Complete Solutions to Exercises 4.3

We have used vectors $\mathbf{w}_1$ and $\mathbf{w}_2$ to represent the orthogonal vectors rather than $\mathbf{p}_1$ and $\mathbf{p}_2$ respectively.

1. (a) We are given the vectors $\mathbf{v}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\mathbf{v}_2 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$ which we need to transform to an orthonormal basis by using the Gram Schmidt Process (4-10):

We have $\mathbf{w}_1 = \mathbf{v}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and

$$\mathbf{w}_2 = \mathbf{v}_2 - \frac{\langle \mathbf{v}_2, \ \mathbf{w}_1 \rangle}{\|\mathbf{w}_1\|^2}\mathbf{w}_1 \qquad (*)$$

What is $\langle \mathbf{v}_2, \ \mathbf{w}_1 \rangle$ *equal to?*

$$\langle \mathbf{v}_2, \ \mathbf{w}_1 \rangle = \begin{pmatrix} 2 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} = (2\times 1) + (1\times 0) = 2$$

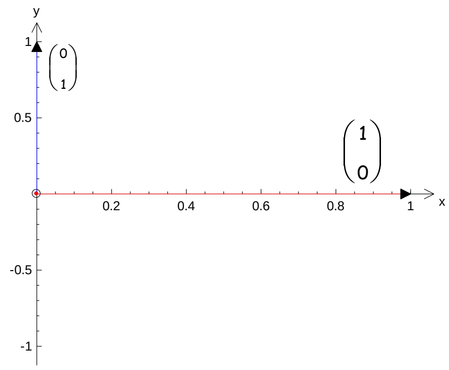
What is $\|\mathbf{w}_1\|^2$ *equal to?*

$$\|\mathbf{w}_1\|^2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} = 1^2 + 0^2 = 1$$

Substituting these, $\langle \mathbf{v}_2, \ \mathbf{w}_1 \rangle = 2$ and $\|\mathbf{w}_1\|^2 = 1$ into (*) gives

$$\begin{aligned} \mathbf{w}_2 &= \mathbf{v}_2 - \frac{2}{1}\mathbf{w}_1 = \mathbf{v}_2 - 2\mathbf{w}_1 \\ &= \begin{pmatrix} 2 \\ 1 \end{pmatrix} - 2\begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad \left[\text{Substituting } \mathbf{w}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \text{ and } \mathbf{v}_2 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}\right] \\ &= \begin{pmatrix} 2-2 \\ 1-0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \end{aligned}$$

Hence we have the vectors $\mathbf{w}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\mathbf{w}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ which are the standard orthonormal basis for $\mathbb{R}^2$ because $\mathbf{w}_1 = \mathbf{e}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\mathbf{w}_2 = \mathbf{e}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$:



(b) We are given the vectors $\mathbf{v}_1 = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$ and $\mathbf{v}_2 = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$ which we need to transform to an orthonormal basis by using the Gram Schmidt Process (4-10):

We have $\mathbf{w}_1 = \mathbf{v}_1 = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$ and

$$\mathbf{w}_2 = \mathbf{v}_2 - \frac{\langle \mathbf{v}_2,\ \mathbf{w}_1 \rangle}{\|\mathbf{w}_1\|^2}\mathbf{w}_1 \qquad (*)$$

What is $\langle \mathbf{v}_2,\ \mathbf{w}_1 \rangle$ *equal to?*

$$\langle \mathbf{v}_2,\ \mathbf{w}_1 \rangle = \begin{pmatrix} 2 \\ -1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 3 \end{pmatrix} = (2\times 1) + (-1\times 3) = -1$$

What is $\|\mathbf{w}_1\|^2$ *equal to?*

$$\|\mathbf{w}_1\|^2 = \begin{pmatrix} 1 \\ 3 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 3 \end{pmatrix} = 1^2 + 3^2 = 10$$

Substituting these, $\langle \mathbf{v}_2,\ \mathbf{w}_1 \rangle = -1$ and $\|\mathbf{w}_1\|^2 = 10$ into (*) gives

$$\begin{aligned}
\mathbf{w}_2 = \mathbf{v}_2 - \frac{-1}{10}\mathbf{w}_1 = \mathbf{v}_2 + \frac{1}{10}\mathbf{w}_1 \\
&= \begin{pmatrix} 2 \\ -1 \end{pmatrix} + \frac{1}{10}\begin{pmatrix} 1 \\ 3 \end{pmatrix} \qquad \left[\text{Substituting } \mathbf{w}_1 = \begin{pmatrix} 1 \\ 3 \end{pmatrix} \text{ and } \mathbf{v}_2 = \begin{pmatrix} 2 \\ -1 \end{pmatrix}\right] \\
&= \begin{pmatrix} 2 + 1/10 \\ -1 + 3/10 \end{pmatrix} \\
&= \begin{pmatrix} 21/10 \\ -7/10 \end{pmatrix} = \frac{1}{10}\begin{pmatrix} 21 \\ -7 \end{pmatrix} = \frac{7}{10}\begin{pmatrix} 3 \\ -1 \end{pmatrix}
\end{aligned}$$

Clearing the fraction gives us the new vector $\mathbf{w}'_2 = \begin{pmatrix} 3 \\ -1 \end{pmatrix}$.

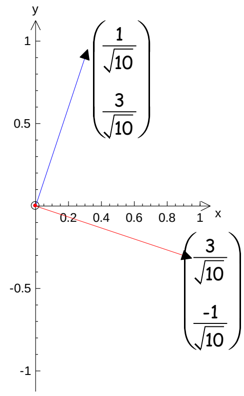
Normalizing the 2 orthogonal vectors $\mathbf{w}_1 = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$ and $\mathbf{w}'_2 = \begin{pmatrix} 3 \\ -1 \end{pmatrix}$ by dividing each vector by its norm, $\mathbf{u} = \dfrac{1}{\|\mathbf{w}\|}\mathbf{w}$. We have already evaluated $\|\mathbf{w}_1\|^2 = 10$ therefore $\|\mathbf{w}_1\| = \sqrt{10}$ and similarly

$$\|\mathbf{w}'_2\|^2 = \begin{pmatrix} 3 \\ -1 \end{pmatrix} \cdot \begin{pmatrix} 3 \\ -1 \end{pmatrix} = 3^2 + (-1)^2 = 10$$

Taking the square root gives $\|\mathbf{w}'_2\| = \sqrt{10}$. Hence our orthonormal basis are

$$\mathbf{u}_1 = \frac{1}{\|\mathbf{w}_1\|}\mathbf{w}_1 = \frac{1}{\sqrt{10}}\begin{pmatrix} 1 \\ 3 \end{pmatrix} \text{ and } \mathbf{u}_2 = \frac{1}{\|\mathbf{w}_2\|}\mathbf{w}_2 = \frac{1}{\sqrt{10}}\begin{pmatrix} 3 \\ -1 \end{pmatrix}$$

Plotting these on $\mathbb{R}^2$:



(c) We are given the vectors $\mathbf{v}_1 = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$ and $\mathbf{v}_2 = \begin{pmatrix} 4 \\ 5 \end{pmatrix}$ which we need to transform to an orthonormal basis by using the Gram Schmidt Process (4-10):

We have $\mathbf{w}_1 = \mathbf{v}_1 = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$ and

$$\mathbf{w}_2 = \mathbf{v}_2 - \frac{\langle \mathbf{v}_2,\ \mathbf{w}_1 \rangle}{\|\mathbf{w}_1\|^2}\mathbf{w}_1 \qquad (*)$$

What is $\langle \mathbf{v}_2,\ \mathbf{w}_1 \rangle$ *equal to?*

$$\langle \mathbf{v}_2,\ \mathbf{w}_1 \rangle = \begin{pmatrix} 4 \\ 5 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ 3 \end{pmatrix} = (4 \times 2) + (5 \times 3) = 23$$

What is $\|\mathbf{w}_1\|^2$ *equal to?*

$$\|\mathbf{w}_1\|^2 = \begin{pmatrix} 2 \\ 3 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ 3 \end{pmatrix} = 2^2 + 3^2 = 13$$

Substituting these, $\langle \mathbf{v}_2,\ \mathbf{w}_1 \rangle = 23$ and $\|\mathbf{w}_1\|^2 = 13$, into (*) gives

$$\begin{aligned} \mathbf{w}_2 &= \mathbf{v}_2 - \frac{23}{13}\mathbf{w}_1 \\ &= \begin{pmatrix} 4 \\ 5 \end{pmatrix} - \frac{23}{13}\begin{pmatrix} 2 \\ 3 \end{pmatrix} && \left[\text{Substituting } \mathbf{w}_1 = \begin{pmatrix} 2 \\ 3 \end{pmatrix} \text{ and } \mathbf{v}_2 = \begin{pmatrix} 4 \\ 5 \end{pmatrix}\right] \\ &= \begin{pmatrix} 4 - 46/13 \\ 5 - 69/13 \end{pmatrix} \\ &= \begin{pmatrix} 6/13 \\ -4/13 \end{pmatrix} = \frac{2}{13}\begin{pmatrix} 3 \\ -2 \end{pmatrix} \end{aligned}$$

Clearing the fraction gives us the new vector $\mathbf{w}'_2 = \begin{pmatrix} 3 \\ -2 \end{pmatrix}$.

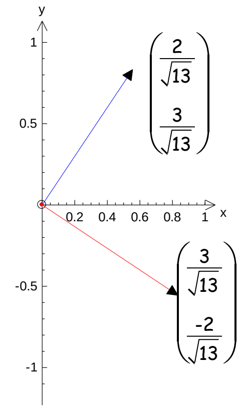
Normalizing the 2 orthogonal vectors $\mathbf{w}_1 = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$ and $\mathbf{w}'_2 = \begin{pmatrix} 3 \\ -2 \end{pmatrix}$ by dividing each vector by its norm, $\mathbf{u} = \frac{1}{\|\mathbf{w}\|}\mathbf{w}$. We have already evaluated $\|\mathbf{w}_1\|^2 = 13$ therefore $\|\mathbf{w}_1\| = \sqrt{13}$ and similarly

$$\|\mathbf{w}'_2\|^2 = \begin{pmatrix} 3 \\ -2 \end{pmatrix} \cdot \begin{pmatrix} 3 \\ -2 \end{pmatrix} = 3^2 + (-2)^2 = 13$$

Taking the square root gives $\|\mathbf{w}'_2\| = \sqrt{13}$. Hence our orthonormal basis are

$$\mathbf{u}_1 = \frac{1}{\|\mathbf{w}_1\|}\mathbf{w}_1 = \frac{1}{\sqrt{13}}\begin{pmatrix} 2 \\ 3 \end{pmatrix} \text{ and } \mathbf{u}_2 = \frac{1}{\|\mathbf{w}'_2\|}\mathbf{w}'_2 = \frac{1}{\sqrt{13}}\begin{pmatrix} 3 \\ -2 \end{pmatrix}$$

Plotting these on $\mathbb{R}^2$:



(d) We are given the basis vectors $\mathbf{v}_1 = \begin{pmatrix} -2 \\ -5 \end{pmatrix}$ and $\mathbf{v}_2 = \begin{pmatrix} -3 \\ -1 \end{pmatrix}$ which we need to transform to an orthonormal basis by using the Gram Schmidt Process (4-10):

