 & 2 & 0 & -1 \\ 2 & 0 & 3 & 0 \\ 0 & -1 & 0 & 4 \end{pmatrix}$$

Multiplying the first row by 3 gives

$$\begin{array}{c} R_1^* = 3R_1 \\ R_2 \\ R_3 \\ R_4 \end{array} \begin{pmatrix} 3 & 0 & 6 & 0 \\ 0 & 2 & 0 & -1 \\ 2 & 0 & 3 & 0 \\ 0 & -1 & 0 & 4 \end{pmatrix}$$

Carrying out the row operation $R_1^* - 2R_3$ gives

$$\begin{array}{c} R_1^{**} = R_1^* - 2R_3 \\ R_2 \\ R_3 \\ R_4 \end{array} \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 2 & 0 & -1 \\ 2 & 0 & 3 & 0 \\ 0 & -1 & 0 & 4 \end{pmatrix}$$

Multiply the second row by 4, that is $4R_2$:

$$\begin{array}{c} R_1^{**} \\ R_2^* = 4R_2 \\ R_3 \\ R_4 \end{array} \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 8 & 0 & -4 \\ 2 & 0 & 3 & 0 \\ 0 & -1 & 0 & 4 \end{pmatrix}$$

Execute the row operation $R_2^* + R_4$:

$$\begin{array}{c} R_1^{**} \\ R_2^{*}+R_4 \\ R_3 \\ R_4 \end{array}\begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 7 & 0 & 0 \\ 2 & 0 & 3 & 0 \\ 0 & -1 & 0 & 4 \end{pmatrix}$$

We have a (lower) triangular matrix and the determinant of this matrix is the product of all the entries along the leading diagonal, that is $-1\times 7\times 3\times 4=-84$. What is the determinant of the given matrix?
It is -84 divided by $4\times 3=12$ because we have multiplied rows by 3 and 4 in the above row operations. We have $D_1=-7$.

How is the determinant D_2 related to D_1?

Since D_2 is the determinant D_1 middle rows interchanged therefore $D_2=-\left(-7\right)=7$.

We can use row operations to find the determinant D_3. We have

$$\begin{array}{c} R_1 \\ R_2 \\ R_3 \\ R_4 \end{array}\begin{pmatrix} 1 & 0 & 2 & 0 \\ 0 & 2 & 0 & -1 \\ 2 & 0 & 3 & 0 \\ 3 & 2 & 5 & -1 \end{pmatrix}$$

Carrying out R_1+R_2 we have

$$\begin{array}{c} R_1^{*}=R_1+R_2 \\ R_2 \\ R_3 \\ R_4 \end{array}\begin{pmatrix} 1 & 2 & 2 & -1 \\ 0 & 2 & 0 & -1 \\ 2 & 0 & 3 & 0 \\ 3 & 2 & 5 & -1 \end{pmatrix}$$

Executing $R_1^{*}+R_3$ we have

$$\begin{array}{c} R_1^{*}+R_3 \\ R_2 \\ R_3 \\ R_4 \end{array}\begin{pmatrix} 3 & 2 & 5 & -1 \\ 0 & 2 & 0 & -1 \\ 2 & 0 & 3 & 0 \\ 3 & 2 & 5 & -1 \end{pmatrix}$$

Since the top and bottom rows are identical therefore the determinant of this matrix is 0. Thus $D_3=0$.

You may spot this from the outset that the bottom row of D_3 is the addition of the remaining rows which means that determinant D_3 is zero.

17. (a) Since the given matrix is a triangular (upper) matrix therefore the determinant is the product of all the entries on the leading diagonal, that is

$$\det\left(\mathbf{A}\right)=3\times\left(-2\right)\times 1\times 4=-24$$

(b) *What do you notice about the entries of matrix **B**?*
The second and fourth rows have identical entries so the determinant of the matrix is 0, that is $\det\left(\mathbf{B}\right)=0$.

(c) To find the determinant of the given matrix, it is easier to expand along the third column:

$$\det(\mathbf{C})=\det\begin{bmatrix}4&-2&1&5\\2&9&0&3\\0&2&0&0\\5&8&0&7\end{bmatrix}=1\det\begin{bmatrix}2&9&3\\0&2&0\\5&8&7\end{bmatrix}$$

$$=2\det\begin{bmatrix}2&3\\5&7\end{bmatrix}=2(14-15)=-2$$

We have $\det(\mathbf{C})=-2$.

18. (a) Expanding along the bottom row we have

$$\det\begin{pmatrix}-1&-2&-3&1\\1&-2&0&3\\2&-3&1&-1\\0&1&0&0\end{pmatrix}=1\det\begin{pmatrix}-1&-3&1\\1&0&3\\2&1&-1\end{pmatrix}$$

$$\underbrace{=}_{\substack{\text{Expanding along}\\\text{the middle row}}}-1\det\begin{pmatrix}-3&1\\1&-1\end{pmatrix}-3\det\begin{pmatrix}-1&-3\\2&1\end{pmatrix}$$

$$=-(3-1)-3(-1+6)=-17$$

Hence the determinant of the matrix **A** is -17.
*What type of matrix is **B**?*
It is a lower triangular matrix therefore the determinant of this matrix is the product of the entries on the leading diagonal, that is

$$\det\begin{pmatrix}2&0&0&0\\5&1&0&0\\-1&1&2&0\\2&-3&1&-3\end{pmatrix}=2\times1\times2\times(-3)=-12$$

Thus the determinant of matrix **B** is -12.
Also $\det(\mathbf{AB})=\det(\mathbf{A})\times\det(\mathbf{B})=-17\times-12=204$ and

$$\det(\mathbf{A}^3)=[\det(\mathbf{A})]^3=[-17]^3=-4913.$$

(b) We need to prove that $\det(\mathbf{A})\neq0$ provided **A** is an invertible $n\times n$ matrix.
Proof.
Suppose $\det(\mathbf{A})=0$. We can assume that **A** is an invertible matrix. Therefore

$$\mathbf{AA}^{-1}=\mathbf{I}$$

Taking the determinant of both sides gives

$$\det(\mathbf{AA}^{-1})=\det(\mathbf{I})=1$$

$$\det(\mathbf{A})\det(\mathbf{A}^{-1})=1$$

Thus $\det(\mathbf{A})\neq0$. Hence we have contradiction therefore our supposition $\det(\mathbf{A})=0$ must be wrong so $\det(\mathbf{A})\neq0$.

