Complete Solutions to Exercises 6.3

1. All the matrices in this question are elementary matrices so we can use

$$\text{(6.7)} \qquad \det(\mathbf{E}) = \begin{cases} 1 & \text{if a multiple of one row is added to another} \\ -1 & \text{if two rows have been interchanged} \\ k & \text{if a row has been multiplied by non-zero } k \end{cases}$$

(a) The matrix **A** is obtained from the identity matrix by multiplying the second row by -10, therefore by the last line of (6.7) we have $\det(\mathbf{A}) = -10$.

(b) The matrix **B** is obtained from the identity matrix **I** by interchanging second and last rows, therefore by the second line of (6.7) we have $\det(\mathbf{B}) = -1$.

(c) *How is matrix **C** obtained from the identity matrix **I**?*
The matrix **C** is obtained from the identity matrix by adding the bottom row to the second row, therefore by the top line of (6.7) we have $\det(\mathbf{C}) = 1$.

(d) *How is matrix **D** obtained from the identity matrix **I**?*
The matrix **D** is obtained from the identity matrix by interchanging the top and bottom rows, therefore by the second line of (6.7) we have $\det(\mathbf{D}) = -1$.

(e) *How is matrix **E** obtained from the identity matrix **I**?*
The matrix **E** is obtained from the identity matrix by multiplying the third row by -0.6, therefore by the bottom line of (6.7) we have $\det(\mathbf{E}) = -0.6$.

(f) *How is matrix **F** obtained from the identity matrix **I**?*
The matrix **F** is obtained from the identity matrix by adding -7 times the third row to the second row, therefore by the top line of (6.7) we have $\det(\mathbf{F}) = 1$.

2. Apart from matrix **G** all the others are triangular or diagonal matrices. H*ow do we find the determinant of these matrices?*
By Proposition $(6\text{-}8)$ we have that the determinant of a triangular or diagonal matrix is a product of the entries along the leading diagonal.

(a) Matrix **A** is a diagonal matrix therefore by $(6\text{-}8)$ we have $\det(\mathbf{A}) = 1\times 2\times 3 = 6$.

(b) Matrix **B** is an upper triangular matrix therefore by $(6\text{-}8)$ we have

$$\det(\mathbf{B}) = 1\times 2\times 3 = 6$$

(c) Matrix **C** is a lower triangular matrix therefore by $(6\text{-}8)$ we have

$$\det(\mathbf{C}) = -2\times(-3)\times 1 = 6$$

(d) Matrix **D** is an upper triangular matrix therefore by $(6\text{-}8)$ we have

$$\det(\mathbf{D}) = 1\times(-3)\times 8\times 3 = -72$$

(e) Matrix **E** is a diagonal matrix therefore by $(6\text{-}8)$ we have

$$\det(\mathbf{E}) = 10\times 20\times 30\times 40 = 240\,000$$

(f) Matrix **F** is a lower triangular matrix therefore by $(6\text{-}8)$ we have

$$\det(\mathbf{F}) = -9\times 3\times 7\times 5 = -945$$

(g) *What do you notice about matrix **G**?*
It is **not** a square matrix therefore we cannot find it's determinant. The determinant of matrix **G** does **not** exist.

3. (a) Matrix **A** is an upper triangular matrix, therefore the determinant is the product of the entries along the leading diagonal. Hence

$$\det(\mathbf{A}) = \sim\ ¥$$

(b) Matrix **B** is a lower triangular matrix and the determinant of this matrix is the product of its entries along the leading diagonal:

$$\det(\mathbf{B}) = 2\sin(¯)\cos(¯) = \sin(2¯)$$

(c) Matrix **C** is an upper triangular matrix, therefore the determinant is the product of the entries along the leading diagonal. Hence

$$\det(\mathbf{C}) = xyz$$

4. Expanding the given determinant:

$$\det\begin{pmatrix} 1 & 2 & x \\ 3 & x & 4 \\ 5 & 5 & x \end{pmatrix} = 1\det\begin{pmatrix} x & 4 \\ 5 & x \end{pmatrix} - 2\det\begin{pmatrix} 3 & 4 \\ 5 & x \end{pmatrix} + x\det\begin{pmatrix} 3 & x \\ 5 & 5 \end{pmatrix}$$

$$= \left(x^2 - 20\right) - 2(3x - 20) + x(15 - 5x)$$

$$= x^2 - 20 - 6x + 40 + 15x - 5x^2$$

$$= -4x^2 + 9x + 20$$

The matrix is non-invertible when the determinant is zero, therefore we need to solve the quadratic equation:

$$-4x^2 + 9x + 20 = 0$$

We use the quadratic formula with $a = -4,\ b = 9$ and $c = 20$:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{-9 \pm \sqrt{9^2 - 4(-4)20}}{2 \times (-4)}$$

$$= \frac{-9 \pm \sqrt{81 + 320}}{-8}$$

$$= \frac{-9 \pm 20.025}{-8} = \frac{-29.025}{-8},\ \frac{11.025}{-8} = 3.63,\ -1.38$$

Therefore the x values which make the determinant non-invertible (singular) is -1.38 (2 dp) and 3.63 (2 dp).