We have $\mathbf{w}_1 = \mathbf{v}_1 = \begin{pmatrix} -2 \\ -5 \end{pmatrix}$ and

$$\mathbf{w}_2 = \mathbf{v}_2 - \frac{\langle \mathbf{v}_2, \mathbf{w}_1 \rangle}{\|\mathbf{w}_1\|^2}\mathbf{w}_1 \qquad (*)$$

What is $\langle \mathbf{v}_2, \mathbf{w}_1 \rangle$ *equal to?*

$$\langle \mathbf{v}_2, \mathbf{w}_1 \rangle = \begin{pmatrix} -3 \\ -1 \end{pmatrix} \cdot \begin{pmatrix} -2 \\ -5 \end{pmatrix} = 6 + 5 = 11$$

What is $\|\mathbf{w}_1\|^2$ *equal to?*

$$\|\mathbf{w}_1\|^2 = \begin{pmatrix} -2 \\ -5 \end{pmatrix} \cdot \begin{pmatrix} -2 \\ -5 \end{pmatrix} = (-2)^2 + (-5)^2 = 29$$

Substituting these, $\langle \mathbf{v}_2, \mathbf{w}_1 \rangle = 11$ and $\|\mathbf{w}_1\|^2 = 29$ into (*) gives

$$\begin{aligned} \mathbf{w}_2 &= \mathbf{v}_2 - \frac{11}{29}\mathbf{w}_1 \\ &= \begin{pmatrix} -3 \\ -1 \end{pmatrix} - \frac{11}{29}\begin{pmatrix} -2 \\ -5 \end{pmatrix} \qquad \left[\text{Substituting } \mathbf{w}_1 = \begin{pmatrix} -2 \\ -5 \end{pmatrix} \text{ and } \mathbf{v}_2 = \begin{pmatrix} -3 \\ -1 \end{pmatrix}\right] \\ &= \begin{pmatrix} -3 + 22/29 \\ -1 + 55/29 \end{pmatrix} = \begin{pmatrix} -65/29 \\ 26/29 \end{pmatrix} = \frac{1}{29}\begin{pmatrix} -65 \\ 26 \end{pmatrix} = \frac{13}{29}\begin{pmatrix} -5 \\ 2 \end{pmatrix} \end{aligned}$$

Clearing the fraction gives us the new vector $\mathbf{w}'_2 = \begin{pmatrix} -5 \\ 2 \end{pmatrix}$.

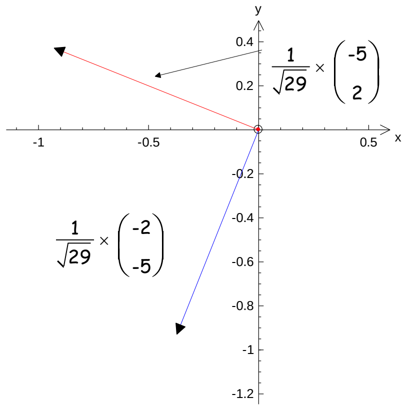
Normalizing the 2 orthogonal vectors $\mathbf{w}_1 = \begin{pmatrix} -2 \\ -5 \end{pmatrix}$ and $\mathbf{w}'_2 = \begin{pmatrix} -5 \\ 2 \end{pmatrix}$ by dividing each vector by its norm, $\mathbf{u} = \frac{1}{\|\mathbf{w}\|}\mathbf{w}$. We have already evaluated $\|\mathbf{w}_1\|^2 = 29$ therefore $\|\mathbf{w}_1\| = \sqrt{29}$ and similarly

$$\|\mathbf{w}'_2\|^2 = \begin{pmatrix} -5 \\ 2 \end{pmatrix} \cdot \begin{pmatrix} -5 \\ 2 \end{pmatrix} = (-5)^2 + (2)^2 = 29$$

Taking the square root gives $\|\mathbf{w}'_2\| = \sqrt{29}$. Hence our orthonormal basis are

$$\mathbf{u}_1 = \frac{1}{\|\mathbf{w}_1\|}\mathbf{w}_1 = \frac{1}{\sqrt{29}}\begin{pmatrix} -2 \\ -5 \end{pmatrix} \text{ and } \mathbf{u}_2 = \frac{1}{\|\mathbf{w}'_2\|}\mathbf{w}'_2 = \frac{1}{\sqrt{29}}\begin{pmatrix} -5 \\ 2 \end{pmatrix}$$

Plotting these on $\mathbb{R}^2$:



2. We are given the basis vectors $\mathbf{v}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $\mathbf{v}_2 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$ and need to transform these to an orthonormal basis by using the Gram Schmidt Process:

First we have $\mathbf{w}_1 = \mathbf{v}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and now we apply the formula:

$$\mathbf{w}_2 = \mathbf{v}_2 - \frac{\langle \mathbf{v}_2, \mathbf{w}_1 \rangle}{\|\mathbf{w}_1\|^2}\mathbf{w}_1 \qquad (*)$$

Evaluating each of the components:

$$\langle \mathbf{v}_2, \mathbf{w}_1 \rangle = \mathbf{v}_2 \cdot \mathbf{w}_1 = \begin{pmatrix} 2 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \end{pmatrix} = 2 + 1 = 3$$

$$\|\mathbf{w}_1\|^2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \end{pmatrix} = 2$$

Substituting these, $\langle \mathbf{v}_2, \mathbf{w}_1 \rangle = 3$ and $\|\mathbf{w}_1\|^2 = 2$, into (*) gives

$$\mathbf{w}_2 = \mathbf{v}_2 - \frac{3}{2}\mathbf{w}_1$$

Substituting the vectors gives

$$\mathbf{w}_2 = \mathbf{v}_2 - \frac{3}{2}\mathbf{w}_1 = \begin{pmatrix} 2 \\ 1 \end{pmatrix} - \frac{3}{2}\begin{pmatrix} 1 \\ 1 \end{pmatrix} \qquad \left[\text{Substituting } \mathbf{w}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \text{ and } \mathbf{v}_2 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}\right]$$

$$= \begin{pmatrix} 2-3/2 \\ 1-3/2 \end{pmatrix} = \begin{pmatrix} 1/2 \\ -1/2 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

Thus removing the scalar $\frac{1}{2}$ we have $\mathbf{w}'_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$. Our orthogonal vectors are

$$\mathbf{w}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \text{ and } \mathbf{w}'_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

Normalizing these gives

$$\mathbf{u}_1 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix} \text{ and } \mathbf{u}_2 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

We obtain a different set of orthonormal basis. In Example 12 the orthonormal basis was $\mathbf{u}_1 = \frac{1}{\sqrt{5}}\begin{pmatrix} 2 \\ 1 \end{pmatrix}$ and $\mathbf{u}_2 = \frac{1}{\sqrt{5}}\begin{pmatrix} -1 \\ 2 \end{pmatrix}$. Note that the only difference in the question was that we switched the vectors around.

3. Using the Gram Schmidt Process (4-10) above means that we need to find vectors $\mathbf{p}_1$, $\mathbf{p}_2$ and $\mathbf{p}_3$ which are orthogonal (perpendicular) to each other. The first vector is straightforward because we only need to write down the given vector $\mathbf{p}_1 = \mathbf{v}_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$.

How do we find the vector $\mathbf{p}_2$ which is perpendicular to $\mathbf{p}_1$?

By the above Gram Schmidt Proces (4-10):

$$\mathbf{p}_2 = \mathbf{v}_2 - \frac{\langle \mathbf{v}_2,\ \mathbf{p}_1 \rangle}{\|\mathbf{p}_1\|^2}\mathbf{p}_1 \qquad (\dagger)$$

What is $\|\mathbf{p}_1\|^2$ equal to?

By the definition of the norm we have

$$\|\mathbf{p}_1\|^2 = \langle \mathbf{p}_1,\ \mathbf{p}_1 \rangle = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = 1^2 + 1^2 + 1^2 = 3$$

What is $\|\mathbf{p}_1\|$ equal to?

Taking the square root of the above gives $\|\mathbf{p}_1\| = \sqrt{3}$.

What is $\langle \mathbf{v}_2,\ \mathbf{p}_1 \rangle$ equal to?

$$\langle \mathbf{v}_2,\ \mathbf{p}_1 \rangle = \mathbf{v}_2 \cdot \mathbf{p}_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = 1^2 + 1^2 + (0 \times 1) = 2$$

Substituting these, $\|\mathbf{p}_1\|^2 = 3$ and $\langle \mathbf{v}_2,\ \mathbf{p}_1 \rangle = 2$, into $(\dagger)$ gives $\mathbf{p}_2 = \mathbf{v}_2 - \frac{2}{3}\mathbf{p}_1$.

Substituting vectors into this yields:

$$\mathbf{p}_2=\begin{pmatrix}1\\1\\0\end{pmatrix}-\frac{2}{3}\begin{pmatrix}1\\1\\1\end{pmatrix}=\begin{pmatrix}1/3\\1/3\\-2/3\end{pmatrix}=\frac{1}{3}\begin{pmatrix}1\\1\\-2\end{pmatrix}$$

Clear the fraction to write the vector $\mathbf{p}'_2=\begin{pmatrix}1\\1\\-2\end{pmatrix}$. (We have nominated this new vector $\mathbf{p}'_2$). *What else do we need to find?*

The vector $\mathbf{p}_3$ which is perpendicular to both vectors $\mathbf{p}_1$ and $\mathbf{p}'_2$. *How?*

By Gram Schmidt Process (4-10):

$$\mathbf{p}_3=\mathbf{v}_3-\frac{\langle\mathbf{v}_3,\ \mathbf{p}_1\rangle}{\|\mathbf{p}_1\|^2}\mathbf{p}_1-\frac{\langle\mathbf{v}_3,\ \mathbf{p}'_2\rangle}{\|\mathbf{p}'_2\|^2}\mathbf{p}'_2 \qquad (\dagger\dagger)$$

Note that we have written $\mathbf{p}'_2$ rather than $\mathbf{p}_2$ because we are ignoring the fraction. We need to evaluate each of these components:

$$\langle\mathbf{v}_3,\ \mathbf{p}_1\rangle=\mathbf{v}_3\cdot\mathbf{p}_1=\begin{pmatrix}2\\0\\0\end{pmatrix}\cdot\begin{pmatrix}1\\1\\1\end{pmatrix}=(2\times1)+(0\times1)+(0\times1)=2$$

$$\langle\mathbf{v}_3,\ \mathbf{p}'_2\rangle=\mathbf{v}_3\cdot\mathbf{p}'_2=\begin{pmatrix}2\\0\\0\end{pmatrix}\cdot\begin{pmatrix}1\\1\\-2\end{pmatrix}=(2\times1)+(0\times1)+(0\times(-2))=2$$

What else do we need to work out?

The norm squared $\|\mathbf{p}'_2\|^2$ which is

$$\|\mathbf{p}'_2\|^2=\mathbf{p}'_2\cdot\mathbf{p}'_2=\begin{pmatrix}1\\1\\-2\end{pmatrix}\cdot\begin{pmatrix}1\\1\\-2\end{pmatrix}=1^2+1^2+(-2)^2=6$$

Taking the square root gives $\|\mathbf{p}'_2\|=\sqrt{6}$.