■

19. (a) Evaluating the determinant of the given matrix:

$$\det\begin{bmatrix}1&0&0&u_1\\0&1&0&u_2\\0&0&1&u_3\\v_1&v_2&v_3&0\end{bmatrix}=1\det\begin{bmatrix}1&0&u_2\\0&1&u_3\\v_2&v_3&0\end{bmatrix}-u_1\det\begin{bmatrix}0&1&0\\0&0&1\\v_1&v_2&v_3\end{bmatrix}$$

$$\begin{aligned}&=1\det\begin{bmatrix}1&u_3\\v_3&0\end{bmatrix}+u_2\det\begin{bmatrix}0&1\\v_2&v_3\end{bmatrix}-u_1\left(-1\det\begin{pmatrix}0&1\\v_1&v_2\end{pmatrix}\right)\\&=\left(0-u_3v_3\right)+u_2\left(0-v_2\right)+u_1\left(0-v_1\right)\\&=-u_3v_3-u_2v_2-u_1v_1\\&=-\left(u_1v_1+u_2v_2+u_3v_3\right)\end{aligned}$$

(b) If the zero in the bottom right hand corner is replaced by 100 then the deduction of the determinant is very similar to the above.

$$\det\begin{bmatrix}1&0&0&u_1\\0&1&0&u_2\\0&0&1&u_3\\v_1&v_2&v_3&100\end{bmatrix}=1\det\begin{bmatrix}1&0&u_2\\0&1&u_3\\v_2&v_3&100\end{bmatrix}-u_1\det\begin{bmatrix}0&1&0\\0&0&1\\v_1&v_2&v_3\end{bmatrix}$$

$$\begin{aligned}&=1\det\begin{bmatrix}1&u_3\\v_3&100\end{bmatrix}+u_2\det\begin{bmatrix}0&1\\v_2&v_3\end{bmatrix}-u_1\left(-1\det\begin{pmatrix}0&1\\v_1&v_2\end{pmatrix}\right)\\&=\left(100-u_3v_3\right)+u_2\left(0-v_2\right)+u_1\left(0-v_1\right)\\&=100-u_3v_3-u_2v_2-u_1v_1\\&=100-\left(u_1v_1+u_2v_2+u_3v_3\right)\end{aligned}$$

20. We first carry out an elementary row operation on the given matrix:

$$\begin{array}{c}R_1\\R_2\\R_3\end{array}\begin{pmatrix}1&a^2&b+c\\1&b^2&a+c\\1&c^2&a+b\end{pmatrix}$$

Carrying out the row operations R_2-R_1 and R_3-R_1:

$$\begin{array}{c}R_1\\R_2{*}=R_2-R_1\\R_3{*}=R_3-R_1\end{array}\begin{pmatrix}1&a^2&b+c\\0&b^2-a^2&a-b\\0&c^2-a^2&a-c\end{pmatrix}$$

Finding the determinant of this matrix is pretty straightforward because we can just expand the first column:

$$\det\begin{pmatrix}1 & a^2 & b+c\\ 0 & b^2-a^2 & a-b\\ 0 & c^2-a^2 & a-c\end{pmatrix} = 1\det\begin{pmatrix}b^2-a^2 & a-b\\ c^2-a^2 & a-c\end{pmatrix}$$
$$= \left(b^2-a^2\right)(a-c)-\left(c^2-a^2\right)(a-b)$$
$$= (b-a)(b+a)(a-c)-(c-a)(c+a)(a-b)$$
$$= -(a-b)(b+a)(a-c)+(a-c)(c+a)(a-b)$$
$$= (a-b)(a-c)\left[-(b+a)+(c+a)\right]$$
$$= (a-b)(a-c)(c-b)$$

Since the only row operations involved adding a multiple of one row to another therefore this is value is the determinant of the given matrix, that is

$$\begin{vmatrix}1 & a^2 & b+c\\ 1 & b^2 & a+c\\ 1 & c^2 & a+b\end{vmatrix} = (a-b)(a-c)(c-b)$$

21. We use the generic formula to find the determinant, that is

(6.6) $$\det(\mathbf{A}) = a_{i1}C_{i1} + a_{i2}C_{i2} + a_{i3}C_{i3} + \cdots + a_{in}C_{in} = \sum_{k=1}^{n} a_{ik}C_{ik}$$

What does this mean?

Expanding along the ith row of the matrix you multiply the entries a_{ij} by its cofactor and sum the entire row. Remember $adj(\mathbf{A})$ is the matrix of cofactors, $\mathbf{C}^T$, transposed. If we transpose again we will get the matrix of cofactors because $\left(\mathbf{C}^T\right)^T = \mathbf{C}$. We have

$$\mathbf{C} = \left[adj(\mathbf{A})\right]^T = \begin{bmatrix}a & 3 & b\\ -1 & 1 & 2\\ c & -2 & d\end{bmatrix}^T = \begin{bmatrix}a & -1 & c\\ 3 & 1 & -2\\ b & 2 & d\end{bmatrix}$$

We are also given $\mathbf{A} = \begin{bmatrix}u & v & w\\ 3 & 3 & -2\\ x & y & z\end{bmatrix}$. *What is* $\det(\mathbf{A})$ *equal to?*

Expanding along the middle row of **A** we have

$$\det(\mathbf{A}) = 3(3) + 3(1) - 2(-2) = 16$$

Thus the determinant of the given matrix is 16.

22. *How do we work out* $\det\left((2\mathbf{A})^{-1}(3\mathbf{A})^T\right)$ *?*

By Proposition (6-14) we have $\det(\mathbf{XY}) = \det(\mathbf{X})\det(\mathbf{Y})$. Applying this to the above we have

$$\det\left((2\mathbf{A})^{-1}(3\mathbf{A})^T\right) = \det\left[(2\mathbf{A})^{-1}\right]\det\left[(3\mathbf{A})^T\right] \qquad (*)$$

What is $(2\mathbf{A})^{-1}$ *and* $(3\mathbf{A})^T$ *equal to?*

$$(2\mathbf{A})^{-1}=\frac{1}{2}\mathbf{A}^{-1} \text{ and } (3\mathbf{A})^{T}=3\mathbf{A}^{T}$$

Substituting these into (*) we have

$$\det\left((2\mathbf{A})^{-1}(3\mathbf{A})^{T}\right)=\det\left(\frac{1}{2}\mathbf{A}^{-1}\right)\times\det\left(3\mathbf{A}^{T}\right)$$

$$=\frac{1}{2^2}\det\left(\mathbf{A}^{-1}\right)\times 3^2\det\left(\mathbf{A}^{T}\right) \qquad \left[\begin{array}{l}\text{Because } \det(k\mathbf{A})=k^n\det(\mathbf{A}) \\ \text{where } \mathbf{A} \text{ is a } n\times n \text{ matrix}\end{array}\right]$$

$$=\frac{3^2}{2^2}\det\left(\mathbf{A}^{-1}\right)\det(\mathbf{A}) \qquad \left[\text{Because } \det\left(\mathbf{A}^{T}\right)=\det(\mathbf{A})\right]$$

$$=\frac{3^2}{2^2}(1)=\frac{3^2}{2^2} \qquad \left[\text{Because } \det\left(\mathbf{A}^{-1}\right)\det(\mathbf{A})=1\right]$$

Thus the determinant of the given matrix $\frac{9}{4}$.

23. (a) (i) We can find the determinant of the given matrix by using our normal technique

$$\begin{vmatrix}1&2&2\\3&1&3\\1&3&1\end{vmatrix}=\begin{vmatrix}1&3\\3&1\end{vmatrix}-2\begin{vmatrix}3&3\\1&1\end{vmatrix}+2\begin{vmatrix}3&1\\1&3\end{vmatrix}=(1-9)-0+2(9-1)=8$$

(ii) We have

$$\begin{vmatrix}3&0&0&2\\0&0&3&0\\0&0&2&2\\3&1&0&0\end{vmatrix}=1\begin{vmatrix}3&0&2\\0&3&0\\0&2&2\end{vmatrix}=3\begin{vmatrix}3&0\\2&2\end{vmatrix}=3(6)=18$$

(b) We use the following properties of determinants to answer this part of the question.

Proposition (6-7) $\quad \det(k\mathbf{A})=k^n\det(\mathbf{A})$

where $\mathbf{A}$ is a square n by n matrix and k a scalar.