5. (a) *How do we find the determinant of the given matrix?*
By using row operations. First we label the rows:

$$\begin{array}{c} R_1 \\ R_2 \\ R_3 \\ R_4 \end{array}\begin{pmatrix} 0 & 0 & 0 & 9 \\ 0 & 1 & 4 & 5 \\ 0 & 0 & 3 & 5 \\ 1 & 2 & 3 & 4 \end{pmatrix}$$

Interchanging the top and bottom rows gives

$$\begin{array}{c} R_4 \\ R_2 \\ R_3 \\ R_1 \end{array}\begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 4 & 5 \\ 0 & 0 & 3 & 5 \\ 0 & 0 & 0 & 9 \end{pmatrix}$$

What is the determinant of this last matrix?
It is a triangular matrix so the determinant is the product of the entries on the leading diagonal, that is $1\times 1\times 3\times 9=27$.
*What is the determinant of the given matrix **A** equal to?*
The only row operation carried out was interchanging of rows. *What affect does this have on the determinant?*
Multiplies by -1. Thus we have

$$-1\times\det\left(\mathbf{A}\right)=27 \text{ which gives } \det\left(\mathbf{A}\right)=-27$$

(b) We label the rows of the given matrix

$$\begin{array}{c} R_1 \\ R_2 \\ R_3 \\ R_4 \end{array}\begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 3 & 5 & 6 \\ 1 & 4 & 3 & 7 \\ 1 & 6 & 1 & 9 \end{pmatrix}$$

Carrying out the row operations R_2-R_1, R_3-R_1 and R_4-R_1:

$$\begin{array}{r} R_1 \\ R_2{*}=R_2-R_1 \\ R_3{*}=R_3-R_1 \\ R_4{*}=R_4-R_1 \end{array}\begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 2 \\ 0 & 2 & 0 & 3 \\ 0 & 4 & -2 & 5 \end{pmatrix}$$

Executing the row operation $R_4{*}+R_2{*}$:

$$\begin{array}{r} R_1 \\ R_2{*} \\ R_3{*} \\ R_4{**}=R_4{*}+R_2{*} \end{array}\begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 2 \\ 0 & 2 & 0 & 3 \\ 0 & 5 & 0 & 7 \end{pmatrix}$$

Interchanging columns 2 and 3 we have

$$\begin{array}{c} \\ R_1 \\ R_2{*} \\ R_3{*} \\ R_4{**} \end{array}\begin{array}{c} \begin{array}{cccc} C_1 & C_3 & C_2 & C_4 \end{array} \\ \begin{pmatrix} 1 & 3 & 2 & 4 \\ 0 & 2 & 1 & 2 \\ 0 & 0 & 2 & 3 \\ 0 & 0 & 5 & 7 \end{pmatrix} \end{array}$$

Multiply the third row by 5, that is $5R_3{*}$ gives

$$\begin{array}{r} R_1 \\ R_2{*} \\ R_3{**}=5R_3{*} \\ R_4{**} \end{array}\begin{pmatrix} 1 & 3 & 2 & 4 \\ 0 & 2 & 1 & 2 \\ 0 & 0 & 10 & 15 \\ 0 & 0 & 5 & 7 \end{pmatrix}$$

Carrying out the row operation $2R_4{**}-R_3{**}$:

$$\begin{array}{r} R_1 \\ R_2{*} \\ R_3{**} \\ 2R_4{**}-R_3{**} \end{array}\begin{pmatrix} 1 & 3 & 2 & 4 \\ 0 & 2 & 1 & 2 \\ 0 & 0 & 10 & 15 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

What is the determinant of this last matrix?

Since we have an upper triangular matrix therefore the determinant is the product of the entries on the leading diagonal, that is $1\times 2\times 10\times(-1)=-20$. *What is the determinant of the given matrix* ***A****?*
Remembering adding or subtracting a multiple of one row to another does **not** change the determinant. However the row operations $5R_3*$ and $2R_4**-R_3**$ multiplies the determinant by 5 and 2 respectively. *Are there any other operations in the above which affect the determinant?*
Yes interchanging columns 2 and 3 multiplies the determinant by -1. Thus we have

$$(-1)\times 5\times 2\times\det(\mathbf{A})=-20 \text{ which gives } \det(\mathbf{A})=2$$

(c) Labelling the rows of the given matrix we have

$$\begin{array}{c} R_1 \\ R_2 \\ R_3 \\ R_4 \end{array}\begin{pmatrix} 1 & 2 & 5 & 7 \\ 3 & 6 & 2 & 8 \\ -1 & -2 & 8 & 7 \\ -4 & -5 & 1 & 2 \end{pmatrix}$$

Carrying out the row operations R_2-3R_1, R_3+R_1 and R_4+4R_1:

$$\begin{array}{r} R_1 \\ R_2*=R_2-3R_1 \\ R_3*=R_3+R_1 \\ R_4*=R_4+4R_1 \end{array}\begin{pmatrix} 1 & 2 & 5 & 7 \\ 0 & 0 & -13 & -13 \\ 0 & 0 & 13 & 14 \\ 0 & 3 & 21 & 30 \end{pmatrix}$$

Carrying out the row operation R_3*+R_2* gives

$$\begin{array}{r} R_1 \\ R_2* \\ R_3**=R_3*+R_2* \\ R_4* \end{array}\begin{pmatrix} 1 & 2 & 5 & 7 \\ 0 & 0 & -13 & -13 \\ 0 & 0 & 0 & 1 \\ 0 & 3 & 21 & 30 \end{pmatrix}$$

Swapping over rows R_2* and R_4*:

$$\begin{array}{r} R_1 \\ R_4* \\ R_3** \\ R_2* \end{array}\begin{pmatrix} 1 & 2 & 5 & 7 \\ 0 & 3 & 21 & 30 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -13 & -13 \end{pmatrix}$$

Swapping rows R_2* and R_3**:

$$\begin{array}{r} R_1 \\ R_4* \\ R_2* \\ R_3** \end{array}\begin{pmatrix} 1 & 2 & 5 & 7 \\ 0 & 3 & 21 & 30 \\ 0 & 0 & -13 & -13 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