Substituting these, $\langle\mathbf{v}_3,\ \mathbf{p}_1\rangle=2$, $\|\mathbf{p}_1\|^2=3$, $\langle\mathbf{v}_3,\ \mathbf{p}'_2\rangle=2$ and $\|\mathbf{p}'_2\|^2=6$ into $(\dagger\dagger)$ gives

$$\begin{aligned}\mathbf{p}_3&=\mathbf{v}_3-\frac{\langle\mathbf{v}_3,\ \mathbf{p}_1\rangle}{\|\mathbf{p}_1\|^2}\mathbf{p}_1-\frac{\langle\mathbf{v}_3,\ \mathbf{p}'_2\rangle}{\|\mathbf{p}'_2\|^2}\mathbf{p}'_2 \qquad (\dagger\dagger)\\&=\mathbf{v}_3-\frac{2}{3}\mathbf{p}_1-\frac{2}{6}\mathbf{p}'_2\\&=\mathbf{v}_3-\frac{2}{3}\mathbf{p}_1-\frac{1}{3}\mathbf{p}'_2 \qquad \left[\text{Because } \frac{2}{6}=\frac{1}{3}\right]\end{aligned}$$

Substituting the vectors into this yields:

$$\mathbf{p}_3 = \mathbf{v}_3 - \frac{2}{3}\mathbf{p}_1 - \frac{1}{3}\mathbf{p}'_2$$

$$= \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix} - \frac{2}{3}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} - \frac{1}{3}\begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix} \quad \left[\text{Substituting } \mathbf{v}_3 = \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix},\ \mathbf{p}_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix},\ \mathbf{p}'_2 = \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix}\right]$$

$$= \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix} - \begin{pmatrix} 2/3 \\ 2/3 \\ 2/3 \end{pmatrix} - \begin{pmatrix} 1/3 \\ 1/3 \\ -2/3 \end{pmatrix} = \begin{pmatrix} 2-2/3-1/3 \\ 0-2/3-1/3 \\ 0-2/3+2/3 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$$

Our orthogonal (perpendicular) vectors are $\mathbf{p}_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$, $\mathbf{p}'_2 = \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix}$ and $\mathbf{p}_3 = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$.

To convert these into an orthonormal (perpendicular unit) basis, what do we need to do?

Normalize these vectors, which means convert these into unit vectors. *How?*

Dividing by its length because (4-7) $\dfrac{\mathbf{p}_j}{\|\mathbf{p}_j\|}$:

We have $\|\mathbf{p}_1\| = \sqrt{1^2+1^2+1^2} = \sqrt{3}$ therefore the unit vector is $\mathbf{u}_1 = \dfrac{1}{\sqrt{3}}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$.

Similarly we have $\|\mathbf{p}'_2\| = \sqrt{6}$ and the unit vector in the direction of $\mathbf{p}'_2$ is

$\mathbf{u}_2 = \dfrac{1}{\sqrt{6}}\begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix}$.

Lastly we have

$$\|\mathbf{p}_3\| = \sqrt{\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}} = \sqrt{1^2 + (-1)^2 + 0^2} = \sqrt{2}$$

Thus last unit vector is $\mathbf{u}_3 = \dfrac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$. Our orthonormal (perpendicular unit) basis is

$$\mathbf{u}_1 = \frac{1}{\sqrt{3}}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix},\ \mathbf{u}_2 = \frac{1}{\sqrt{6}}\begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix} \text{ and } \mathbf{u}_3 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$$

4. (a) We are given the basis vectors $\mathbf{v}_1 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$, $\mathbf{v}_2 = \begin{pmatrix} 3 \\ 1 \\ 1 \end{pmatrix}$ and $\mathbf{v}_3 = \begin{pmatrix} -1 \\ -1 \\ -1 \end{pmatrix}$ which need to be transformed to an orthonormal basis by using the Gram Schmidt Process (4-10).

First we have $\mathbf{w}_1 = \mathbf{v}_1 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$. *What is* $\mathbf{w}_2$ *equal to?*

By the Gram Schmidt Process we have

$$\mathbf{w}_2 = \mathbf{v}_2 - \frac{\langle \mathbf{v}_2,\ \mathbf{w}_1 \rangle}{\|\mathbf{w}_1\|^2}\mathbf{w}_1 \qquad (\$)$$

What is $\langle \mathbf{v}_2,\ \mathbf{w}_1 \rangle$ *equal to?*

$$\langle \mathbf{v}_2,\ \mathbf{w}_1 \rangle = \begin{pmatrix} 3 \\ 1 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} = 3+0+1 = 4$$

What is $\|\mathbf{w}_1\|^2$ *equal to?*

$$\|\mathbf{w}_1\|^2 = \mathbf{w}_1 \cdot \mathbf{w}_1 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} = 1+1 = 2$$

Substituting these, $\langle \mathbf{v}_2,\ \mathbf{w}_1 \rangle = 4$ and $\|\mathbf{w}_1\|^2 = 2$, into ($) gives

$$\mathbf{w}_2 = \mathbf{v}_2 - \frac{4}{2}\mathbf{w}_1 = \mathbf{v}_2 - 2\mathbf{w}_1$$

Substituting the vectors gives

$$\begin{aligned} \mathbf{w}_2 &= \mathbf{v}_2 - 2\mathbf{w}_1 \\ &= \begin{pmatrix} 3 \\ 1 \\ 1 \end{pmatrix} - 2\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 3-2 \\ 1-0 \\ 1-2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} \end{aligned}$$

What else do we need to find?

The last vector $\mathbf{w}_3$. *How?*

$$\mathbf{w}_3 = \mathbf{v}_3 - \frac{\langle \mathbf{v}_3,\ \mathbf{w}_1 \rangle}{\|\mathbf{w}_1\|^2}\mathbf{w}_1 - \frac{\langle \mathbf{v}_3,\ \mathbf{w}_2 \rangle}{\|\mathbf{w}_2\|^2}\mathbf{w}_2 \qquad (£)$$

We need to evaluate each of the components of (£). *What is* $\langle \mathbf{v}_3,\ \mathbf{w}_1 \rangle$ *equal to?*

$$\langle \mathbf{v}_3,\ \mathbf{w}_1 \rangle = \begin{pmatrix} -1 \\ -1 \\ -1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} = -1+0-1 = -2$$

What is $\langle \mathbf{v}_3,\ \mathbf{w}_2 \rangle$ *equal to?*

$$\langle \mathbf{v}_3,\ \mathbf{w}_2 \rangle = \begin{pmatrix} -1 \\ -1 \\ -1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} = -1-1+1 = -1$$

What is $\|\mathbf{w}_2\|^2$ *equal to?*

$$\|\mathbf{w}_2\|^2 = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} = 1+1+1 = 3$$

Substituting these, $\langle \mathbf{v}_3,\ \mathbf{w}_1 \rangle = -2$, $\langle \mathbf{v}_3,\ \mathbf{w}_2 \rangle = -1$, $\|\mathbf{w}_1\|^2 = 2$ and $\|\mathbf{w}_2\|^2 = 3$ into (£) above gives

$$\mathbf{w}_3 = \mathbf{v}_3 - \frac{\langle \mathbf{v}_3,\ \mathbf{w}_1 \rangle}{\|\mathbf{w}_1\|^2}\mathbf{w}_1 - \frac{\langle \mathbf{v}_3,\ \mathbf{w}_2 \rangle}{\|\mathbf{w}_2\|^2}\mathbf{w}_2$$

$$= \mathbf{v}_3 - \frac{(-2)}{2}\mathbf{w}_1 - \frac{(-1)}{3}\mathbf{w}_2 = \mathbf{v}_3 + \mathbf{w}_1 + \frac{1}{3}\mathbf{w}_2$$

Substituting the vectors gives

$$\mathbf{w}_3 = \mathbf{v}_3 + \mathbf{w}_1 + \frac{1}{3}\mathbf{w}_2$$

$$= \begin{pmatrix} -1 \\ -1 \\ -1 \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + \frac{1}{3}\begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} \qquad \left[\text{Substituting } \mathbf{v}_3 = \begin{pmatrix} -1 \\ -1 \\ -1 \end{pmatrix},\ \mathbf{w}_1 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix},\ \mathbf{w}_2 = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}\right]$$

$$= \begin{pmatrix} -1 \\ -1 \\ -1 \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + \begin{pmatrix} 1/3 \\ 1/3 \\ -1/3 \end{pmatrix} = \begin{pmatrix} -1+1+1/3 \\ -1+0+1/3 \\ -1+1-1/3 \end{pmatrix} = \begin{pmatrix} 1/3 \\ -2/3 \\ -1/3 \end{pmatrix} = \frac{1}{3}\begin{pmatrix} 1 \\ -2 \\ -1 \end{pmatrix}$$

Clearing the fraction gives $\mathbf{w}'_3 = \begin{pmatrix} 1 \\ -2 \\ -1 \end{pmatrix}$.

Thus the vectors $\mathbf{w}_1 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$, $\mathbf{w}_2 = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$ and $\mathbf{w}'_3 = \begin{pmatrix} 1 \\ -2 \\ -1 \end{pmatrix}$ are 3 orthogonal vectors.

Normalizing these vectors gives an orthonormal basisf:

$$\mathbf{u}_1 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix},\ \mathbf{u}_2 = \frac{1}{\sqrt{3}}\begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} \text{ and } \mathbf{u}_3 = \frac{1}{\sqrt{6}}\begin{pmatrix} 1 \\ -2 \\ -1 \end{pmatrix}$$

(b) We are given the vectors $\mathbf{v}_1 = \begin{pmatrix} 2 \\ 2 \\ 2 \end{pmatrix}$, $\mathbf{v}_2 = \begin{pmatrix} -1 \\ 0 \\ -1 \end{pmatrix}$ and $\mathbf{v}_3 = \begin{pmatrix} -1 \\ 2 \\ 3 \end{pmatrix}$. Remember by question 16 of Exercises 4.2 which says that $\{\mathbf{v}_1,\ \mathbf{v}_2,\ \cdots,\ \mathbf{v}_n\}$ are orthogonal if and only if $\{k_1\mathbf{v}_1,\ k_2\mathbf{v}_2,\ \cdots,\ k_n\mathbf{v}_n\}$. We can take out the scalars from the first two vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ to make our computation easier, that is we have

$$\mathbf{v}'_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix},\ \mathbf{v}'_2 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \text{ and } \mathbf{v}_3 = \begin{pmatrix} -1 \\ 2 \\ 3 \end{pmatrix}$$

We have removed the 2 from the first vector $\mathbf{v}_1$ and -1 from the second vector $\mathbf{v}_2$ to give us $\mathbf{v}'_1$ and $\mathbf{v}'_2$ respectively.