Proposition (6-14) $\quad \det(\mathbf{XY})=\det(\mathbf{X})\det(\mathbf{Y})$

Proposition (6-17) $\quad \det\left(\mathbf{A}^{-1}\right)=\dfrac{1}{\det(\mathbf{A})}$

(i) We have

$$\det(3\mathbf{A})=3^4\det(\mathbf{A})=3^4\times 2=162$$

(ii) Likewise

$$\det\left(\mathbf{C}^{-1}\mathbf{B}\right)=\det\left(\mathbf{C}^{-1}\right)\det(\mathbf{B})$$

$$=\frac{1}{\det(\mathbf{C})}\det(\mathbf{B})=\frac{1}{5}(-3)=-\frac{3}{5}$$

(iii) We also have

$$\begin{aligned}\det\left(\mathbf{A}^2\mathbf{C}^{-1}\mathbf{B}^T\right) &= \det\left(\mathbf{A}^2\right)\det\left(\mathbf{C}^{-1}\right)\det\left(\mathbf{B}^T\right)\\ &= \left[\det(\mathbf{A})\right]^2 \frac{1}{\det(\mathbf{C})}\det(\mathbf{B}) \qquad \left[\text{Because } \det\left(\mathbf{B}^T\right)=\det(\mathbf{B})\right]\\ &= 2^2\frac{1}{5}(-3) = -\frac{12}{5}\end{aligned}$$

24. (a) (i) We can use row operations to convert the given matrix into a triangular matrix:

$$\begin{array}{c} R_1 \\ R_2 \\ R_3 \\ R_4 \end{array}\begin{pmatrix} 0 & 0 & 1 & 3 \\ 0 & 0 & 3 & 1 \\ 0 & 1 & a & b \\ 3 & 1 & c & d \end{pmatrix}$$

Multiplying the top row by 3 we have

$$\begin{array}{c} R_1^* = 3R_1 \\ R_2 \\ R_3 \\ R_4 \end{array}\begin{pmatrix} 0 & 0 & 3 & 9 \\ 0 & 0 & 3 & 1 \\ 0 & 1 & a & b \\ 3 & 1 & c & d \end{pmatrix}$$

Carrying out the row operation $R_1^* - R_2$ gives

$$\begin{array}{c} R_1^{**} = R_1^* - R_2 \\ R_2 \\ R_3 \\ R_4 \end{array}\begin{pmatrix} 0 & 0 & 0 & 8 \\ 0 & 0 & 3 & 1 \\ 0 & 1 & a & b \\ 3 & 1 & c & d \end{pmatrix}$$

Interchanging rows R_1^{**} and R_4, R_2 and R_3 gives

$$\begin{array}{c} R_4 \\ R_3 \\ R_2 \\ R_1^{**} \end{array}\begin{pmatrix} 3 & 1 & c & d \\ 0 & 1 & a & b \\ 0 & 0 & 3 & 1 \\ 0 & 0 & 0 & 8 \end{pmatrix}$$

This is an upper triangular matrix so the determinant of this matrix is the product of the entries on the leading diagonal, that is $3\times1\times3\times8=72$. *What is the determinant of the given matrix above?*

The row operations carried out were

(I) Multiplying by 3.

(II) $R_1^* - R_2$

(III) Interchanging rows R_1^{**} and R_4, R_2 and R_3

We need to divide 72 by 3 because of row operation (I). The second row operation does **not change** the determinant. Row operation (III) has the multiplying the determinant by $(-1)\times(-1)=1$. We have

$$\begin{vmatrix} 0 & 0 & 1 & 3 \\ 0 & 0 & 3 & 1 \\ 0 & 1 & a & b \\ 3 & 1 & c & d \end{vmatrix} = \frac{72}{3} = 24$$

(ii) We have

$$\begin{vmatrix} 0 & 0 & a & 0 & 0 \\ 0 & b & 0 & 0 & 0 \\ c & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 2 & 0 \end{vmatrix} = -1 \begin{vmatrix} 0 & 0 & a & 0 \\ 0 & b & 0 & 0 \\ c & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 \end{vmatrix}$$

$$= -2 \begin{vmatrix} 0 & 0 & a \\ 0 & b & 0 \\ c & 0 & 0 \end{vmatrix} = -2a \begin{vmatrix} 0 & b \\ c & 0 \end{vmatrix} = -2a(-bc) = 2abc$$

Hence the determinant is $2abc$.

(iii) We have

$$\begin{vmatrix} 0 & 0 & 3 & 0 & 0 \\ 0 & 2 & 4 & 0 & 0 \\ 1 & 0 & 5 & 0 & 0 \\ 0 & 0 & 6 & 0 & 9 \\ 0 & 0 & 7 & 8 & 0 \end{vmatrix} = 3 \begin{vmatrix} 0 & 2 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 9 \\ 0 & 0 & 8 & 0 \end{vmatrix}$$

$$= 3(-2) \begin{vmatrix} 1 & 0 & 0 \\ 0 & 0 & 9 \\ 0 & 8 & 0 \end{vmatrix} = -6\left[-(9 \times 8)\right] = 432$$

(b) This is similar to question 22. We need to apply the properties of the determinants and then use $\det \mathbf{A} = 3$ and $\det \mathbf{B} = -2$:

(i) We have

$$\begin{aligned} \det\left(\mathbf{B}\mathbf{B}^T\right) &= \det(\mathbf{B})\det\left(\mathbf{B}^T\right) \\ &= \det(\mathbf{B})\det(\mathbf{B}) \qquad \left[\text{Because } \det\left(\mathbf{B}^T\right) = \det(\mathbf{B})\right] \\ &= (-2)(-2) = 4 \end{aligned}$$

The determinant is 4.

(ii) Similarly we have

$$\begin{aligned} \det\left(\mathbf{A}^{-1}\mathbf{B}\mathbf{A}\right) &= \det\left(\mathbf{A}^{-1}\right)\det(\mathbf{B})\det(\mathbf{A}) \\ &= \underbrace{\det\left(\mathbf{A}^{-1}\right)\det(\mathbf{A})}_{=1}\det(\mathbf{B}) \qquad \left[\text{Because } \det\left(\mathbf{A}^{-1}\right)\det(\mathbf{A}) = 1\right] \\ &= 1 \times (-2) = -2 \end{aligned}$$

Hence we have -2.

(iii) *What is* $\det\left(\mathbf{A}\mathbf{B}^{-1}\right)$ *equal to?*

$$\det\left(\mathbf{A}\mathbf{B}^{-1}\right) = \det(\mathbf{A})\det\left(\mathbf{B}^{-1}\right) = 3\left(-\frac{1}{2}\right) = -\frac{3}{2}$$

25. We need to use the properties of determinants to evaluate the given combinations.
Which properties?
By using the following properties of determinants:

Proposition (6-7) $\det(k\mathbf{A}) = k^n \det(\mathbf{A})$

where $\mathbf{A}$ is a square n by n matrix and k a scalar.

Proposition (6-14) $\det(\mathbf{XY}) = \det(\mathbf{X})\det(\mathbf{Y})$

Proposition (6-17) $\det(\mathbf{A}^{-1}) = \dfrac{1}{\det(\mathbf{A})}$

(i) *How do we find* $\det(-\mathbf{A}^3\mathbf{B}^{-2})$ *?*

Applying the above to this we get

$$\begin{aligned}
\det(-\mathbf{A}^3\mathbf{B}^{-2}) &= \det\left((-1)\mathbf{A}^3\mathbf{B}^{-1}\mathbf{B}^{-1}\right) \\
&= (-1)^2 \det(\mathbf{A}^3)\det(\mathbf{B}^{-1})\det(\mathbf{B}^{-1}) \\
&= 2^3 \frac{1}{3}\frac{1}{3} \qquad \left[\text{Because } \det(\mathbf{A}) = 2 \text{ and } \det(\mathbf{B}) = 3\right] \\
&\frac{8}{9}
\end{aligned}$$

Hence the determinant of the given combination of matrices is 8/9.