What is the determinant of this matrix?
Since we have an upper triangular matrix therefore the determinant is the product of the entries on the leading diagonal, that is $1\times 3\times(-13)\times 1=-39$. *What is the determinant of the given matrix* ***A****?*
Examining the row operations we know that adding a multiple of one row to another does **not** change the determinant. However we have a couple of row interchanges, R_2* and R_4*, R_2* and R_3** which means each time we multiply by -1. Thus

$$-1\times(-1)\det(\mathbf{A})=-39 \text{ gives } \det(\mathbf{A})=-39$$

6. (a) Labelling the rows we have

$$\begin{array}{c} R_1 \\ R_2 \\ R_3 \\ R_4 \\ R_5 \end{array} \begin{pmatrix} 1 & 1 & 0 & 0 & 1 \\ -1 & 2 & 0 & 0 & 0 \\ 0 & 0 & 1 & 4 & 16 \\ 0 & 0 & 1 & 5 & 25 \\ 0 & 0 & 1 & 2 & 4 \end{pmatrix} = \mathbf{A}$$

Carrying out the row operations R_2+R_1, R_4-R_3 and R_5-R_3 gives

$$\begin{array}{c} R_1 \\ R_2{*}=R_2+R_1 \\ R_3 \\ R_4{*}=R_4-R_3 \\ R_5{*}=R_5-R_3 \end{array} \begin{pmatrix} 1 & 1 & 0 & 0 & 1 \\ 0 & 3 & 0 & 0 & 1 \\ 0 & 0 & 1 & 4 & 16 \\ 0 & 0 & 0 & 1 & 9 \\ 0 & 0 & 0 & -2 & -12 \end{pmatrix}$$

Carrying out the row operation $R_5{*}+2R_4{*}$:

$$\begin{array}{c} R_1 \\ R_2{*} \\ R_3 \\ R_4{*} \\ R_5{**}=R_5{*}+2R_4{*} \end{array} \begin{pmatrix} 1 & 1 & 0 & 0 & 1 \\ 0 & 3 & 0 & 0 & 1 \\ 0 & 0 & 1 & 4 & 16 \\ 0 & 0 & 0 & 1 & 9 \\ 0 & 0 & 0 & 0 & 6 \end{pmatrix}$$

The determinant of this last matrix is given by the product of the entries on the leading diagonal, that is $1\times3\times1\times1\times6=18$. *What is the determinant of the given matrix* $\boldsymbol{A}$*?* The only row operations carried out have been adding a multiple of one row to another so the determinant of the given matrix does **not** change. Thus $\det(\mathbf{A})=18$.

(b) Labelling the rows we have

$$\begin{array}{c} R_1 \\ R_2 \\ R_3 \\ R_4 \\ R_5 \end{array} \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 5 & 8 & 11 & 1 \\ 7 & 6 & 1 & 9 & 8 \\ 4 & 10 & 16 & 22 & 2 \\ 2 & 3 & 7 & 9 & 5 \end{pmatrix} = \mathbf{A}$$

What do you notice about this matrix?

The fourth row, R_4, is 2 times the second row R_2. Executing the row operation $2R_2$ gives

$$\begin{array}{c} R_1 \\ 2R_2 \\ R_3 \\ R_4 \\ R_5 \end{array} \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 4 & 10 & 16 & 22 & 2 \\ 7 & 6 & 1 & 9 & 8 \\ 4 & 10 & 16 & 22 & 2 \\ 2 & 3 & 7 & 9 & 5 \end{pmatrix}$$

What is the determinant of this bottom matrix?

Since we have two identical rows R_4 and $2R_2$ therefore by Proposition (6-1) we have the determinant is 0. *What is the determinant of the given matrix* $\mathbf{A}$*?*
This is also zero, that is $\det(\mathbf{A})=0$.

7. (a)We need a counter example, that is we need one example where

$$\det(\mathbf{A}+\mathbf{B})\neq\det(\mathbf{A})+\det(\mathbf{B}) \qquad [\text{Not Equal}]$$

Let $\mathbf{A}=\begin{pmatrix}1&2\\3&4\end{pmatrix}$ and $\mathbf{B}=\begin{pmatrix}-1&1\\2&2\end{pmatrix}$ then

$$\mathbf{A}+\mathbf{B}=\begin{pmatrix}1-1&2+1\\3+2&4+2\end{pmatrix}=\begin{pmatrix}0&3\\5&6\end{pmatrix}$$

The determinant of this matrix is given by

$$\begin{aligned}\det(\mathbf{A}+\mathbf{B})&=\det\begin{pmatrix}0&3\\5&6\end{pmatrix}\\&=0-15=-15\end{aligned}$$

The determinants of matrices $\mathbf{A}$ and $\mathbf{B}$ is

$$\det(\mathbf{A})=\begin{pmatrix}1&2\\3&4\end{pmatrix}=4-6=-2 \text{ and } \det(\mathbf{B})=\begin{pmatrix}-1&1\\2&2\end{pmatrix}=-2-2=-4$$

We have

$$\begin{aligned}\det(\mathbf{A})+\det(\mathbf{B})&=-2-4\\&=-6\neq-15=\det(\mathbf{A}+\mathbf{B})\end{aligned}$$

(b) Let $\mathbf{u}=\begin{pmatrix}1&0&0\end{pmatrix}^T$, $\mathbf{v}=\begin{pmatrix}0&1&0\end{pmatrix}^T$ and $\mathbf{w}=\begin{pmatrix}0&0&1\end{pmatrix}^T$. Then

$$\det\begin{pmatrix}\mathbf{u}&\mathbf{v}&\mathbf{w}\end{pmatrix}=\det\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}=\det(\mathbf{I})=1$$

$$\det\begin{pmatrix}\mathbf{v}&\mathbf{u}&\mathbf{w}\end{pmatrix}=\det\begin{pmatrix}0&1&0\\1&0&0\\0&0&1\end{pmatrix}=-1$$

Hence $\det\begin{pmatrix}\mathbf{u}&\mathbf{v}&\mathbf{w}\end{pmatrix}\neq\det\begin{pmatrix}\mathbf{v}&\mathbf{u}&\mathbf{w}\end{pmatrix}$.