Using the Gram Schmidt Process (4-10) we have

$$\mathbf{w}_1 = \mathbf{v}'_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

What is $\mathbf{w}_2$ *equal to?*

By the Gram Schmidt Process we have

$$\mathbf{w}_2 = \mathbf{v}'_2 - \frac{\langle \mathbf{v}'_2,\ \mathbf{w}_1 \rangle}{\|\mathbf{w}_1\|^2}\mathbf{w}_1 \qquad (*)$$

What is $\langle \mathbf{v}'_2,\ \mathbf{w}_1 \rangle$ *equal to?*

$$\langle \mathbf{v}'_2,\ \mathbf{w}_1 \rangle = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = 1 + 0 + 1 = 2$$

What is $\|\mathbf{w}_1\|^2$ *equal to?*

$$\|\mathbf{w}_1\|^2 = \mathbf{w}_1 \cdot \mathbf{w}_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = 1 + 1 + 1 = 3$$

Substituting these, $\langle \mathbf{v}'_2,\ \mathbf{w}_1 \rangle = 2$ and $\|\mathbf{w}_1\|^2 = 3$, into (*) gives

$$\mathbf{w}_2 = \mathbf{v}'_2 - \frac{\langle \mathbf{v}'_2,\ \mathbf{w}_1 \rangle}{\|\mathbf{w}_1\|^2}\mathbf{w}_1 = \mathbf{v}'_2 - \frac{2}{3}\mathbf{w}_1$$

Substituting the vectors gives

$$\begin{aligned} \mathbf{w}_2 &= \mathbf{v}'_2 - \frac{2}{3}\mathbf{w}_1 \\ &= \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} - \frac{2}{3}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 - 2/3 \\ 0 - 2/3 \\ 1 - 2/3 \end{pmatrix} = \frac{1}{3}\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} \end{aligned}$$

Removing the fraction gives $\mathbf{w}'_2 = \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$. *What else do we need to find?*

The last vector $\mathbf{w}_3$. *How?*

$$\mathbf{w}_3 = \mathbf{v}_3 - \frac{\langle \mathbf{v}_3,\ \mathbf{w}_1 \rangle}{\|\mathbf{w}_1\|^2}\mathbf{w}_1 - \frac{\langle \mathbf{v}_3,\ \mathbf{w}'_2 \rangle}{\|\mathbf{w}'_2\|^2}\mathbf{w}'_2 \qquad (**)$$

We need to evaluate each of the components of (**). *What is* $\langle \mathbf{v}_3,\ \mathbf{w}_1 \rangle$ *equal to?*

$$\langle \mathbf{v}_3, \ \mathbf{w}_1 \rangle = \begin{pmatrix} -1 \\ 2 \\ 3 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = -1+2+3 = 4$$

What is $\langle \mathbf{v}_3, \ \mathbf{w}'_2 \rangle$ *equal to?*

$$\langle \mathbf{v}_3, \ \mathbf{w}'_2 \rangle = \begin{pmatrix} -1 \\ 2 \\ 3 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} = -1-4+3 = -2$$

What is $\|\mathbf{w}'_2\|^2$ *equal to?*

$$\|\mathbf{w}'_2\|^2 = \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} = 1+4+1 = 6$$

Substituting these, $\langle \mathbf{v}_3, \ \mathbf{w}_1 \rangle = 4$, $\langle \mathbf{v}_3, \ \mathbf{w}'_2 \rangle = -2$, $\|\mathbf{w}_1\|^2 = 3$ and $\|\mathbf{w}'_2\|^2 = 6$ into (**) above gives

$$\mathbf{w}_3 = \mathbf{v}_3 - \frac{\langle \mathbf{v}_3, \ \mathbf{w}_1 \rangle}{\|\mathbf{w}_1\|^2}\mathbf{w}_1 - \frac{\langle \mathbf{v}_3, \ \mathbf{w}'_2 \rangle}{\|\mathbf{w}'_2\|^2}\mathbf{w}'_2$$

$$= \mathbf{v}_3 - \frac{4}{3}\mathbf{w}_1 - \frac{(-2)}{6}\mathbf{w}'_2 = \mathbf{v}_3 - \frac{4}{3}\mathbf{w}_1 + \frac{1}{3}\mathbf{w}'_2 \qquad \left[\text{Because } -\frac{(-2)}{6} = \frac{1}{3}\right]$$

Substituting the vectors gives

$$\mathbf{w}_3 = \mathbf{v}_3 - \frac{4}{3}\mathbf{w}_1 + \frac{1}{3}\mathbf{w}'_2$$

$$= \begin{pmatrix} -1 \\ 2 \\ 3 \end{pmatrix} - \frac{4}{3}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + \frac{1}{3}\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} \qquad \left[\text{Substituting } \mathbf{v}_3 = \begin{pmatrix} -1 \\ 2 \\ 3 \end{pmatrix}, \ \mathbf{w}_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \ \mathbf{w}'_2 = \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}\right]$$

$$= \begin{pmatrix} -1 \\ 2 \\ 3 \end{pmatrix} + \begin{pmatrix} -4/3 \\ -4/3 \\ -4/3 \end{pmatrix} + \begin{pmatrix} 1/3 \\ -2/3 \\ 1/3 \end{pmatrix} = \begin{pmatrix} -1-4/3+1/3 \\ 2-4/3-2/3 \\ 3-4/3+1/3 \end{pmatrix} = \begin{pmatrix} -2 \\ 0 \\ 2 \end{pmatrix} = 2\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$$

Clearing the scalar gives $\mathbf{w}'_3 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$.

Thus the vectors $\mathbf{w}_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$, $\mathbf{w}'_2 = \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$ and $\mathbf{w}'_3 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$ are 3 orthogonal vectors.

Normalizing these vectors gives an orthonormal basis:

$$\mathbf{u}_1 = \frac{1}{\sqrt{3}}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \ \mathbf{u}_2 = \frac{1}{\sqrt{6}}\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} \text{ and } \mathbf{u}_3 = \frac{1}{\sqrt{2}}\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$$

(c) We are given the basis vectors $\mathbf{v}_1 = \begin{pmatrix}1\\2\\0\end{pmatrix}$, $\mathbf{v}_2 = \begin{pmatrix}2\\0\\2\end{pmatrix}$ and $\mathbf{v}_3 = \begin{pmatrix}1\\0\\3\end{pmatrix}$. Remember by question 16 of Exercises 4.2 which says that $\{\mathbf{v}_1,\ \mathbf{v}_2,\ \cdots,\ \mathbf{v}_n\}$ are orthogonal if and only if $\{k_1\mathbf{v}_1,\ k_2\mathbf{v}_2,\ \cdots,\ k_n\mathbf{v}_n\}$. We can take out the scalar from the second vector $\mathbf{v}_2$ to make our computation easier, that is we have

$$\mathbf{v}_1 = \begin{pmatrix}1\\2\\0\end{pmatrix},\ \mathbf{v}'_2 = \begin{pmatrix}1\\0\\1\end{pmatrix} \text{ and } \mathbf{v}_3 = \begin{pmatrix}1\\0\\3\end{pmatrix}$$

We have removed the 2 from the second vector $\mathbf{v}_2$.

Using the Gram Schmidt Process (4-10) we have

$$\mathbf{w}_1 = \mathbf{v}_1 = \begin{pmatrix}1\\2\\0\end{pmatrix}$$

What is $\mathbf{w}_2$ equal to?

By the Gram Schmidt Process we have

$$\mathbf{w}_2 = \mathbf{v}'_2 - \frac{\langle \mathbf{v}'_2,\ \mathbf{w}_1\rangle}{\|\mathbf{w}_1\|^2}\mathbf{w}_1 \qquad (*)$$

What is $\langle \mathbf{v}'_2,\ \mathbf{w}_1\rangle$ equal to?

$$\langle \mathbf{v}'_2,\ \mathbf{w}_1\rangle = \begin{pmatrix}1\\0\\1\end{pmatrix}\cdot\begin{pmatrix}1\\2\\0\end{pmatrix} = 1$$

What is $\|\mathbf{w}_1\|^2$ equal to?

$$\|\mathbf{w}_1\|^2 = \mathbf{w}_1\cdot\mathbf{w}_1 = \begin{pmatrix}1\\2\\0\end{pmatrix}\cdot\begin{pmatrix}1\\2\\0\end{pmatrix} = 1^2 + 2^2 = 5$$

Substituting these, $\langle \mathbf{v}'_2,\ \mathbf{w}_1\rangle = 1$ and $\|\mathbf{w}_1\|^2 = 5$, into (*) gives

$$\mathbf{w}_2 = \mathbf{v}'_2 - \frac{\langle \mathbf{v}'_2,\ \mathbf{w}_1\rangle}{\|\mathbf{w}_1\|^2}\mathbf{w}_1 = \mathbf{v}'_2 - \frac{1}{5}\mathbf{w}_1$$

Substituting the vectors gives

$$\begin{aligned}\mathbf{w}_2 &= \mathbf{v}'_2 - \frac{1}{5}\mathbf{w}_1\\ &= \begin{pmatrix}1\\0\\1\end{pmatrix} - \frac{1}{5}\begin{pmatrix}1\\2\\0\end{pmatrix} = \begin{pmatrix}1-1/5\\0-2/5\\1-0\end{pmatrix} = \begin{pmatrix}4/5\\-2/5\\1\end{pmatrix} = \frac{1}{5}\begin{pmatrix}4\\-2\\5\end{pmatrix}\end{aligned}$$

Removing the fraction gives $\mathbf{w}'_2 = \begin{pmatrix}4\\-2\\5\end{pmatrix}$. *What else do we need to find?*

The last vector $\mathbf{w}_3$. *How?*

$$\mathbf{w}_3 = \mathbf{v}_3 - \frac{\langle \mathbf{v}_3,\ \mathbf{w}_1 \rangle}{\|\mathbf{w}_1\|^2}\mathbf{w}_1 - \frac{\langle \mathbf{v}_3,\ \mathbf{w}'_2 \rangle}{\|\mathbf{w}'_2\|^2}\mathbf{w}'_2 \qquad (**)$$

We need to evaluate each of the components of (**). *What is* $\langle \mathbf{v}_3,\ \mathbf{w}_1 \rangle$ *equal to?*

$$\langle \mathbf{v}_3,\ \mathbf{w}_1 \rangle = \begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} = 1$$

What is $\langle \mathbf{v}_3,\ \mathbf{w}'_2 \rangle$ *equal to?*

$$\langle \mathbf{v}_3,\ \mathbf{w}'_2 \rangle = \begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix} \cdot \begin{pmatrix} 4 \\ -2 \\ 5 \end{pmatrix} = 4 + 15 = 19$$

What is $\|\mathbf{w}'_2\|^2$ *equal to?*

$$\|\mathbf{w}'_2\|^2 = \begin{pmatrix} 4 \\ -2 \\ 5 \end{pmatrix} \cdot \begin{pmatrix} 4 \\ -2 \\ 5 \end{pmatrix} = 16 + 4 + 25 = 45$$

Substituting these, $\langle \mathbf{v}_3,\ \mathbf{w}_1 \rangle = 1$, $\langle \mathbf{v}_3,\ \mathbf{w}'_2 \rangle = 19$, $\|\mathbf{w}_1\|^2 = 5$ and $\|\mathbf{w}'_2\|^2 = 45$, into (**) above gives

$$\begin{aligned}\mathbf{w}_3 &= \mathbf{v}_3 - \frac{\langle \mathbf{v}_3,\ \mathbf{w}_1 \rangle}{\|\mathbf{w}_1\|^2}\mathbf{w}_1 - \frac{\langle \mathbf{v}_3,\ \mathbf{w}'_2 \rangle}{\|\mathbf{w}'_2\|^2}\mathbf{w}'_2 \\ &= \mathbf{v}_3 - \frac{1}{5}\mathbf{w}_1 - \frac{19}{45}\mathbf{w}'_2\end{aligned}$$

Substituting the vectors gives

$$\begin{aligned}\mathbf{w}_3 &= \mathbf{v}_3 - \frac{1}{5}\mathbf{w}_1 - \frac{19}{45}\mathbf{w}'_2 \\ &= \begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix} - \frac{1}{5}\begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} - \frac{19}{45}\begin{pmatrix} 4 \\ -2 \\ 5 \end{pmatrix} \qquad \left[\text{Substituting } \mathbf{w}_1 = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix},\ \mathbf{w}'_2 = \begin{pmatrix} 4 \\ -2 \\ 5 \end{pmatrix} \text{ and } \mathbf{v}_3 = \begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix}\right] \text{Cl} \\ &= \begin{pmatrix} 1 - 1/5 - 76/45 \\ 0 - 2/5 + 38/45 \\ 3 - 0 - 95/45 \end{pmatrix} = \begin{pmatrix} -8/9 \\ 4/9 \\ 8/9 \end{pmatrix} = \frac{1}{9}\begin{pmatrix} -8 \\ 4 \\ 8 \end{pmatrix} = \frac{4}{9}\begin{pmatrix} -2 \\ 1 \\ 2 \end{pmatrix}\end{aligned}$$

earing the scalar gives $\mathbf{w}'_3 = \begin{pmatrix} -2 \\ 1 \\ 2 \end{pmatrix}$.