(ii) Similarly we have

$$\begin{aligned}
\det(2\mathbf{A}^{-1}\mathbf{BA}) &= 2^2 \det(\mathbf{A}^{-1}\mathbf{BA}) \\
&= 4\det(\mathbf{A}^{-1})\det(\mathbf{B})\det(\mathbf{A}) \\
&= 4\det(\mathbf{B}) \qquad \left[\text{Because } \det(\mathbf{A}^{-1})\det(\mathbf{A}) = 1\right] \\
&= 4\times 3 = 12
\end{aligned}$$

Thus determinant is 12.

(iii) Likewise we have

$$\begin{aligned}
\det(\mathbf{A}^{-1}\mathbf{A}^T) &= \det(\mathbf{A}^{-1})\det(\mathbf{A}^T) \\
&= \det(\mathbf{A}^{-1})\det(\mathbf{A}) \qquad \left[\text{Because } \det(\mathbf{A}^T) = \det(\mathbf{A})\right] \\
&= 1 \qquad \left[\text{Because } \det(\mathbf{A}^{-1})\det(\mathbf{A}) = 1\right]
\end{aligned}$$

Thus the determinant of $\mathbf{A}^{-1}\mathbf{A}^T$ is equal to 1.

(b) We are given that $\begin{vmatrix} a & d & g \\ b & e & h \\ c & f & i \end{vmatrix} = 1$ and need to find $\begin{vmatrix} a+d & d+g & g+a \\ b+e & e+h & h+b \\ c+f & f+i & i+c \end{vmatrix}$. Expanding this determinant along the first column we have

$$\begin{vmatrix} a+d & d+g & g+a \\ b+e & e+h & h+b \\ c+f & f+i & i+c \end{vmatrix} = (a+d)\begin{vmatrix} e+h & h+b \\ f+i & i+c \end{vmatrix} - (b+e)\begin{vmatrix} d+g & g+a \\ f+i & i+c \end{vmatrix} + (c+f)\begin{vmatrix} d+g & g+a \\ e+h & h+b \end{vmatrix}$$

$$= a\begin{vmatrix} e+h & h+b \\ f+i & i+c \end{vmatrix} - b\begin{vmatrix} d+g & g+a \\ f+i & i+c \end{vmatrix} + c\begin{vmatrix} d+g & g+a \\ e+h & h+b \end{vmatrix}$$

$$+d\begin{vmatrix} e+h & h+b \\ f+i & i+c \end{vmatrix} - e\begin{vmatrix} d+g & g+a \\ f+i & i+c \end{vmatrix} + f\begin{vmatrix} d+g & g+a \\ e+h & h+b \end{vmatrix}$$

$$= \begin{vmatrix} a & d+g & g+a \\ b & e+h & h+b \\ c & f+i & i+c \end{vmatrix} + \begin{vmatrix} d & d+g & g+a \\ e & e+h & h+b \\ f & f+i & i+c \end{vmatrix} \qquad (\dagger)$$

The first determinant in (†) on the Right Hand Side has determinant 1 because this can be obtained from the given matrix $\begin{vmatrix} a & d & g \\ b & e & h \\ c & f & i \end{vmatrix}$ by adding column 3 to column 2 and adding column 1 to column 3. Remember this column (row) operation does **not change** the determinant.

What is the second determinant on the Right Hand Side of (†)?

Let us write the given determinant as a matrix and label the columns:

$$\begin{array}{ccc} C_1 & C_2 & C_3 \end{array}$$
$$\begin{pmatrix} a & d & g \\ b & e & h \\ c & f & i \end{pmatrix}$$

Interchanging the columns C_1 and C_2 gives

$$\begin{array}{ccc} C_2 & C_1 & C_3 \end{array}$$
$$\begin{pmatrix} d & a & g \\ e & b & h \\ f & c & i \end{pmatrix}$$

Interchanging columns C_1 and C_3 gives

$$\begin{array}{ccc} C_2 & C_3 & C_1 \end{array}$$
$$\begin{pmatrix} d & g & a \\ e & h & b \\ f & i & c \end{pmatrix}$$

Carrying out the column operations $C_1 + C_3$ and $C_3 + C_2$:

$$\begin{array}{ccc} C_2 & C_3 + C_2 & C_1 + C_3 \end{array}$$
$$\begin{pmatrix} d & g+d & a+g \\ e & h+e & b+h \\ f & i+f & c+i \end{pmatrix}$$

The determinant of this last matrix is the second determinant we have on the Right Hand Side of (†). *What is determinant of this, that is* $\begin{vmatrix} d & g+d & a+g \\ e & h+e & b+h \\ f & i+f & c+i \end{vmatrix}$ *equal to?*

The column operations carried out were interchanging columns C_1 and C_2, C_1 and C_3 which means we have to multiply by $(-1)(-1)=1$. The remaining column operations C_1+C_3 and C_3+C_2 also does **not** change the determinant. Thus

$$\begin{vmatrix} d & g+d & a+g \\ e & h+e & b+h \\ f & i+f & c+i \end{vmatrix} = \begin{vmatrix} a & d & g \\ b & e & h \\ c & f & i \end{vmatrix} = 1$$

Substituting $\begin{vmatrix} a & d+g & g+a \\ b & e+h & h+b \\ c & f+i & i+c \end{vmatrix} = 1$ and $\begin{vmatrix} d & g+d & a+g \\ e & h+e & b+h \\ f & i+f & c+i \end{vmatrix} = 1$ into (†) gives

$$\begin{vmatrix} a+d & d+g & g+a \\ b+e & e+h & h+b \\ c+f & f+i & i+c \end{vmatrix} = \begin{vmatrix} a & d+g & g+a \\ b & e+h & h+b \\ c & f+i & i+c \end{vmatrix} + \begin{vmatrix} d & d+g & g+a \\ e & e+h & h+b \\ f & f+i & i+c \end{vmatrix} = 1+1=2$$

The determinant of the given matrix is 2.

26. We are given $\det(\mathbf{A})=3$.

(a) $\det\begin{bmatrix} a-2 & 1 & 2 \\ b-4 & 3 & 4 \\ c-6 & 5 & 6 \end{bmatrix} = \det(\mathbf{A}) = 3$ because this is matrix **A** with the first and last columns subtracted, that is $C_1 - C_3$. Remember subtracting a multiple of one row (column) to another does **not** change the determinant.

(b) *How is* $\begin{bmatrix} 7a & 7 & 14 \\ b & 3 & 4 \\ c & 5 & 6 \end{bmatrix}$ *related to* $\mathbf{A} = \begin{bmatrix} a & 1 & 2 \\ b & 3 & 4 \\ c & 5 & 6 \end{bmatrix}$*?*

The first row of matrix **A** has been multiplied by 7. Thus

$$\det\begin{bmatrix} 7a & 7 & 14 \\ b & 3 & 4 \\ c & 5 & 6 \end{bmatrix} = 7\det\begin{bmatrix} a & 1 & 2 \\ b & 3 & 4 \\ c & 5 & 6 \end{bmatrix} = 7\det(\mathbf{A}) = 7\times 3 = 21$$

(c) What is the determinant of $2\mathbf{A}^{-1}\mathbf{A}^T$?