(c) We need to show that $\det(k\mathbf{A})\neq k\det(\mathbf{A})$. Let $\mathbf{A}=\begin{pmatrix}1&0\\0&1\end{pmatrix}=\mathbf{I}$ then

$$\det(2\mathbf{A})=\det(2\mathbf{I})=\det\begin{pmatrix}2&0\\0&2\end{pmatrix}=4$$

However $2\det(\mathbf{A})=2\det(\mathbf{I})=2\times1=2$. Hence $\det(k\mathbf{A})\neq k\det(\mathbf{A})$.

8. (a) We use

$$(6.8) \qquad \det(\mathbf{B})=\begin{cases}\det(\mathbf{A}) & \text{if a multiple of one row is added to another}\\-\det(\mathbf{A}) & \text{if two rows have been interchanged}\\k\det(\mathbf{A}) & \text{if a row has been multiplied by non-zero } k\end{cases}$$

to find the determinant of the matrix $\mathbf{A}$. *Why?*

Because we can write the first row of matrix **A** as $\frac{1}{2}$ times the entries 1, 1 and -1.

Therefore using the last line of result (6.8) we have

$$\det(\mathbf{A})=\frac{1}{2}\det\begin{pmatrix}1 & 1 & -1\\ 2 & 3 & 4\\ 1 & -1 & 1\end{pmatrix}$$

$$=\frac{1}{2}\left[\det\begin{pmatrix}3 & 4\\ -1 & 1\end{pmatrix}-\det\begin{pmatrix}2 & 4\\ 1 & 1\end{pmatrix}-\det\begin{pmatrix}2 & 3\\ 1 & -1\end{pmatrix}\right]$$

$$=\frac{1}{2}\left[(3+4)-(2-4)-(-2-3)\right]=7$$

(b) We multiply the first row by 6, the second row by 21 and the bottom row by 3 of matrix **B**. We have

$$(6\times 21\times 3)\det(\mathbf{B})=378\det(\mathbf{B})=\det\begin{pmatrix}3 & 2 & 6\\ 3 & 2 & 1\\ 2 & 1 & -4\end{pmatrix}\qquad (*)$$

How do we find the determinant of the matrix on the Right Hand Side?

$$\det\begin{pmatrix}3 & 2 & 6\\ 3 & 2 & 1\\ 2 & 1 & -4\end{pmatrix}=3\det\begin{pmatrix}2 & 1\\ 1 & -4\end{pmatrix}-2\det\begin{pmatrix}3 & 1\\ 2 & -4\end{pmatrix}+6\det\begin{pmatrix}3 & 2\\ 2 & 1\end{pmatrix}$$

$$=3(-8-1)-2(-12-2)+6(3-4)=-5$$

Substituting this into (*) gives

$$378\det(\mathbf{B})=-5$$

$$\det(\mathbf{B})=-\frac{5}{378}$$

(c) The top and bottom rows are multiples of 10 so therefore we have

$$\det(\mathbf{C})=(10\times 10)\det\begin{pmatrix}1 & 2 & -3\\ -4 & 5 & -6\\ -7 & 8 & -9\end{pmatrix}\qquad (*)$$

We need to find the determinant of the Right Hand Side matrix:

$$\det\begin{pmatrix}1 & 2 & -3\\ -4 & 5 & -6\\ -7 & 8 & -9\end{pmatrix}=\det\begin{pmatrix}5 & -6\\ 8 & -9\end{pmatrix}-2\det\begin{pmatrix}-4 & -6\\ -7 & -9\end{pmatrix}-3\det\begin{pmatrix}-4 & 5\\ -7 & 8\end{pmatrix}$$

$$=(-45+48)-2(36-42)-3(-32+35)$$

$$=3-2(-6)-3(3)=6$$

Substituting this into (*) gives

$$\det(\mathbf{C})=100\times 6=600$$

9. (a) We need to find $\det(-2\mathbf{AB})$ where $\det(\mathbf{A})=3$, $\det(\mathbf{B})=-4$:

$$\det(-2\mathbf{AB}) = \det((-2\mathbf{A})\mathbf{B})$$
$$= -2^3 \det(\mathbf{AB}) = -8\det(\mathbf{A})\det(\mathbf{B}) = -8\times 3\times(-4) = 96$$

(b) We can use the result of question 19 to find $\det\left(\mathbf{A}^5\mathbf{B}^6\right)$:

$$\det\left(\mathbf{A}^5\mathbf{B}^6\right) = \det\left(\mathbf{A}^5\right)\det\left(\mathbf{B}^6\right)$$
$$= \left[\det(\mathbf{A})\right]^5\left[\det(\mathbf{B})\right]^6 \qquad \left[\text{Because } \det\left(\mathbf{A}^n\right) = \left[\det(\mathbf{A})\right]^n\right]$$
$$= 3^5(-4)^6 = 995328$$

(c) We need to find $\det\left(\mathbf{A}^{-1}\mathbf{A}^T\right)$. Using the following properties of determinants:

> Proposition $(6\text{-}5)$. Let $\mathbf{A}$ be a square matrix then $\det\left(\mathbf{A}^T\right) = \det(\mathbf{A})$.