Thus the vectors $\mathbf{w}_1 = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}$, $\mathbf{w}'_2 = \begin{pmatrix} 4 \\ -2 \\ 5 \end{pmatrix}$, and $\mathbf{w}'_3 = \begin{pmatrix} -2 \\ 1 \\ 2 \end{pmatrix}$ are 3 orthogonal vectors.

Normalizing these vectors gives an orthonormal basis:

$$\mathbf{u}_1 = \frac{1}{\sqrt{5}}\begin{pmatrix}1\\2\\0\end{pmatrix},\ \mathbf{u}_2 = \frac{1}{\sqrt{45}}\begin{pmatrix}4\\-2\\5\end{pmatrix}, \text{ and } \mathbf{u}_3 = \frac{1}{3}\begin{pmatrix}-2\\1\\2\end{pmatrix}$$

5. (a) We are given the vectors $\mathbf{v}_1 = \begin{pmatrix}1\\0\\3\\0\end{pmatrix}$, $\mathbf{v}_2 = \begin{pmatrix}1\\2\\1\\0\end{pmatrix}$ and $\mathbf{v}_3 = \begin{pmatrix}2\\1\\0\\0\end{pmatrix}$ which we need to transform to an orthonormal basis. Using the Gram Schmidt Process (4-10) we have $\mathbf{w}_1 = \mathbf{v}_1 = \begin{pmatrix}1\\0\\3\\0\end{pmatrix}$. We need to find the vectors $\mathbf{w}_2$ and $\mathbf{w}_3$ which are orthogonal to each other and $\mathbf{w}_1$. Again using the Gram Schmidt Process we have

$$\mathbf{w}_2 = \mathbf{v}_2 - \frac{\langle \mathbf{v}_2,\ \mathbf{w}_1\rangle}{\|\mathbf{w}_1\|^2}\mathbf{w}_1 \qquad (\dagger)$$

We need to find each part of this (†):

$$\langle \mathbf{v}_2,\ \mathbf{w}_1\rangle = \mathbf{v}_2 \cdot \mathbf{w}_1 = \begin{pmatrix}1\\2\\1\\0\end{pmatrix}\cdot\begin{pmatrix}1\\0\\3\\0\end{pmatrix} = 1+3 = 4$$

$$\|\mathbf{w}_1\|^2 = \begin{pmatrix}1\\0\\3\\0\end{pmatrix}\cdot\begin{pmatrix}1\\0\\3\\0\end{pmatrix} = 1^2+3^2 = 10$$

Substituting these, $\langle \mathbf{v}_2,\ \mathbf{w}_1\rangle = 4$ and $\|\mathbf{w}_1\|^2 = 10$, into (†) gives

$$\mathbf{w}_2 = \mathbf{v}_2 - \frac{4}{10}\mathbf{w}_1 = \mathbf{v}_2 - \frac{2}{5}\mathbf{w}_1 \qquad \left[\text{Because } \frac{4}{10} = \frac{2}{5}\right]$$

Substituting the vectors into this:

$$\begin{aligned}\mathbf{w}_2 &= \mathbf{v}_2 - \frac{2}{5}\mathbf{w}_1\\ &= \begin{pmatrix}1\\2\\1\\0\end{pmatrix} - \frac{2}{5}\begin{pmatrix}1\\0\\3\\0\end{pmatrix} \qquad \left[\text{Because } \mathbf{w}_1 = \begin{pmatrix}1\\0\\3\\0\end{pmatrix} \text{ and } \mathbf{v}_2 = \begin{pmatrix}1\\2\\1\\0\end{pmatrix}\right]\\ &= \begin{pmatrix}1-2/5\\2-0\\1-6/5\\0-0\end{pmatrix} = \begin{pmatrix}3/5\\2\\-1/5\\0\end{pmatrix} = \frac{1}{5}\begin{pmatrix}3\\10\\-1\\0\end{pmatrix}\end{aligned}$$

Clearing the scalar 1/5 gives $\mathbf{w}'_2 = \begin{pmatrix} 3 \\ 10 \\ -1 \\ 0 \end{pmatrix}$. We also need to find $\mathbf{w}_3$. *What is* $\mathbf{w}_3$ *equal to?*

Using the Gram Schmidt Process we have

$$\mathbf{w}_3 = \mathbf{v}_3 - \frac{\langle \mathbf{v}_3,\ \mathbf{w}_1 \rangle}{\|\mathbf{w}_1\|^2}\mathbf{w}_1 - \frac{\langle \mathbf{v}_3,\ \mathbf{w}'_2 \rangle}{\|\mathbf{w}'_2\|^2}\mathbf{w}'_2 \qquad (\dagger\dagger)$$

Evaluating each component gives

$$\langle \mathbf{v}_3,\ \mathbf{w}_1 \rangle = \mathbf{v}_3 \cdot \mathbf{w}_1 = \begin{pmatrix} 2 \\ 1 \\ 0 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 0 \\ 3 \\ 0 \end{pmatrix} = 2$$

Similarly we have

$$\langle \mathbf{v}_3,\ \mathbf{w}'_2 \rangle = \mathbf{v}_3 \cdot \mathbf{w}'_2 = \begin{pmatrix} 2 \\ 1 \\ 0 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 3 \\ 10 \\ -1 \\ 0 \end{pmatrix} = (2\times 3) + (1\times 10) + 0 + 0 = 16$$

We also need to work out

$$\|\mathbf{w}'_2\|^2 = \begin{pmatrix} 3 \\ 10 \\ -1 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 3 \\ 10 \\ -1 \\ 0 \end{pmatrix} = 3^2 + 10^2 + (-1)^2 + 0^2 = 110$$

Substituting the scalars, $\langle \mathbf{v}_3,\ \mathbf{w}_1 \rangle = 2$, $\|\mathbf{w}_1\|^2 = 10$, $\langle \mathbf{v}_3,\ \mathbf{w}'_2 \rangle = 16$ and $\|\mathbf{w}'_2\|^2 = 110$, into $(\dagger\dagger)$ gives

$$\begin{aligned} \mathbf{w}_3 &= \mathbf{v}_3 - \frac{\langle \mathbf{v}_3,\ \mathbf{w}_1 \rangle}{\|\mathbf{w}_1\|^2}\mathbf{w}_1 - \frac{\langle \mathbf{v}_3,\ \mathbf{w}'_2 \rangle}{\|\mathbf{w}'_2\|^2}\mathbf{w}'_2 \\ &= \mathbf{v}_3 - \frac{2}{10}\mathbf{w}_1 - \frac{16}{110}\mathbf{w}'_2 = \mathbf{v}_3 - \frac{1}{5}\mathbf{w}_1 - \frac{8}{55}\mathbf{w}'_2 \qquad \left[\text{Because } \frac{2}{10} = \frac{1}{5} \text{ and } \frac{16}{110} = \frac{8}{55}\right] \end{aligned}$$

Substituting the vectors into this

$$\mathbf{w}_3 = \mathbf{v}_3 - \frac{1}{5}\mathbf{w}_1 - \frac{8}{55}\mathbf{w}'_2$$

$$= \begin{pmatrix} 2 \\ 1 \\ 0 \\ 0 \end{pmatrix} - \frac{1}{5}\begin{pmatrix} 1 \\ 0 \\ 3 \\ 0 \end{pmatrix} - \frac{8}{55}\begin{pmatrix} 3 \\ 10 \\ -1 \\ 0 \end{pmatrix} \qquad \left[\text{Because } \mathbf{w}_1 = \begin{pmatrix} 1 \\ 0 \\ 3 \\ 0 \end{pmatrix},\ \mathbf{w}'_2 = \begin{pmatrix} 3 \\ 10 \\ -1 \\ 0 \end{pmatrix} \text{ and } \mathbf{v}_3 = \begin{pmatrix} 2 \\ 1 \\ 0 \\ 0 \end{pmatrix}\right]$$

$$= \begin{pmatrix} 2-1/5-24/55 \\ 1-0-80/55 \\ 0-3/5+8/55 \\ 0-0-0 \end{pmatrix} = \begin{pmatrix} 15/11 \\ -5/11 \\ -5/11 \\ 0 \end{pmatrix} = \frac{5}{11}\begin{pmatrix} 3 \\ -1 \\ -1 \\ 0 \end{pmatrix}$$

Removing the scalar we have $\mathbf{w}'_3 = \begin{pmatrix} 3 \\ -1 \\ -1 \\ 0 \end{pmatrix}$. *What else do we need to do?*

We need to normalize each of the vectors $\mathbf{w}_1 = \begin{pmatrix} 1 \\ 0 \\ 3 \\ 0 \end{pmatrix}$, $\mathbf{w}'_2 = \begin{pmatrix} 3 \\ 10 \\ -1 \\ 0 \end{pmatrix}$ and $\mathbf{w}'_3 = \begin{pmatrix} 3 \\ -1 \\ -1 \\ 0 \end{pmatrix}$:

$$\mathbf{u}_1 = \frac{1}{\sqrt{10}}\begin{pmatrix} 1 \\ 0 \\ 3 \\ 0 \end{pmatrix},\ \mathbf{u}_2 = \frac{1}{\sqrt{110}}\begin{pmatrix} 3 \\ 10 \\ -1 \\ 0 \end{pmatrix} \text{ and } \mathbf{u}_3 = \frac{1}{\sqrt{11}}\begin{pmatrix} 3 \\ -1 \\ -1 \\ 0 \end{pmatrix}$$

These are our orthonormal basis.