The determinant of $\mathbf{A}^{-1}$ is equal to $\dfrac{1}{\det(\mathbf{A})} = \dfrac{1}{3}$. *What is the determinant of* $\mathbf{A}^T$*?*

$$\det\left(\mathbf{A}^T\right) = \det(\mathbf{A}) = 3$$

Thus

$$\det\left(2\mathbf{A}^{-1}\mathbf{A}^T\right) = 2\det\left(\mathbf{A}^{-1}\right)\det\left(\mathbf{A}^T\right)$$

$$= 2\frac{1}{3}3 = 2$$

(d) *How do we find the determinant of* $\begin{bmatrix} a-2 & 1 & 2 \\ b & 3 & 4 \\ c & 5 & 6 \end{bmatrix}$*?*

We can expand this in the conventional manner and then substitute the value of $\det(\mathbf{A}) = 3$:

$$\det\begin{bmatrix} a-2 & 1 & 2 \\ b & 3 & 4 \\ c & 5 & 6 \end{bmatrix} = (a-2)\det\begin{bmatrix} 3 & 4 \\ 5 & 6 \end{bmatrix} - 1\det\begin{bmatrix} b & 4 \\ c & 5 \end{bmatrix} + 2\det\begin{bmatrix} b & 3 \\ c & 5 \end{bmatrix}$$

$$= a\det\begin{bmatrix} 3 & 4 \\ 5 & 6 \end{bmatrix} - 2\det\begin{bmatrix} 3 & 4 \\ 5 & 6 \end{bmatrix} - \det\begin{bmatrix} b & 4 \\ c & 5 \end{bmatrix} + 2\det\begin{bmatrix} b & 3 \\ c & 5 \end{bmatrix}$$

$$= \underbrace{a\det\begin{bmatrix} 3 & 4 \\ 5 & 6 \end{bmatrix} - \det\begin{bmatrix} b & 4 \\ c & 5 \end{bmatrix} + 2\det\begin{bmatrix} b & 3 \\ c & 5 \end{bmatrix}}_{=\det(\mathbf{A})} - 2\det\begin{bmatrix} 3 & 4 \\ 5 & 6 \end{bmatrix}$$

$$= 3 - 2(18-20) = 7$$

Thus the determinant of the given matrix is 7.

27. By Proposition (6-14) we have $\det(\mathbf{XY}) = \det(\mathbf{X})\det(\mathbf{Y})$. Using this on the given matrix we have

$$\det(\mathbf{A}) = \det\begin{pmatrix} a & b & c \\ 0 & d & e \\ 0 & 0 & f \end{pmatrix}\det\begin{pmatrix} p & 0 & 0 \\ q & r & 0 \\ s & t & u \end{pmatrix}$$

How do we find the determinant of the matrices on the Right Hand Side?
Both matrices are triangular matrices therefore their determinant is the product of the entries on the leading diagonal. We have

$$\det(\mathbf{A}) = \det\begin{pmatrix} a & b & c \\ 0 & d & e \\ 0 & 0 & f \end{pmatrix}\det\begin{pmatrix} p & 0 & 0 \\ q & r & 0 \\ s & t & u \end{pmatrix}$$
$$= (a \times d \times f)(p \times r \times u)$$

We have that the determinant of the given matrix is $adfpru$.

28. We use Cramer's rule:
The formula for Cramer's rule (6-16) is given by

$$\boxed{x_1 = \frac{\det(\mathbf{A}_1(\mathbf{b}))}{\det(\mathbf{A})}, \quad x_2 = \frac{\det(\mathbf{A}_2(\mathbf{b}))}{\det(\mathbf{A})}, \quad x_3 = \frac{\det(\mathbf{A}_3(\mathbf{b}))}{\det(\mathbf{A})}}$$

In our case we let $\mathbf{A} = \begin{bmatrix} a & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 3 & 6 \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, $x_1 = x$, $x_2 = y$ and $x_3 = z$. Therefore

$$\det(\mathbf{A}) = \det\begin{bmatrix} a & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 3 & 6 \end{bmatrix} = a\det\begin{bmatrix} 2 & 3 \\ 3 & 6 \end{bmatrix} - \det\begin{bmatrix} 1 & 3 \\ 1 & 6 \end{bmatrix} + \det\begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix}$$
$$= a(12-9) - (6-3) + (3-2) = 3a - 2$$

We have solutions provided $3a - 2 \neq 0$. Evaluating each of the determinants in the above formula (6-16):

$$\det(\mathbf{A}_1(\mathbf{b})) = \det\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 3 & 6 \end{bmatrix} = \det\begin{bmatrix} 2 & 3 \\ 3 & 6 \end{bmatrix} - \det\begin{bmatrix} 1 & 3 \\ 1 & 6 \end{bmatrix} + \det\begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix}$$
$$= (12-9) - (6-3) + (3-2) = 1$$

Similarly we have

$$\det(\mathbf{A}_2(\mathbf{b})) = \det\begin{bmatrix} a & 1 & 1 \\ 1 & 1 & 3 \\ 1 & 1 & 6 \end{bmatrix} = a\det\begin{bmatrix} 1 & 3 \\ 1 & 6 \end{bmatrix} - \det\begin{bmatrix} 1 & 3 \\ 1 & 6 \end{bmatrix} + \det\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$
$$= a(6-3) - (6-3) + 0 = 3a - 3$$

Also

$$\det(\mathbf{A}_3(\mathbf{b})) = \det\begin{bmatrix} a & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 3 & 1 \end{bmatrix} = a\det\begin{bmatrix} 2 & 1 \\ 3 & 1 \end{bmatrix} - \det\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} + \det\begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix}$$
$$= a(2-3) - 0 + (3-2) = -a + 1$$

Substituting each of these determinants

$$\det(\mathbf{A}) = 3a - 2,\ \det(\mathbf{A}_1(\mathbf{b})) = 1,\ \det(\mathbf{A}_2(\mathbf{b})) = 3a - 3 \text{ and } \det(\mathbf{A}_3(\mathbf{b})) = 1 - a$$

into

$$x = \frac{\det(\mathbf{A}_1(\mathbf{b}))}{\det(\mathbf{A})},\quad y = \frac{\det(\mathbf{A}_2(\mathbf{b}))}{\det(\mathbf{A})},\quad z = \frac{\det(\mathbf{A}_3(\mathbf{b}))}{\det(\mathbf{A})}$$

Gives

$$x = \frac{1}{3a-2},\ y = \frac{3a-3}{3a-2},\ z = \frac{1-a}{3a-2}$$

29. We need to prove that:

If $\mathbf{A}$ is a $n \times n$ matrix satisfying $\mathbf{A}^5 = \mathbf{O}$, then $\det(\mathbf{A}) = 0$.

Proof.

By Proposition (6-15) which says

$$\det(\mathbf{A}_1\mathbf{A}_2\mathbf{A}_3\cdots\mathbf{A}_n) = \det(\mathbf{A}_1)\det(\mathbf{A}_2)\det(\mathbf{A}_3)\cdots\det(\mathbf{A}_n)$$

we have

$$\det(\mathbf{A}^5) = \det(\mathbf{AAAAA})$$
$$= \underbrace{\det(\mathbf{A})\det(\mathbf{A})\cdots\det(\mathbf{A})}_{\text{5 copies}}$$

We know that $\det(\mathbf{O}) = 0$ so we have

$$\det\left(\mathbf{A}^5\right)=\det(\mathbf{A})\det(\mathbf{A})\cdots\det(\mathbf{A})=0 \quad\Rightarrow\quad \det(\mathbf{A})=0$$

Hence we have our required result.