And

> Proposition (6-16). If $\mathbf{A}$ is an invertible (non-singular) matrix then
> $$\det\left(\mathbf{A}^{-1}\right) = \frac{1}{\det(\mathbf{A})}$$

We have

$$\det\left(\mathbf{A}^{-1}\mathbf{A}^T\right) = \det\left(\mathbf{A}^{-1}\right)\det\left(\mathbf{A}^T\right)$$
$$= \frac{1}{\det(\mathbf{A})}\det(\mathbf{A}) \qquad [\text{By (6-5) and (6-16)}]$$
$$= 1$$

10. We evaluate the determinant of each matrix first:

$$\det(\mathbf{A}) = \begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix} = 4 - 6 = -2$$
$$\det(\mathbf{B}) = \begin{vmatrix} 5 & 6 \\ 7 & 8 \end{vmatrix} = 40 - 42 = -2$$
$$\det(\mathbf{C}) = \det\begin{pmatrix} 9 & 10 \\ 11 & 12 \end{pmatrix} = 108 - 110 = -2$$

Since $\det(\mathbf{A})\times\det(\mathbf{B})\times\det(\mathbf{C}) = -2\times(-2)\times(-2) = -8 \neq 0$ so the matrix **ABC** is invertible.

11. (a) Swapping the first and last rows gives

$$\begin{pmatrix} 21 & 0 & 0 & 0 \\ 0 & 89 & 0 & 0 \\ 0 & 0 & 98 & 0 \\ 0 & 0 & 0 & 61 \end{pmatrix}$$

The determinant of this matrix is positive. Since we have swapped rows so the determinant of the given matrix is negative.

(b) Since the third row is 6 times the first row in

$$\begin{pmatrix} 2 & 5 & 7 & 8 \\ 7 & 8 & 3 & 1 \\ 12 & 30 & 42 & 48 \\ 10 & 51 & 41 & 44 \end{pmatrix}$$

So the determinant of this matrix is zero.
(c) Interchanging the top and bottom rows and the two middle rows of the given matrix yields:

$$\begin{array}{c} R_1 \\ R_2 \\ R_3 \\ R_4 \end{array} \begin{pmatrix} 23 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & -5 & 0 \\ 0 & 0 & 25 & 1 \end{pmatrix}$$

$R_4 + 5R_3$ makes no difference to the determinant because we are adding a multiple of one row to another. The determinant of this matrix is negative because of the -5 along the leading diagonal. Hence the determinant of the given matrix is negative.

12. We apply proof by mathematical induction.
Proof. We first prove it for $n = 2$, that is 2 by 2 lower triangular matrix:

$$\det\begin{pmatrix} a_{11} & 0 \\ a_{21} & a_{22} \end{pmatrix} = a_{11}a_{22} - 0 = a_{11}a_{22}$$

Hence the result is true for $n = 2$. Assume the result is true for $n = k$, that is a k by k lower triangular matrix:

$$\det\begin{pmatrix} a_{11} & 0 & \cdots & 0 \\ a_{21} & a_{22} & \cdots & \vdots \\ \vdots & \vdots & \ddots & 0 \\ a_{k1} & a_{k2} & \cdots & a_{kk} \end{pmatrix} = a_{11}a_{22}a_{33}\cdots a_{kk} \qquad (\dagger)$$

Required to prove this result for $n = k+1$. For $k+1$ by $k+1$ lower triangular matrix we expand along the last column to find the determinant:

$$\det\begin{pmatrix} a_{11} & 0 & \cdots & 0 \\ a_{21} & a_{22} & \cdots & \vdots \\ \vdots & \vdots & \ddots & 0 \\ a_{(k+1)1} & a_{(k+1)2} & \cdots & a_{(k+1)(k+1)} \end{pmatrix} = (-1)^{(k+1)+(k+1)} a_{(k+1)(k+1)} \underbrace{\det\begin{pmatrix} a_{11} & 0 & \cdots & 0 \\ a_{21} & a_{22} & \cdots & \vdots \\ \vdots & \vdots & \ddots & 0 \\ a_{k1} & a_{k2} & \cdots & a_{kk} \end{pmatrix}}_{=a_{11}a_{22}a_{33}\cdots a_{kk} \text{ by } (\dagger)}$$

$$= (-1)^{2(k+1)} a_{(k+1)(k+1)}a_{11}a_{22}a_{33}\cdots a_{kk}$$

$$= a_{11}a_{22}a_{33}\cdots a_{kk}a_{(k+1)(k+1)}$$

Hence by induction we have our result.

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13. (a)We first prove it by using induction.
Proof.
We first prove it for $n = 2$ that is 2 by 2 diagonal matrix:

$$\det\begin{pmatrix} a_{11} & 0 \\ 0 & a_{22} \end{pmatrix} = a_{11}a_{22} - 0 = a_{11}a_{22}$$

Hence the result is true for $n=2$. Assume the result is true for $n=k$, that is a *k* by *k* diagonal matrix:

$$\det\begin{pmatrix} a_{11} & 0 & \cdots & 0 \\ 0 & a_{22} & \cdots & \vdots \\ \vdots & \vdots & \ddots & 0 \\ 0 & 0 & \cdots & a_{kk} \end{pmatrix} = a_{11}a_{22}a_{33}\cdots a_{kk}$$

Required to prove this result for $n=k+1$. For $k+1$ by $k+1$ diagonal matrix we expand along the bottom row to find the determinant:

$$\det\begin{pmatrix} a_{11} & 0 & \cdots & 0 \\ 0 & a_{22} & \cdots & \vdots \\ \vdots & \vdots & \ddots & 0 \\ 0 & 0 & \cdots & a_{(k+1)(k+1)} \end{pmatrix} = (-1)^{(k+1)+(k+1)}\, a_{(k+1)(k+1)} \underbrace{\det\begin{pmatrix} a_{11} & 0 & \cdots & 0 \\ 0 & a_{22} & \cdots & \vdots \\ \vdots & \vdots & \ddots & 0 \\ 0 & 0 & \cdots & a_{kk} \end{pmatrix}}_{=a_{11}a_{22}a_{33}\cdots a_{kk}}$$

$$= (-1)^{2(k+1)}\, a_{(k+1)(k+1)} a_{11}a_{22}a_{33}\cdots a_{kk}$$

$$= a_{11}a_{22}a_{33}\cdots a_{kk}a_{(k+1)(k+1)}$$

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(b) We now prove it without using induction:

Proof.