(b) We are given the basis vectors $\mathbf{v}_1 = \begin{pmatrix} 1 \\ 1 \\ 5 \\ 2 \end{pmatrix}$, $\mathbf{v}_2 = \begin{pmatrix} -3 \\ 3 \\ 4 \\ -2 \end{pmatrix}$ and $\mathbf{v}_3 = \begin{pmatrix} -1 \\ -2 \\ 2 \\ 5 \end{pmatrix}$ which we transform to an orthonormal basis by using the Gram Schmidt Process. The first vector is straightforward, we just copy the first vector $\mathbf{w}_1 = \mathbf{v}_1 = \begin{pmatrix} 1 \\ 1 \\ 5 \\ 2 \end{pmatrix}$. Applying the Gram Schmidt Process we have

$$\mathbf{w}_2 = \mathbf{v}_2 - \frac{\langle \mathbf{v}_2,\ \mathbf{w}_1 \rangle}{\|\mathbf{w}_1\|^2}\mathbf{w}_1 \qquad (*)$$

Determining the inner product

$$\langle \mathbf{v}_2,\ \mathbf{w}_1\rangle = \mathbf{v}_2\cdot\mathbf{w}_1 = \begin{pmatrix}-3\\3\\4\\-2\end{pmatrix}\cdot\begin{pmatrix}1\\1\\5\\2\end{pmatrix} = (-3\times1)+(3\times1)+(4\times5)+(-2\times2)=16$$

Finding the norm squared

$$\|\mathbf{w}_1\|^2 = \begin{pmatrix}1\\1\\5\\2\end{pmatrix}\cdot\begin{pmatrix}1\\1\\5\\2\end{pmatrix} = 1^2+1^2+5^2+2^2 = 31$$

Substituting these, $\langle \mathbf{v}_2,\ \mathbf{w}_1\rangle = 16$ and $\|\mathbf{w}_1\|^2 = 31$, into (*) yields

$$\mathbf{w}_2 = \mathbf{v}_2 - \frac{16}{31}\mathbf{w}_1$$

Placing the vectors into this

$$\begin{aligned}\mathbf{w}_2 &= \mathbf{v}_2 - \frac{16}{31}\mathbf{w}_1\\ &= \begin{pmatrix}-3\\3\\4\\-2\end{pmatrix} - \frac{16}{31}\begin{pmatrix}1\\1\\5\\2\end{pmatrix} \qquad \left[\text{Because } \mathbf{w}_1 = \begin{pmatrix}1\\1\\5\\2\end{pmatrix} \text{ and } \mathbf{v}_2 = \begin{pmatrix}-3\\3\\4\\-2\end{pmatrix}\right]\\ &= \begin{pmatrix}-3-16/31\\3-16/31\\4-80/31\\-2-32/31\end{pmatrix} = \begin{pmatrix}-109/31\\77/31\\44/31\\-94/31\end{pmatrix} = \frac{1}{31}\begin{pmatrix}-109\\77\\44\\-94\end{pmatrix}\end{aligned}$$

Removing the scalar gives $\mathbf{w}'_2 = \begin{pmatrix}-109\\77\\44\\-94\end{pmatrix}$. *What else do we need to determine?*

The last orthogonal vector $\mathbf{w}_3$ by using the Gram Schmidt Process:

$$\mathbf{w}_3 = \mathbf{v}_3 - \frac{\langle \mathbf{v}_3,\ \mathbf{w}_1\rangle}{\|\mathbf{w}_1\|^2}\mathbf{w}_1 - \frac{\langle \mathbf{v}_3,\ \mathbf{w}'_2\rangle}{\|\mathbf{w}'_2\|^2}\mathbf{w}'_2 \qquad (**)$$

Evaluating the first inner product

$$\langle \mathbf{v}_3,\ \mathbf{w}_1\rangle = \mathbf{v}_3\cdot\mathbf{w}_1 = \begin{pmatrix}-1\\-2\\2\\5\end{pmatrix}\cdot\begin{pmatrix}1\\1\\5\\2\end{pmatrix} = (-1\times1)+(-2\times1)+(2\times5)+(5\times2)=17$$

Working out the second inner product

$$\langle \mathbf{v}_3,\ \mathbf{w}'_2\rangle = \mathbf{v}_3\cdot\mathbf{w}'_2 = \begin{pmatrix}-1\\-2\\2\\5\end{pmatrix}\cdot\begin{pmatrix}-109\\77\\44\\-94\end{pmatrix} = 109-154+88-470 = -427$$

Also need to find $\|\mathbf{w}'_2\|^2$:

$$\|\mathbf{w}'_2\|^2 = \begin{pmatrix}-109\\77\\44\\-94\end{pmatrix}\cdot\begin{pmatrix}-109\\77\\44\\-94\end{pmatrix} = 109^2+77^2+44^2+94^2 = 28582$$

Putting these, $\langle \mathbf{v}_3,\ \mathbf{w}_1\rangle = 17$, $\|\mathbf{w}_1\|^2 = 31$, $\langle \mathbf{v}_3,\ \mathbf{w}'_2\rangle = -427$ and $\|\mathbf{w}'_2\|^2 = 28582$ into (**) gives

$$\begin{aligned}\mathbf{w}_3 &= \mathbf{v}_3 - \frac{\langle \mathbf{v}_3,\ \mathbf{w}_1\rangle}{\|\mathbf{w}_1\|^2}\mathbf{w}_1 - \frac{\langle \mathbf{v}_3,\ \mathbf{w}'_2\rangle}{\|\mathbf{w}'_2\|^2}\mathbf{w}'_2\\ &= \mathbf{v}_3 - \frac{17}{31}\mathbf{w}_1 - \frac{(-427)}{28582}\mathbf{w}'_2 = \mathbf{v}_3 - \frac{17}{31}\mathbf{w}_1 + \frac{427}{28582}\mathbf{w}'_2\end{aligned}$$

Placing the vectors Because $\mathbf{w}_1 = \begin{pmatrix}1\\1\\5\\2\end{pmatrix}$, $\mathbf{w}'_2 = \begin{pmatrix}-109\\77\\44\\-94\end{pmatrix}$ and $\mathbf{v}_3 = \begin{pmatrix}-1\\-2\\2\\5\end{pmatrix}$ into this

$$\begin{aligned}\mathbf{w}_3 &= \mathbf{v}_3 - \frac{17}{31}\mathbf{w}_1 + \frac{427}{28582}\mathbf{w}'_2\\ &= \begin{pmatrix}-1\\-2\\2\\5\end{pmatrix} - \frac{17}{31}\begin{pmatrix}1\\1\\5\\2\end{pmatrix} + \frac{427}{28582}\begin{pmatrix}-109\\77\\44\\-94\end{pmatrix}\\ &= \begin{pmatrix}-1-17/31-46543/28582\\-2-17/31+32879/28582\\2-85/31+18788/28582\\5-34/31-40138/28582\end{pmatrix} = \begin{pmatrix}-2929/922\\-1289/922\\-78/922\\2304/922\end{pmatrix} = \frac{1}{922}\begin{pmatrix}-2929\\-1289\\-78\\2304\end{pmatrix}\end{aligned}$$

Clearing the scalar we have $\mathbf{w}'_3 = \begin{pmatrix}-2929\\-1289\\-78\\2304\end{pmatrix}$. We have our orthogonal vectors

$\mathbf{w}_1 = \begin{pmatrix}1\\1\\5\\2\end{pmatrix}$, $\mathbf{w}'_2 = \begin{pmatrix}-109\\77\\44\\-94\end{pmatrix}$ and $\mathbf{w}'_3 = \begin{pmatrix}-2929\\-1289\\-78\\2304\end{pmatrix}$. We need to normalize these. *How?*

Dividing by the norm of each. From above we have $\|\mathbf{w}_1\|^2 = 31$ which gives

$\|\mathbf{w}_1\| = \sqrt{31}$ and $\|\mathbf{w}'_2\|^2 = 28582$ implies $\|\mathbf{w}'_2\| = \sqrt{28582}$. We need to evaluate the norm of the last vector $\mathbf{w}'_3$ which is given by

$$\|\mathbf{w}'_3\|^2 = \begin{pmatrix} -2929 \\ -1289 \\ -78 \\ 2304 \end{pmatrix} \cdot \begin{pmatrix} -2929 \\ -1289 \\ -78 \\ 2304 \end{pmatrix} = (-2929)^2 + (-1289)^2 + (-78)^2 + 2304^2 = 15555062$$

What is $\|\mathbf{w}'_3\|$ *equal to?*

The square root of the above $\|\mathbf{w}'_3\| = \sqrt{15555062}$. Our normalized vectors are

$$\mathbf{w}_1 = \frac{1}{\sqrt{31}}\begin{pmatrix} 1 \\ 1 \\ 5 \\ 2 \end{pmatrix}, \ \mathbf{w}'_2 = \frac{1}{\sqrt{28582}}\begin{pmatrix} -109 \\ 77 \\ 44 \\ -94 \end{pmatrix} \text{ and } \mathbf{w}'_3 = \frac{1}{\sqrt{15555062}}\begin{pmatrix} -2929 \\ -1289 \\ -78 \\ 2304 \end{pmatrix}$$

This is our orthonormal basis for $\mathbb{R}^4$.

We can also apply the Gram Schmidt Process for subspaces as the next example demonstrates.

(c) The first vector is given $\mathbf{p}_1 = \mathbf{v}_1 = \begin{pmatrix} 1 & 2 & 3 & 4 \end{pmatrix}^T$. Next we find a vector which is orthogonal (perpendicular) to this by using (4-10) which is:

$$\mathbf{p}_2 = \mathbf{v}_2 - \frac{\langle \mathbf{v}_2, \ \mathbf{p}_1 \rangle}{\|\mathbf{p}_1\|^2}\mathbf{p}_1$$

We need to find each of these components. *What is the inner product* $\langle \mathbf{v}_2, \ \mathbf{p}_1 \rangle$ *equal to?*

$$\langle \mathbf{v}_2, \ \mathbf{p}_1 \rangle = \begin{pmatrix} 2 \\ 1 \\ 1 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix} = (2 \times 1) + (1 \times 2) + (1 \times 3) + (0 \times 4) = 7$$

We also need to find $\|\mathbf{p}_1\|^2$:

$$\|\mathbf{p}_1\|^2 = \mathbf{p}_1 \cdot \mathbf{p}_1 = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix} = 1^2 + 2^2 + 3^2 + 4^2 = 30$$

Substituting these, $\langle \mathbf{v}_2, \ \mathbf{p}_1 \rangle = 7$ and $\|\mathbf{p}_1\|^2 = 30$, into the above formula gives

$$\mathbf{p}_2 = \mathbf{v}_2 - \frac{\langle \mathbf{v}_2, \ \mathbf{p}_1 \rangle}{\|\mathbf{p}_1\|^2}\mathbf{p}_1 = \mathbf{v}_2 - \frac{7}{30}\mathbf{p}_1$$

Substituting the vectors into this yields:

$$\mathbf{p}_2 = \mathbf{v}_2 - \frac{7}{30}\mathbf{p}_1 = \begin{pmatrix}2\\1\\1\\0\end{pmatrix} - \frac{7}{30}\begin{pmatrix}1\\2\\3\\4\end{pmatrix} \qquad \left[\text{Substituting } \mathbf{v}_2 = \begin{pmatrix}2\\1\\1\\0\end{pmatrix} \text{ and } \mathbf{p}_1 = \begin{pmatrix}1\\2\\3\\4\end{pmatrix}\right]$$

$$= \begin{pmatrix}2-7/30\\1-14/30\\1-21/30\\0-28/30\end{pmatrix} = \begin{pmatrix}53/30\\16/30\\9/30\\-28/30\end{pmatrix} = \frac{1}{30}\begin{pmatrix}53\\16\\9\\-28\end{pmatrix}$$

Ignore the scalar (fraction) to make the computation easier so that
$\mathbf{p}'_2 = \begin{pmatrix}53 & 16 & 9 & -28\end{pmatrix}^T$.