■

30. (a) We have

$$\mathbf{A}_2=\begin{bmatrix}1&2\\-1&0\end{bmatrix},\quad \mathbf{A}_3=\begin{bmatrix}1&2&3\\-1&0&3\\-1&-2&0\end{bmatrix}\text{ and }\mathbf{A}_4=\begin{bmatrix}1&2&3&4\\-1&0&3&4\\-1&-2&0&4\\-1&-2&-3&0\end{bmatrix}$$

The determinants of these is given by the following evaluations.

Using row operations to evaluate the determinant of $\mathbf{A}_2$ we have

$$\begin{array}{c}R_1\\R_2\end{array}\begin{bmatrix}1&2\\-1&0\end{bmatrix}$$

Carrying out the row operation R_2+R_1 gives

$$\begin{array}{c}R_1\\R_2+R_1\end{array}\begin{bmatrix}1&2\\0&2\end{bmatrix}$$

We have a (upper) triangular matrix. *How do we find the determinant of a triangular matrix?*

Multiplying the entries on the leading diagonal, that is $1\times 2=2!=2$. Thus $\det\left(\mathbf{A}_2\right)=2!$.

We use row operations to evaluate the determinant of $\mathbf{A}_3$.

$$\begin{array}{c}R_1\\R_2\\R_3\end{array}\begin{bmatrix}1&2&3\\-1&0&3\\-1&-2&0\end{bmatrix}$$

Performing the row operations R_2+R_1 and R_3+R_1 gives

$$\begin{array}{c}R_1\\R_2+R_1\\R_3+R_1\end{array}\begin{bmatrix}1&2&3\\0&2&6\\0&0&3\end{bmatrix}$$

We have an upper triangular matrix so the determinant of this matrix is the product of all the entries along the leading diagonal, that is $1\times 2\times 3=3!=6$.

Hence we have $\det\left(\mathbf{A}_3\right)=3!=6$.

In a similar manner we find the determinant of $\mathbf{A}_4$.

$$\begin{array}{c}R_1\\R_2\\R_3\\R_4\end{array}\begin{bmatrix}1&2&3&4\\-1&0&3&4\\-1&-2&0&4\\-1&-2&-3&0\end{bmatrix}$$

Executing the row operations R_2+R_1, R_3+R_1 and R_4+R_1:

$$\begin{array}{c}R_1\\R_2+R_1\\R_3+R_1\\R_4+R_1\end{array}\begin{bmatrix}1&2&3&4\\0&2&6&8\\0&0&3&8\\0&0&0&4\end{bmatrix}$$

What is the determinant of this matrix?

$1\times 2\times 3\times 4 = 4! = 24$ because it is a triangular matrix. Thus we have $\det\left(\mathbf{A}_4\right) = 4! = 24$.

(b) Labelling the rows of the general given matrix we have

$$\begin{array}{c} R_1 \\ R_2 \\ R_3 \\ \vdots \\ R_n \end{array}\begin{bmatrix} 1 & 2 & 3 & 4 & \cdots & n \\ -1 & 0 & 3 & 4 & \cdots & n \\ -1 & -2 & 0 & 4 & \cdots & n \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ -1 & -2 & -3 & -4 & \cdots & 0 \end{bmatrix}$$

Carrying out the row operations $R_2 + R_1,\ R_3 + R_1,\ \cdots\ R_n + R_1$:

$$\begin{array}{c} R_1 \\ R_2 + R_1 \\ R_3 + R_1 \\ \vdots \\ R_n + R_1 \end{array}\begin{bmatrix} 1 & 2 & 3 & 4 & \cdots & n \\ 0 & 2 & 3 & 4 & \cdots & n \\ 0 & 0 & 3 & 4 & \cdots & n \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & n \end{bmatrix}$$

From part (a) we have that the determinant of this matrix is given by

$$1\times 2\times 3\times 4\times\cdots\times n = n!$$

Hence $\det\left(\mathbf{A}_n\right) = n!$

31. First we need to find the determinant of the given matrix $\begin{bmatrix} A & B & D \\ 0 & C & E \\ 0 & 0 & F \end{bmatrix} = \mathbf{X}$.

$$\det\left(\mathbf{X}\right) = \det\left(\begin{bmatrix} A & B & D \\ 0 & C & E \\ 0 & 0 & F \end{bmatrix}\right) = ACF \qquad \left[\begin{array}{l}\text{Because we have a} \\ \text{triangular matrix}\end{array}\right]$$

We need to evaluate the cofactor of each entry.

Cofactor of A:

$$\det\begin{pmatrix} C & E \\ 0 & F \end{pmatrix} = CF$$

Cofactor of B:

$$\det\begin{pmatrix} 0 & E \\ 0 & F \end{pmatrix} = 0$$

Cofactor of D:

$$\det\begin{pmatrix} 0 & C \\ 0 & 0 \end{pmatrix} = 0$$

Cofactor of 0:

$$-\det\begin{pmatrix} B & D \\ 0 & F \end{pmatrix} = -BF$$

Cofactor of C:

$$\det\begin{pmatrix} A & D \\ 0 & F \end{pmatrix} = AF$$

Cofactor of E:

$$\det\begin{pmatrix} A & B \\ 0 & 0 \end{pmatrix} = 0$$

Cofactor of 0:

$$\det\begin{pmatrix} B & D \\ C & E \end{pmatrix} = BE - CD$$

Cofactor of 0:

$$-\det\begin{pmatrix} A & D \\ 0 & E \end{pmatrix} = -AE$$

Cofactor of F:

$$\det\begin{pmatrix} A & B \\ 0 & C \end{pmatrix} = AC$$

Hence the cofactor matrix $\mathbf{C}$ is given by

$$\mathbf{C} = \begin{pmatrix} CF & 0 & 0 \\ -BF & AF & 0 \\ BE - CD & -AE & AC \end{pmatrix}$$

The inverse matrix is given by (6-4):

$$\mathbf{X}^{-1} = \frac{1}{\det(\mathbf{X})}\mathbf{C}^T$$

The transpose of the cofactor matrix is

$$\mathbf{C}^T = \begin{pmatrix} CF & -BF & BE - CD \\ 0 & AF & -AE \\ 0 & 0 & AC \end{pmatrix}$$

Hence the inverse matrix is

$$\mathbf{X}^{-1} = \frac{1}{\det(\mathbf{X})}\mathbf{C}^T = \frac{1}{ACF}\begin{pmatrix} CF & -BF & BE - CD \\ 0 & AF & -AE \\ 0 & 0 & AC \end{pmatrix} \text{ provided } ACF \neq 0$$

32. The formula for Cramer's rule (6-16) is given by

$$\boxed{x_1 = \frac{\det(\mathbf{X}_1(\mathbf{b}))}{\det(\mathbf{X})}, \quad x_2 = \frac{\det(\mathbf{X}_2(\mathbf{b}))}{\det(\mathbf{X})}, \quad x_3 = \frac{\det(\mathbf{X}_3(\mathbf{b}))}{\det(\mathbf{X})}}$$

We have the same matrix as the previous question. Therefore

$$\det(\mathbf{X}) = ACF \text{ provided } ACF \neq 0$$

Let $x_2 = y$ in the above formula and $\mathbf{b} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ which is the given vector:

$$\det(\mathbf{X}_2(\mathbf{b})) = \det\begin{bmatrix} A & 1 & D \\ 0 & 1 & E \\ 0 & 0 & F \end{bmatrix} = AF \qquad \begin{bmatrix} \text{Because we have} \\ \text{a triangular matrix} \end{bmatrix}$$

Substituting $\det\left(\mathbf{X}_2\left(\mathbf{b}\right)\right)=AF$ and $\det\left(\mathbf{X}\right)=ACF$ into $y=\dfrac{\det\left(\mathbf{X}_2\left(\mathbf{b}\right)\right)}{\det\left(\mathbf{X}\right)}$ gives

$$y=\frac{AF}{ACF}=\frac{1}{C}$$

33. We need to find the determinant of $\begin{bmatrix} 1 & 0 & 0 & 2 & 4 & 6 & 8 \\ 0 & 1 & 0 & 5 & 12 & 13 & 9 \\ 0 & 0 & 1 & -1 & 31 & 5 & 23 \\ 0 & 0 & 0 & 4 & 2 & 7 & 1 \\ 0 & 0 & 0 & -2 & 1 & 3 & -2 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 2 & 5 & 3 \end{bmatrix}=\mathbf{X}$.