Consider the general *n* by *n* diagonal matrix and each time we expand along the first row to find the determinant:

$$\det\begin{pmatrix} a_{11} & & \mathbf{0} \\ & \ddots & \\ \mathbf{0} & & a_{nn} \end{pmatrix} = a_{11}\det\begin{pmatrix} a_{22} & & \mathbf{0} \\ & \ddots & \\ \mathbf{0} & & a_{nn} \end{pmatrix}$$

$$= a_{11}a_{22}\det\begin{pmatrix} a_{33} & & \mathbf{0} \\ & \ddots & \\ \mathbf{0} & & a_{nn} \end{pmatrix}$$

$$= a_{11}a_{22}\cdots a_{(n-2)(n-2)}\det\begin{pmatrix} a_{(n-1)(n-1)} & 0 \\ 0 & a_{nn} \end{pmatrix}$$

$$= a_{11}a_{22}\cdots a_{(n-2)(n-2)}a_{(n-1)(n-1)}a_{nn}$$

This is our required result.

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14. We need to prove the following:

Proposition (6-10). Let **B** be a matrix obtained from the matrix **A** by

(b) adding (or subtracting) a multiple of one row to another. In this case $\det(\mathbf{B})=\det(\mathbf{A})$.

(c) interchanging two rows (or columns). In this case $\det(\mathbf{B})=-\det(\mathbf{A})$.

Proof of (b).

Let **E** be the elementary matrix obtained from the identity by adding (or subtracting) one row to another. By Result (6.7):

$$(6.7) \qquad \det(\mathbf{E}) = \begin{cases} 1 & \text{if a multiple of one row is added to another} \\ -1 & \text{if two rows have been interchanged} \\ k & \text{if a row has been multiplied by non-zero } k \end{cases}$$

We have $\det(\mathbf{E}) = 1$.

The matrix **B** in part (b) is equal to **EA** because matrix multiplication **EA** performs the same row operation of adding (or subtracting) one row to another. Therefore we have

$$\begin{aligned}\det(\mathbf{B}) &= \det(\mathbf{EA}) \\ &= \det(\mathbf{E})\det(\mathbf{A}) = 1\times\det(\mathbf{A}) = \det(\mathbf{A})\end{aligned}$$

■

Proof of (c). Let **E** be the elementary matrix obtained from the identity by interchanging rows (or columns) then by (6.7) we have $\det(\mathbf{E}) = -1$.

The matrix **B** in part (c) is equal to **EA** because matrix multiplication **EA** performs the same row operation. Therefore we have

$$\begin{aligned}\det(\mathbf{B}) &= \det(\mathbf{EA}) \\ &= \det(\mathbf{E})\det(\mathbf{A}) = -1\times\det(\mathbf{A}) = -\det(\mathbf{A})\end{aligned}$$

15. *Proof.*

Since **A** is an invertible (non-singular) matrix therefore $\mathbf{A}^{-1}$ is invertible (non-singular) and $\mathbf{AA}^{-1} = \mathbf{I}$ where **I** is the identity matrix. *What is the determinant of the identity matrix **I** equal to?*

$$\det(\mathbf{I}) = 1$$

Therefore we have

$$\begin{aligned}\det(\mathbf{AA}^{-1}) &= \det(\mathbf{A})\det(\mathbf{A}^{-1}) \qquad \left[\text{By Proposition (6-15)}\right] \\ &= \det(\mathbf{I}) = 1\end{aligned}$$

We have $\det(\mathbf{A})\det(\mathbf{A}^{-1}) = 1$. Since matrix **A** is invertible therefore $\det(\mathbf{A}) \neq 0$ so we can divide both sides of this by $\det(\mathbf{A})$:

$$\det(\mathbf{A})\det(\mathbf{A}^{-1}) = 1$$

$$\det(\mathbf{A}^{-1}) = \frac{1}{\det(\mathbf{A})} \qquad \left[\text{Dividing Both Sides by } \det(\mathbf{A})\right]$$

This is our required result.

■

16. *Proof.*

($\Rightarrow$). First we assume matrix **A** is invertible and from this we deduce that $\mathbf{A}^T\mathbf{A}$ is invertible.

Since matrix **A** is invertible therefore by Proposition (6-13) we have $\det(\mathbf{A}) \neq 0$:

$$\begin{aligned}\det(\mathbf{A}^T\mathbf{A}) &= \det(\mathbf{A}^T)\det(\mathbf{A}) \\ &= \det(\mathbf{A})\det(\mathbf{A}) \neq 0 \qquad \left[\text{By Proposition (6-5) } \det\left(\mathbf{A}^T\right) = \det(\mathbf{A})\right]\end{aligned}$$

Hence $\mathbf{A}^T\mathbf{A}$ is invertible.

$(\Leftarrow)$. Now we go the other way, that is we assume $\mathbf{A}^T\mathbf{A}$ is invertible and deduce that $\mathbf{A}$ is invertible.
Since $\mathbf{A}^T\mathbf{A}$ is invertible therefore $\det\left(\mathbf{A}^T\mathbf{A}\right)\neq 0$:

$$\det\left(\mathbf{A}^T\mathbf{A}\right)=\det\left(\mathbf{A}^T\right)\det\left(\mathbf{A}\right)\neq 0$$

Hence $\det(\mathbf{A})\neq 0$ which means that matrix **A** is invertible.
We have proven our result both ways which means that we have proven the given 'if and only if' statement.