What else do we need to find?

The perpendicular vector $\mathbf{p}_3$by applying Gram Schmidt Process (4-10) which is

$$\mathbf{p}_3 = \mathbf{v}_3 - \frac{\langle \mathbf{v}_3,\ \mathbf{p}_1\rangle}{\|\mathbf{p}_1\|^2}\mathbf{p}_1 - \frac{\langle \mathbf{v}_3,\ \mathbf{p}'_2\rangle}{\|\mathbf{p}'_2\|^2}\mathbf{p}'_2 \qquad (*)$$

Evaluating each of the components gives:

$$\langle \mathbf{v}_3,\ \mathbf{p}_1\rangle = \mathbf{v}_3 \cdot \mathbf{p}_1 = \begin{pmatrix}3\\0\\-1\\3\end{pmatrix} \cdot \begin{pmatrix}1\\2\\3\\4\end{pmatrix} = (3\times 1) + (0\times 2) + (-1\times 3) + (3\times 4) = 12$$

$$\langle \mathbf{v}_3,\ \mathbf{p}'_2\rangle = \mathbf{v}_3 \cdot \mathbf{p}'_2 = \begin{pmatrix}3\\0\\-1\\3\end{pmatrix} \cdot \begin{pmatrix}53\\16\\9\\-28\end{pmatrix} = (3\times 53) + (0\times 16) + (-1\times 9) + (3\times(-28)) = 66$$

$$\|\mathbf{p}'_2\|^2 = \mathbf{p}'_2 \cdot \mathbf{p}'_2 = \begin{pmatrix}53\\16\\9\\-28\end{pmatrix} \cdot \begin{pmatrix}53\\16\\9\\-28\end{pmatrix} = 53^2 + 16^2 + 9^2 + (-28)^2 = 3930$$

Substituting all these, $\langle \mathbf{v}_3,\ \mathbf{p}_1\rangle = 12$, $\|\mathbf{p}_1\|^2 = 30$, $\langle \mathbf{v}_3,\ \mathbf{p}'_2\rangle = 66$, and $\|\mathbf{p}'_2\|^2 = 3930$ into (*) yields:

$$\mathbf{p}_3 = \mathbf{v}_3 - \frac{\langle \mathbf{v}_3,\ \mathbf{p}_1\rangle}{\|\mathbf{p}_1\|^2}\mathbf{p}_1 - \frac{\langle \mathbf{v}_3,\ \mathbf{p}'_2\rangle}{\|\mathbf{p}'_2\|^2}\mathbf{p}'_2$$

$$= \mathbf{v}_3 - \frac{12}{30}\mathbf{p}_1 - \frac{66}{3930}\mathbf{p}'_2 = \mathbf{v}_3 - \frac{2}{5}\mathbf{p}_1 - \frac{11}{655}\mathbf{p}'_2 \qquad \left[\text{Because } \frac{12}{30} = \frac{2}{5},\quad \frac{66}{3930} = \frac{11}{655}\right]$$

Substituting the vectors $\mathbf{p}_1 = \begin{pmatrix}1\\2\\3\\4\end{pmatrix}$, $\mathbf{p}'_2 = \begin{pmatrix}53\\16\\9\\-28\end{pmatrix}$ and $\mathbf{v}_3 = \begin{pmatrix}3\\0\\-1\\3\end{pmatrix}$ into this

$$\mathbf{p}_3 = \mathbf{v}_3 - \frac{2}{5}\mathbf{p}_1 - \frac{11}{655}\mathbf{p}'_2 = \begin{pmatrix} 3 \\ 0 \\ -1 \\ 3 \end{pmatrix} - \frac{2}{5}\begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix} - \frac{11}{655}\begin{pmatrix} 53 \\ 16 \\ 9 \\ -28 \end{pmatrix}$$

$$= \begin{pmatrix} 3 \\ 0 \\ -1 \\ 3 \end{pmatrix} - \begin{pmatrix} 2/5 \\ 4/5 \\ 6/5 \\ 8/5 \end{pmatrix} - \begin{pmatrix} 583/655 \\ 176/655 \\ 99/655 \\ -308/655 \end{pmatrix} \qquad \begin{bmatrix} \text{Carrying Out} \\ \text{Scalar Multiplication} \end{bmatrix}$$

$$= \begin{pmatrix} 3-2/5-583/655 \\ 0-4/5-176/655 \\ -1-6/5-99/655 \\ 3-8/5+308/655 \end{pmatrix} = \begin{pmatrix} 224/131 \\ -140/131 \\ -308/131 \\ 245/131 \end{pmatrix} = \frac{1}{131}\begin{pmatrix} 224 \\ -140 \\ -308 \\ 245 \end{pmatrix}$$

Ignore the fraction and let $\mathbf{p}'_3 = \begin{pmatrix} 224 & -140 & -308 & 245 \end{pmatrix}^T$. We need to convert each of the 3 perpendicular vectors

$$\mathbf{p}_1 = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix}, \quad \mathbf{p}'_2 = \begin{pmatrix} 53 \\ 16 \\ 9 \\ -28 \end{pmatrix} \text{ and } \mathbf{p}'_3 = \begin{pmatrix} 224 \\ -140 \\ -308 \\ 245 \end{pmatrix}$$

into unit vectors (normalize). *How?*

By dividing each vector by its norm (length). From above the length of $\mathbf{p}_1$ is $\|\mathbf{p}_1\| = \sqrt{30}$:

$$\mathbf{u}_1 = \frac{1}{\|\mathbf{p}_1\|}\mathbf{p}_1 = \frac{1}{\sqrt{30}}\begin{pmatrix} 1 & 2 & 3 & 4 \end{pmatrix}^T$$

Similarly the length of the perpendicular vector $\mathbf{p}'_2$ is $\|\mathbf{p}'_2\| = \sqrt{3930}$:

$$\mathbf{u}_2 = \frac{1}{\|\mathbf{p}'_2\|}\mathbf{p}'_2 = \frac{1}{\sqrt{3930}}\begin{pmatrix} 53 & 16 & 9 & -28 \end{pmatrix}^T$$

The norm (length) of the last perpendicular vector is

$$\|\mathbf{p}'_3\| = \sqrt{\begin{pmatrix} 224 \\ -140 \\ -308 \\ 245 \end{pmatrix} \cdot \begin{pmatrix} 224 \\ -140 \\ -308 \\ 245 \end{pmatrix}} = \sqrt{224^2 + (-140)^2 + (-308)^2 + (245)^2} = \sqrt{224665}$$

Therefore $\mathbf{u}_3 = \frac{1}{\|\mathbf{p}'_3\|}\mathbf{p}'_3 = \frac{1}{\sqrt{224665}}\begin{pmatrix} 224 & -140 & -308 & 245 \end{pmatrix}^T$. Thus an orthonormal basis for the given subspace is $\mathbf{u}_1 = \frac{1}{\sqrt{30}}\begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix}$, $\mathbf{u}_2 = \frac{1}{\sqrt{3930}}\begin{pmatrix} 53 \\ 16 \\ 9 \\ -28 \end{pmatrix}$, $\mathbf{u}_3 = \frac{1}{\sqrt{224665}}\begin{pmatrix} 224 \\ -140 \\ -308 \\ 245 \end{pmatrix}$.

6. We need to normalize the orthogonal polynomials:

$$\mathbf{p}_1 = 1, \quad \mathbf{p}_2 = x \quad \text{and} \quad \mathbf{p}_3 = x^2 - \frac{1}{3}$$

How?
By dividing each vector by its norm. The length of $\mathbf{p}_1$ is:

$$\|\mathbf{p}_1\|^2 = \langle \mathbf{p}_1,\ \mathbf{p}_1 \rangle = \int_{-1}^{1} (1)(1)\,\mathrm{d}x = [x]_{-1}^{1} = [1-(-1)] = 2$$

Hence $\mathbf{p}_1 = \frac{1}{\sqrt{2}}$.

From Example 17 we have $\|\mathbf{p}_2\|^2 = \frac{2}{3}$ therefore $\|\mathbf{p}_2\| = \sqrt{\frac{2}{3}}$. Hence

$$\mathbf{p}_2 = \frac{x}{\sqrt{2/3}} = \sqrt{\frac{3}{2}}x$$

We need to evaluate the norm of the vector $\mathbf{p}_3 = x^2 - \frac{1}{3}$:

$$\begin{aligned}\|\mathbf{p}_3\|^2 = \langle \mathbf{p}_3,\ \mathbf{p}_3 \rangle &= \int_{-1}^{1} \left(x^2 - \frac{1}{3}\right)\left(x^2 - \frac{1}{3}\right)\mathrm{d}x \\ &= \int_{-1}^{1} \left(x^4 - \frac{2}{3}x^2 + \frac{1}{9}\right)\mathrm{d}x \\ &= \frac{1}{9}\int_{-1}^{1} \left(9x^4 - 6x^2 + 1\right)\mathrm{d}x \\ &= \frac{1}{9}\left[\frac{9x^5}{5} - 2x^3 + x\right]_{-1}^{1} = \frac{2}{9}\left[\frac{9}{5} - 2 + 1\right] = \frac{8}{45}\end{aligned}$$

Taking the square root gives $\|\mathbf{p}_3\| = \sqrt{\frac{8}{45}}$. Our orthonormal basis is

$$\mathbf{p}_1 = \frac{1}{\sqrt{2}},\ \ \mathbf{p}_2 = \sqrt{\frac{3}{2}}x \ \text{ and } \ \mathbf{p}_3 = \frac{1}{\sqrt{8/45}}\left(x^2 - \frac{1}{3}\right) = \sqrt{\frac{45}{8}}\left(x^2 - \frac{1}{3}\right)$$

7. We have $\mathbf{p}_1 = \mathbf{v}_1 = x^2$. Applying Gram – Schmidt Process:

$$\mathbf{p}_2 = \mathbf{v}_2 - \frac{\langle \mathbf{v}_2,\ \mathbf{p}_1 \rangle}{\|\mathbf{p}_1\|^2}\mathbf{p}_1 \qquad (*)$$

We need to find each of these components. *What is the inner product* $\langle \mathbf{v}_2,\ \mathbf{p}_1 \rangle$ *equal to?*

$$\langle \mathbf{v}_2,\ \mathbf{p}_1 \rangle = \int_{-1}^{1} x\left(x^2\right)\mathrm{d}x = \int_{-1}^{1} x^3\,\mathrm{d}x = \left[\frac{x^4}{4}\right]_{-1}^{1} = \frac{1}{4}\left[1^4 - (-1)^4\right] = 0$$