Expanding along the penultimate row of **X** gives

$$\det\left(\mathbf{X}\right)=\left(-1\right)^{6+5}\det\begin{bmatrix} 1 & 0 & 0 & 2 & 6 & 8 \\ 0 & 1 & 0 & 5 & 13 & 9 \\ 0 & 0 & 1 & -1 & 5 & 23 \\ 0 & 0 & 0 & 4 & 7 & 1 \\ 0 & 0 & 0 & -2 & 3 & -2 \\ 0 & 0 & 0 & -1 & 5 & 3 \end{bmatrix}=-\det\left(\mathbf{Y}\right)$$ where **Y** is the 6 by 6 matrix

We using row operations we try to convert this matrix into an upper triangular matrix. Labelling the rows we have

$$\begin{array}{c} \mathrm{R}_1 \\ \mathrm{R}_2 \\ \mathrm{R}_3 \\ \mathrm{R}_4 \\ \mathrm{R}_5 \\ \mathrm{R}_6 \end{array}\begin{bmatrix} 1 & 0 & 0 & 2 & 6 & 8 \\ 0 & 1 & 0 & 5 & 13 & 9 \\ 0 & 0 & 1 & -1 & 5 & 23 \\ 0 & 0 & 0 & 4 & 7 & 1 \\ 0 & 0 & 0 & -2 & 3 & -2 \\ 0 & 0 & 0 & -1 & 5 & 3 \end{bmatrix}$$

Executing the row operations $2\mathrm{R}_5+\mathrm{R}_4$ and $4\mathrm{R}_6+\mathrm{R}_4$ gives

$$\begin{array}{c} \mathrm{R}_1 \\ \mathrm{R}_2 \\ \mathrm{R}_3 \\ \mathrm{R}_4 \\ \mathrm{R}_5{*}=2\mathrm{R}_5+\mathrm{R}_4 \\ \mathrm{R}_6{*}=4\mathrm{R}_6+\mathrm{R}_4 \end{array}\begin{bmatrix} 1 & 0 & 0 & 2 & 6 & 8 \\ 0 & 1 & 0 & 5 & 13 & 9 \\ 0 & 0 & 1 & -1 & 5 & 23 \\ 0 & 0 & 0 & 4 & 7 & 1 \\ 0 & 0 & 0 & 0 & 13 & -3 \\ 0 & 0 & 0 & 0 & 27 & 13 \end{bmatrix}$$

Executing the row operation $13\mathrm{R}_6{*}-27\mathrm{R}_5{*}$:

$$\begin{array}{r} R_1 \\ R_2 \\ R_3 \\ R_4 \\ R_5* \\ R_6^{\dagger}=13R_6*-27R_5* \end{array}\begin{bmatrix} 1 & 0 & 0 & 2 & 6 & 8 \\ 0 & 1 & 0 & 5 & 13 & 9 \\ 0 & 0 & 1 & -1 & 5 & 23 \\ 0 & 0 & 0 & 4 & 7 & 1 \\ 0 & 0 & 0 & 0 & 13 & -3 \\ 0 & 0 & 0 & 0 & 0 & 250 \end{bmatrix}=\mathbf{Z}$$

Now we have an upper triangular matrix so the determinant is given by multiplying the entries on the leading diagonal:

$$\det(\mathbf{Z})=1\times1\times1\times4\times13\times250=13000$$

We use formula (6.8) to evaluate the determinant of the given matrix:

$$(6.8)\qquad \det(\mathbf{B})=\begin{cases}\text{(i)}\quad \det(\mathbf{A}) & \text{if a multiple of one row is added to another}\\ \text{(ii)}\quad -\det(\mathbf{A}) & \text{if two rows have been interchanged}\\ \text{(iii)}\quad k\det(\mathbf{A}) & \text{if a row has been multiplied by non-zero } k\end{cases}$$

We have multiplied R_5 by 2, R_6 by 4 and R_6* by 13. Hence we divide

$$\det(\mathbf{Z})=13000 \text{ by } 2\times4\times13 \text{ to find } \det(\mathbf{Y})$$

Hence $\det(\mathbf{Y})=\dfrac{13000}{2\times4\times13}=125$. Therefore

$$\det(\mathbf{X})=-\det(\mathbf{Y})=-125$$

34. The matrix is invertible if and only if the determinant of the matrix does **not equal zero**. We evaluate the determinant of the given matrix in terms of *k* and then find the values of *k* where the determinant $\neq 0$.

Labelling the rows of the given matrix we have

$$\begin{array}{r} R_1 \\ R_2 \\ R_3 \\ R_4 \\ R_5 \end{array}\begin{pmatrix} 1 & 1 & 0 & 0 & 1 \\ -1 & k & 0 & 0 & 0 \\ 0 & 0 & 1 & 3 & 9 \\ 0 & 0 & 1 & 4 & 16 \\ 0 & 0 & 1 & k & k^2 \end{pmatrix}$$

Carrying out the row operations R_1+R_2, R_4-R_3 and R_5-R_3 gives

$$\begin{array}{r} R_1*=R_1+R_2 \\ R_2 \\ R_3 \\ R_4*=R_4-R_3 \\ R_5*=R_5-R_3 \end{array}\begin{pmatrix} 0 & 1+k & 0 & 0 & 1 \\ -1 & k & 0 & 0 & 0 \\ 0 & 0 & 1 & 3 & 9 \\ 0 & 0 & 0 & 1 & 7 \\ 0 & 0 & 0 & k-3 & k^2-9 \end{pmatrix}$$

We find the determinant of this matrix by expanding along the bottom row.

$$\det\begin{pmatrix} 0 & 1+k & 0 & 0 & 1 \\ -1 & k & 0 & 0 & 0 \\ 0 & 0 & 1 & 3 & 9 \\ 0 & 0 & 0 & 1 & 7 \\ 0 & 0 & 0 & k-3 & k^2-9 \end{pmatrix} = -(k-3)\det\begin{pmatrix} 0 & k+1 & 0 & 1 \\ -1 & k & 0 & 0 \\ 0 & 0 & 1 & 9 \\ 0 & 0 & 0 & 7 \end{pmatrix}$$

$$+(k^2-9)\det\begin{pmatrix} 0 & k+1 & 0 & 0 \\ -1 & k & 0 & 0 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$= -7(k-3)\det\begin{pmatrix} 0 & k+1 & 0 \\ -1 & k & 0 \\ 0 & 0 & 1 \end{pmatrix} + (k^2-9)\det\begin{pmatrix} 0 & k+1 & 0 \\ -1 & k & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \left[-7(k-3)+(k^2-9)\right]\det\begin{pmatrix} 0 & k+1 & 0 \\ -1 & k & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= (k-3)\left[-7+(k+3)\right](k+1)$$

$$= (k-3)(k-4)(k+1)$$

We conclude that $(k-3)(k-4)(k+1) \neq 0$ implies that $k \neq 3,\ k \neq 4$ or $k \neq -1$. Thus the given matrix is invertible as long as $k \neq 3,\ k \neq 4$ or $k \neq -1$.