■

17. We need to prove the following:
Let $\mathbf{E}_1$, $\mathbf{E}_2$, $\mathbf{E}_3, \cdots$ and $\mathbf{E}_n$ be elementary matrices and **B** be a square matrix of the same size. Then

$$\det\left(\mathbf{E}_1\mathbf{E}_2\mathbf{E}_3\cdots\mathbf{E}_n\mathbf{B}\right)=\det\left(\mathbf{E}_1\right)\det\left(\mathbf{E}_2\right)\det\left(\mathbf{E}_3\right)\cdots\det\left(\mathbf{E}_n\right)\det\left(\mathbf{B}\right)$$

Proof.
We use proof by induction. Clearly it is true for $n=1$, that is

$$\det\left(\mathbf{E}_1\mathbf{B}\right)=\det\left(\mathbf{E}_1\right)\det\left(\mathbf{B}\right)$$

This follows by Proposition (6-11) which is $\det(\mathbf{EB})=\det(\mathbf{E})\det(\mathbf{B})$. We assume it is true for $n=k$, that is

$$\det\left(\mathbf{E}_1\mathbf{E}_2\mathbf{E}_3\cdots\mathbf{E}_k\mathbf{B}\right)=\det\left(\mathbf{E}_1\right)\det\left(\mathbf{E}_2\right)\det\left(\mathbf{E}_3\right)\cdots\det\left(\mathbf{E}_k\right)\det\left(\mathbf{B}\right) \qquad (*)$$

Required to prove it for $n=k+1$, that is we need to prove

$$\det\left(\mathbf{E}_1\mathbf{E}_2\mathbf{E}_3\cdots\mathbf{E}_k\mathbf{E}_{k+1}\mathbf{B}\right)=\det\left(\mathbf{E}_1\right)\det\left(\mathbf{E}_2\right)\det\left(\mathbf{E}_3\right)\cdots\det\left(\mathbf{E}_k\right)\det\left(\mathbf{E}_{k+1}\right)\det\left(\mathbf{B}\right)$$

Examining the Left Hand Side of this we have

$$\begin{aligned}\det\left(\mathbf{E}_1\mathbf{E}_2\mathbf{E}_3\cdots\mathbf{E}_k\mathbf{E}_{k+1}\mathbf{B}\right)&=\det\left(\left(\mathbf{E}_1\mathbf{E}_2\mathbf{E}_3\cdots\mathbf{E}_k\mathbf{E}_{k+1}\right)\mathbf{B}\right)\\&=\det\left(\mathbf{E}_1\mathbf{E}_2\mathbf{E}_3\cdots\mathbf{E}_k\mathbf{E}_{k+1}\right)\det\left(\mathbf{B}\right)\\&=\underbrace{\det\left(\mathbf{E}_1\right)\det\left(\mathbf{E}_2\right)\det\left(\mathbf{E}_3\right)\cdots\det\left(\mathbf{E}_k\right)\det\left(\mathbf{E}_{k+1}\right)}_{\text{by } (*)}\det\left(\mathbf{B}\right)\end{aligned}$$

Thus by induction we have our required result.

■

18. We need to prove the following:
If $\mathbf{A}_1$, $\mathbf{A}_2$, $\mathbf{A}_3$, $\cdots$ and $\mathbf{A}_n$ are square matrices of the same size then

$$\det\left(\mathbf{A}_1\mathbf{A}_2\mathbf{A}_3\cdots\mathbf{A}_n\right)=\det\left(\mathbf{A}_1\right)\det\left(\mathbf{A}_2\right)\det\left(\mathbf{A}_3\right)\cdots\det\left(\mathbf{A}_n\right)$$

Proof.
Again we use mathematical induction. For $n=2$ we have

$$\det\left(\mathbf{A}_1\mathbf{A}_2\right)=\det\left(\mathbf{A}_1\right)\det\left(\mathbf{A}_2\right) \qquad \left[\text{By (6-14) } \det(\mathbf{AB})=\det(\mathbf{A})\det(\mathbf{B})\right]$$

Assume the result is true for $n=k$, that is we have

$$\det\left(\mathbf{A}_1\mathbf{A}_2\mathbf{A}_3\cdots\mathbf{A}_k\right)=\det\left(\mathbf{A}_1\right)\det\left(\mathbf{A}_2\right)\det\left(\mathbf{A}_3\right)\cdots\det\left(\mathbf{A}_k\right) \qquad (*)$$

We need to prove

$$\det\left(\mathbf{A}_1\mathbf{A}_2\mathbf{A}_3\cdots\mathbf{A}_k\mathbf{A}_{k+1}\right)=\det\left(\mathbf{A}_1\right)\det\left(\mathbf{A}_2\right)\det\left(\mathbf{A}_3\right)\cdots\det\left(\mathbf{A}_k\right)\det\left(\mathbf{A}_{k+1}\right)$$

Using the associative property of matrix multiplication we have

$$\begin{aligned}\det\left(\mathbf{A}_1\mathbf{A}_2\mathbf{A}_3\cdots\mathbf{A}_k\mathbf{A}_{k+1}\right) &= \det\left(\left(\mathbf{A}_1\mathbf{A}_2\mathbf{A}_3\cdots\mathbf{A}_k\right)\mathbf{A}_{k+1}\right)\\ &= \det\left(\mathbf{A}_1\mathbf{A}_2\mathbf{A}_3\cdots\mathbf{A}_k\right)\det\left(\mathbf{A}_{k+1}\right)\\ &= \underbrace{\det\left(\mathbf{A}_1\right)\det\left(\mathbf{A}_2\right)\det\left(\mathbf{A}_3\right)\cdots\det\left(\mathbf{A}_k\right)}_{\text{By }(*)}\det\left(\mathbf{A}_{k+1}\right)\end{aligned}$$

Hence by induction we have our result.