35. We know from that the inverse of a given matrix **B** is given by the formula:

$$\mathbf{B}^{-1} = \frac{1}{\det(\mathbf{B})} adj(\mathbf{B}) = \frac{1}{\det(\mathbf{B})}\mathbf{C} \qquad (\dagger)$$

because we are given that $adj(\mathbf{B}) = \mathbf{C}$. We also know from the theory of determinants that

$$\det\left(2\mathbf{B}^{-1}\left(\mathbf{C}^T\right)^{-2}\right) = 2^4\det\left(\mathbf{B}^{-1}\right)\det\left(\left(\mathbf{C}^T\right)^{-2}\right) \qquad (*)$$

What does $\left(\mathbf{C}^T\right)^{-2}$ *mean?*

$$\left(\mathbf{C}^T\right)^{-2} = \left(\mathbf{C}^T\right)^{-1}\left(\mathbf{C}^T\right)^{-1}$$

By Proposition (6-5) we have $\det\left(\mathbf{C}^T\right) = \det(\mathbf{C})$. We have

$$\det\left(\left(\mathbf{C}^T\right)^{-2}\right) = \det\left[\left(\mathbf{C}^T\right)^{-1}\left(\mathbf{C}^T\right)^{-1}\right]$$

$$= \det\left[\left(\mathbf{C}^T\right)^{-1}\right]\det\left[\left(\mathbf{C}^T\right)^{-1}\right]$$

$$= \det\left[\left(\mathbf{C}^{-1}\right)^T\right]\det\left[\left(\mathbf{C}^{-1}\right)^T\right]$$

$$= \left(\det\left[\left(\mathbf{C}^{-1}\right)\right]\right)^2 = \left[\frac{1}{\det(C)}\right]^2$$

We are given that $\mathbf{C}=\begin{pmatrix} 6 & 3 & 9 & 0 \\ -6 & -3 & 6 & 0 \\ -3 & 6 & 3 & 0 \\ 2 & 1 & 3 & -5 \end{pmatrix}$ by using row operations we have

$$\det(\mathbf{C})=15^3$$

Therefore from above $\det\left(\left(\mathbf{C}^T\right)^{-2}\right)=\left[\frac{1}{\det(C)}\right]^2=\left[\frac{1}{15^3}\right]^2=\frac{1}{15^6}$.

From (†) we have

$$\det(\mathbf{C})=\det\left[\det(\mathbf{B})\mathbf{B}^{-1}\right]$$

$$=\left[\det(\mathbf{B})\right]^4\det\left(\mathbf{B}^{-1}\right)=\left[\det(\mathbf{B})\right]^4\frac{1}{\det(\mathbf{B})}=\left[\det(\mathbf{B})\right]^3$$

Hence $\det(\mathbf{B})=\left[\det(\mathbf{C})\right]^{1/3}=\left[15^3\right]^{1/3}=15$ so $\det\left(\mathbf{B}^{-1}\right)=\frac{1}{15}$.

Substituting $\det\left(\mathbf{B}^{-1}\right)=\frac{1}{15}$ and $\det\left(\left(\mathbf{C}^T\right)^{-2}\right)=\frac{1}{15^6}$ into (*) we have

$$\det\left(2\mathbf{B}^{-1}\left(\mathbf{C}^T\right)^{-2}\right)=2^4\det\left(\mathbf{B}^{-1}\right)\det\left(\left(\mathbf{C}^T\right)^{-2}\right)$$

$$=\frac{2^4}{15\times 15^6}=\frac{2^4}{15^7}$$

36. Since the given matrix $\mathbf{A}=\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}$ is *not* a square matrix so we cannot find the determinant of this matrix.

37. We apply Proposition (6-16):

$$\boxed{\det\left(\mathbf{A}_1\mathbf{A}_2\cdots\mathbf{A}_n\right)=\det\left(\mathbf{A}_1\right)\times\det\left(\mathbf{A}_2\right)\times\cdots\times\det\left(\mathbf{A}_n\right)}$$

To show that $\det\left(\mathbf{P}^{-1}\mathbf{AP}\right)=\det(\mathbf{A})$.

Proof. Since **P** is invertible so

$$\det\left(\mathbf{P}^{-1}\mathbf{AP}\right)=\det(\mathbf{P})\det(\mathbf{A})\det\left(\mathbf{P}^{-1}\right)$$

$$=\det(\mathbf{P})\det(\mathbf{A})\frac{1}{\det(\mathbf{P})}\qquad\left[\text{By (6-17)}\quad\det\left(\mathbf{P}^{-1}\right)=\frac{1}{\det(\mathbf{P})}\right]$$

$$=\det(\mathbf{A})$$

■

38. We need to show that the determinant of the given matrix

$$\mathbf{A}=\begin{pmatrix} 0 & 0 & \cdots & a_{1n} \\ 0 & \cdots & a_{2(n-1)} & a_{2n} \\ \vdots & ⋰ & ⋰ & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}$$

is $\det(\mathbf{A}) = (-1)^{\lfloor n/2 \rfloor} a_{1n} \cdots a_{2(n-1)} a_{n1}$.

Proof.

We can convert the matrix **A** into an upper triangular matrix by using row operations. Labelling the rows:

$$\begin{array}{c} R_1 \\ \vdots \\ \vdots \\ R_n \end{array} \begin{pmatrix} 0 & 0 & \cdots & a_{1n} \\ 0 & \cdots & a_{2(n-1)} & a_{2n} \\ \vdots & \iddots & \iddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}$$

Swapping the first and last rows gives

$$\begin{array}{c} R_n \\ \vdots \\ \vdots \\ R_1 \end{array} \begin{pmatrix} a_{n1} & a_{n2} & \cdots & a_{nn} \\ 0 & \cdots & a_{2(n-1)} & a_{2n} \\ \vdots & \iddots & \iddots & \vdots \\ 0 & 0 & \cdots & a_{1n} \end{pmatrix}$$

The determinant of this last matrix is $-\det(\mathbf{A})$ because

(6.8) (b)	$\det(\mathbf{B}) = -\det(\mathbf{A})$	if two rows have been interchanged

Repeating this process $\lfloor n/2 \rfloor$ times gives an upper triangular matrix:

$$\begin{pmatrix} a_{n1} & a_{n2} & \cdots & a_{nn} \\ 0 & a_{(n-1)2} & \cdots & a_{(n-1)n} \\ \vdots & \iddots & \iddots & \vdots \\ 0 & \cdots & a_{2(n-1)} & a_{2n} \\ 0 & 0 & \cdots & a_{1n} \end{pmatrix} = \mathbf{U}$$

The determinant of matrix **U** is the product of the leading diagonal:

$$\det(\mathbf{U}) = a_{n1} \times a_{(n-1)2} \times \cdots \times a_{2(n-1)} \times a_{1n}$$

Since the number of swaps has been $\lfloor n/2 \rfloor$ so the determinant of the given matrix

$$\det(\mathbf{A}) = (-1)^{\lfloor n/2 \rfloor} a_{n1} \times a_{(n-1)2} \times \cdots \times a_{2(n-1)} \times a_{1n}$$

This is our required result.

■