■

19. *Proof.*
We have

$$\begin{aligned}\det\left(\mathbf{A}^n\right) &= \det\left(\underbrace{\mathbf{A}\times\mathbf{A}\times\mathbf{A}\times\cdots\times\mathbf{A}}_{n\text{ copies}}\right)\\ &= \underbrace{\det(\mathbf{A})\det(\mathbf{A})\det(\mathbf{A})\cdots\det(\mathbf{A})}_{n\text{ copies}} \qquad \left[\text{By Proposition (6-15)}\right]\\ &= \left[\det(\mathbf{A})\right]^n\end{aligned}$$

■

20. *Proof.*
Using Proposition (6-15) which says $\det(\mathbf{AB}) = \det(\mathbf{A})\det(\mathbf{B})$ we have

$$\begin{aligned}\det(\mathbf{AB}) &= \det(\mathbf{A})\det(\mathbf{B})\\ &= \det(\mathbf{B})\det(\mathbf{A}) = \det(\mathbf{BA})\end{aligned}$$

■

21. *Proof.*
Since $\mathbf{A}^T\mathbf{A} = \mathbf{I}$ we have

$$\det\left(\mathbf{A}^T\mathbf{A}\right) = \det(\mathbf{I}) = 1 \qquad (\dagger)$$

(6-5)
By Proposition (6-5) we have $\det\left(\mathbf{A}^T\right) = \det(\mathbf{A})$ therefore by Proposition (6-15) which says $\det(\mathbf{AB}) = \det(\mathbf{A})\det(\mathbf{B})$ we deduce that

$$\begin{aligned}\det\left(\mathbf{A}^T\mathbf{A}\right) &= \det\left(\mathbf{A}^T\right)\det(\mathbf{A})\\ &= \det(\mathbf{A})\det(\mathbf{A})\\ &= \left[\det(\mathbf{A})\right]^2 = 1 \qquad \left[\text{By } (\dagger)\right]\end{aligned}$$

Taking the square root of both sides of this $\left[\det(\mathbf{A})\right]^2 = 1$ gives

$$\det(\mathbf{A}) = \sqrt{1} = \pm 1$$

■

22. *Proof.*
From Chapter 1 we have the following property of matrices:

$$\left(\mathbf{A}^T\right)^{-1} = \left(\mathbf{A}^{-1}\right)^T$$

We also know by Proposition (6-5) that $\det\left[\left(\mathbf{A}^{-1}\right)^T\right] = \det\left(\mathbf{A}^{-1}\right) = \dfrac{1}{\det(\mathbf{A})}$. Hence

$$\det\left[\left(\mathbf{A}^T\right)^{-1}\right] = \det\left[\left(\mathbf{A}^{-1}\right)^T\right] = \frac{1}{\det(\mathbf{A})}$$

■

23. *Proof*.
Since the determinant of matrix which has two identical rows is zero therefore that matrix which has a multiple of a row is also zero. *Why?*
We can take out the multiple and then the determinant of the remaining matrix is zero because we have two identical rows left in the matrix.

■

24. *Proof*.
We have

$$\begin{aligned}\det(\mathbf{B}) &= \det\left(\mathbf{P}^{-1}\mathbf{A}\mathbf{P}\right)\\ &= \det\left(\mathbf{P}^{-1}\right)\det(\mathbf{A}\mathbf{P})\\ &= \det\left(\mathbf{P}^{-1}\right)\det(\mathbf{P}\mathbf{A})\\ &= \det\left(\mathbf{P}^{-1}\right)\det(\mathbf{P})\det(\mathbf{A})\\ &= \det\left(\mathbf{P}^{-1}\mathbf{P}\right)\det(\mathbf{A})\\ &= \det(\mathbf{I})\det(\mathbf{A}) = 1\times\det(\mathbf{A}) = \det(\mathbf{A})\end{aligned}$$

Thus we have our result $\det(\mathbf{B}) = \det(\mathbf{A})$.

■

25. *Proof*.
Since matrix $\mathbf{A}$ is invertible (non-singular) therefore we have

$$\mathbf{A}^{-1} = \frac{1}{\det(\mathbf{A})}adj(\mathbf{A}) \qquad (*)$$

where $\det(\mathbf{A}) \neq 0$. By (6-16) we have

$$\det\left(\mathbf{A}^{-1}\right) = \frac{1}{\det(\mathbf{A})}$$

From (*) we have

$$\begin{aligned}\det\left(\mathbf{A}^{-1}\right) &= \det\left(\frac{1}{\det(\mathbf{A})}adj(\mathbf{A})\right)\\ &= \left(\frac{1}{\det(\mathbf{A})}\right)^n \det\left(adj(\mathbf{A})\right) \qquad \left[\begin{array}{l}\text{By (6.16) with } k = 1/\det(\mathbf{A})\\ \det(k\mathbf{A}) = k^n\det(\mathbf{A})\end{array}\right]\end{aligned}$$

Substituting $\det\left(\mathbf{A}^{-1}\right) = \dfrac{1}{\det(\mathbf{A})}$ into the above gives

$$\frac{1}{\left[\det(\mathbf{A})\right]^n}\det\left(adj(\mathbf{A})\right) = \frac{1}{\det(\mathbf{A})}$$

$$\det\left(adj(\mathbf{A})\right) = \left[\det(\mathbf{A})\right]^n\frac{1}{\det(\mathbf{A})} = \left[\det(\mathbf{A})\right]^{n-1}$$

■

26. *Proof.*
We use (6-14):

The linear system $\mathbf{Ax} = \mathbf{b}$ has a unique solution $\Leftrightarrow$ $\det(\mathbf{A}) \neq 0$.

Remember a linear system has *no* solutions, unique solutions or an infinite number of solutions.
The given linear system $\mathbf{Ax} = \mathbf{O}$ will have a solution because $\mathbf{x} = \mathbf{O}$ is a solution. By (6-14) we have $\mathbf{Ax} = \mathbf{O}$ has an infinite number of solutions $\Leftrightarrow$ $\det(\mathbf{A}) = 0$.

■