Since the inner product is zero so the vectors $\mathbf{v}_2$ and $\mathbf{p}_1$ are already orthogonal. Putting this into (*) yields

$$\mathbf{p}_2 = \mathbf{v}_2 - \frac{0}{\|\mathbf{p}_1\|^2}\mathbf{p}_1 = \mathbf{v}_2 = x \quad [\text{Because we are given } \ \mathbf{v}_2 = x]$$

What else do we need to find?
The perpendicular vector $\mathbf{p}_3$by applying Gram Schmidt Process (4-10) which is

$$\mathbf{p}_3 = \mathbf{v}_3 - \frac{\langle \mathbf{v}_3,\ \mathbf{p}_1 \rangle}{\|\mathbf{p}_1\|^2}\mathbf{p}_1 - \frac{\langle \mathbf{v}_3,\ \mathbf{p}_2 \rangle}{\|\mathbf{p}_2\|^2}\mathbf{p}_2 \qquad (**)$$

Evaluating each of the components gives:

$$\langle \mathbf{v}_3,\ \mathbf{p}_1 \rangle = \int_{-1}^{1} 1(x^2)\,\mathrm{d}x = \int_{-1}^{1} x^2\,\mathrm{d}x = \left[\frac{x^3}{3}\right]_{-1}^{1} = \frac{1}{3}\left[1^3 - (-1)^3\right] = \frac{2}{3}$$

$$\langle \mathbf{v}_3,\ \mathbf{p}_2 \rangle = \int_{-1}^{1} 1(x)\,\mathrm{d}x = \int_{-1}^{1} x\,\mathrm{d}x = \left[\frac{x^2}{2}\right]_{-1}^{1} = \frac{1}{2}\left[1^2 - (-1)^2\right] = 0$$

$$\|\mathbf{p}_1\|^2 = \langle \mathbf{p}_1,\ \mathbf{p}_1 \rangle = \int_{-1}^{1} x^2(x^2)\,\mathrm{d}x = \left[\frac{x^5}{5}\right]_{-1}^{1} = \frac{1}{5}\left[1^5 - (-1)^5\right] = \frac{2}{5}$$

$$\|\mathbf{p}_2\|^2 = \langle \mathbf{p}_2,\ \mathbf{p}_2 \rangle = \int_{-1}^{1} x(x)\,\mathrm{d}x = \int_{-1}^{1} x^2\,\mathrm{d}x = \frac{2}{3}$$

Substituting all these, $\langle \mathbf{v}_3,\ \mathbf{p}_1 \rangle = \frac{2}{3}$, $\|\mathbf{p}_1\|^2 = \frac{2}{5}$, $\langle \mathbf{v}_3,\ \mathbf{p}_2 \rangle = 0$, and $\|\mathbf{p}_2\|^2 = \frac{2}{3}$ into (**) yields:

$$\begin{aligned}\mathbf{p}_3 &= \mathbf{v}_3 - \frac{\langle \mathbf{v}_3,\ \mathbf{p}_1 \rangle}{\|\mathbf{p}_1\|^2}\mathbf{p}_1 - \frac{\langle \mathbf{v}_3,\ \mathbf{p}_2 \rangle}{\|\mathbf{p}_2\|^2}\mathbf{p}_2 \\ &= 1 - \frac{2/3}{2/5}(x^2) - \frac{0}{2/3}x = 1 - \frac{5}{3}x^2\end{aligned}$$

Our 3 orthogonal polynomials (vectors) are

$$\mathbf{p}_1 = x^2,\quad \mathbf{p}_2 = x \quad \text{and} \quad \mathbf{p}_3 = 1 - \frac{5}{3}x^2$$

Note that we have a different orthogonal basis if we change the order of our standard basis for the inner product space P_2.

8. We need to write each polynomial as a linear combination of:

$$\mathbf{p}_1 = 1,\quad \mathbf{p}_2 = x \quad \text{and} \quad \mathbf{p}_3 = \frac{1}{2}(3x^2 - 1)$$

(a) We have

$$\begin{aligned}k_1\mathbf{p}_1 + k_2\mathbf{p}_2 + k_3\mathbf{p}_3 &= k_1(1) + k_2(x) + k_3\left(\frac{1}{2}(3x^2 - 1)\right) \\ &= \frac{3}{2}k_3(x^2) + k_2 x + \left(k_1 - \frac{k_3}{2}\right) = x^2 + x + 1\end{aligned}$$

Equating coefficients of

x^2: $\frac{3}{2}k_3 = 1 \Rightarrow k_3 = \frac{2}{3}$

x: $k_2 = 1$

Const: $k_1 - \frac{k_3}{2} = 1 \Rightarrow k_1 - \frac{2}{3(2)} = 1 \Rightarrow k_1 = \frac{4}{3}$

Our linear combination is $\frac{4}{3}\mathbf{p}_1 + \mathbf{p}_2 + \frac{2}{3}\mathbf{p}_3 = x^2 + x + 1$. (By expanding out check this result.)

(b) Similarly for $2x^2 - 1$ we have

$$k_1\mathbf{p}_1+k_2\mathbf{p}_2+k_3\mathbf{p}_3=\frac{3}{2}k_3\left(x^2\right)+k_2x+\left(k_1-\frac{k_3}{2}\right)=2x^2-1$$

Equating coefficients of

x^2: $\frac{3}{2}k_3=2 \Rightarrow k_3=\frac{4}{3}$

x: $k_2=0$

Const: $k_1-\frac{k_3}{2}=-1 \Rightarrow k_1-\frac{4}{3(2)}=-1 \Rightarrow k_1=-\frac{1}{3}$

The linear combination is $-\frac{1}{3}\mathbf{p}_1+\frac{4}{3}\mathbf{p}_3=2x^2-1$.

(c) We have

$$k_1\mathbf{p}_1+k_2\mathbf{p}_2+k_3\mathbf{p}_3=\frac{3}{2}k_3\left(x^2\right)+k_2x+\left(k_1-\frac{k_3}{2}\right)=3$$

Equating coefficients of

x^2: $\frac{3}{2}k_3=0 \Rightarrow k_3=0$

x: $k_2=0$

Const: $k_1-\frac{k_3}{2}=3 \Rightarrow k_1-0=3 \Rightarrow k_1=3$

Hence $3\mathbf{p}_1=3$.

(d) We only have the quadratic term x^2 so:

$$k_1\mathbf{p}_1+k_2\mathbf{p}_2+k_3\mathbf{p}_3=\frac{3}{2}k_3\left(x^2\right)+k_2x+\left(k_1-\frac{k_3}{2}\right)=x^2:$$

Equating coefficients of

x^2: $\frac{3}{2}k_3=1 \Rightarrow k_3=\frac{2}{3}$

x: $k_2=0$

Const: $k_1-\frac{k_3}{2}=0 \Rightarrow k_1-\frac{2}{3(2)}=0 \Rightarrow k_1=\frac{1}{3}$

We have $\frac{1}{3}\mathbf{p}_1+\frac{2}{3}\mathbf{p}_3=x^2$.

(e) We need to construct $5x+2$ from the linear combination of

$$\mathbf{p}_1=1,\quad \mathbf{p}_2=x \quad\text{and}\quad \mathbf{p}_3=\frac{1}{2}\left(3x^2-1\right)$$

$$k_1\mathbf{p}_1+k_2\mathbf{p}_2+k_3\mathbf{p}_3=\frac{3}{2}k_3\left(x^2\right)+k_2x+\left(k_1-\frac{k_3}{2}\right)=5x+2:$$

Equating coefficients of

x^2: $\frac{3}{2}k_3=0 \Rightarrow k_3=0$

x: $k_2=5$

Const: $k_1-\frac{k_3}{2}=2 \Rightarrow k_1-0=2 \Rightarrow k_1=2$.

Hence $2\mathbf{p}_1+5\mathbf{p}_2=5x+2$.

9. We need to prove Proposition (4-11) which claims:
Let $\{\mathbf{v}_1,\ \mathbf{v}_2,\ \mathbf{v}_3,\ \cdots,\ \mathbf{v}_n\}$ be an *orthonormal* set of vectors in an inner product space *V* of dimension *n*. Let **u** be a vector in V then

$$\mathbf{u} = \langle \mathbf{u},\ \mathbf{v}_1\rangle \mathbf{v}_1 + \langle \mathbf{u},\ \mathbf{v}_2\rangle \mathbf{v}_2 + \langle \mathbf{u},\ \mathbf{v}_3\rangle \mathbf{v}_3 + \cdots + \langle \mathbf{u},\ \mathbf{v}_n\rangle \mathbf{v}_n$$

Proof.
Since we are given that the vectors $\{\mathbf{v}_1,\ \mathbf{v}_2,\ \mathbf{v}_3,\ \cdots,\ \mathbf{v}_n\}$ are orthonormal so they are linearly independent. Hence these vectors form a basis for the *n* dimensional vector space. Let **u** be a vector in *V* then we can write this vector as a linear combination of $\{\mathbf{v}_1, \cdots,\ \mathbf{v}_n\}$:

$$\mathbf{u} = k_1\mathbf{v}_1 + k_2\mathbf{v}_2 + k_3\mathbf{v}_3 + \cdots + k_n\mathbf{v}_n \qquad (*)$$

where *k*'s are scalars. Required to prove that

$$k_1 = \langle \mathbf{u},\ \mathbf{v}_1\rangle,\ k_2 = \langle \mathbf{u},\ \mathbf{v}_2\rangle,\ k_3 = \langle \mathbf{u},\ \mathbf{v}_3\rangle,\ \ \cdots,\ \ k_n = \langle \mathbf{u},\ \mathbf{v}_n\rangle$$

We just prove that $k_j = \langle \mathbf{u},\ \mathbf{v}_j\rangle$ for any $1 \le j \le n$. Consider the inner product:

$$\langle \mathbf{u},\ \mathbf{v}_j\rangle \quad \text{where } 1 \le j \le n$$

Substituting for the vector **u** the linear combination in (*) we have

$$\begin{aligned}
\langle \mathbf{u},\ \mathbf{v}_j\rangle &= \langle k_1\mathbf{v}_1 + k_2\mathbf{v}_2 + k_3\mathbf{v}_3 + \cdots + k_n\mathbf{v}_n,\ \mathbf{v}_j\rangle \\
&= \langle k_1\mathbf{v}_1, \mathbf{v}_j\rangle + \langle k_2\mathbf{v}_2,\ \mathbf{v}_j\rangle + \cdots + \langle k_j\mathbf{v}_j,\ \mathbf{v}_j\rangle + \cdots + \langle k_n\mathbf{v}_n,\ \mathbf{v}_j\rangle \\
&= k_1\underbrace{\langle \mathbf{v}_1,\ \mathbf{v}_j\rangle}_{=0} + k_2\underbrace{\langle \mathbf{v}_2,\ \mathbf{v}_j\rangle}_{=0} + \cdots + k_j\langle \mathbf{v}_j,\ \mathbf{v}_j\rangle + \cdots + k_n\underbrace{\langle \mathbf{v}_n,\ \mathbf{v}_j\rangle}_{=0} \\
&= 0 + \underbrace{0 + \cdots}_{=0} + k_j\langle \mathbf{v}_j,\ \mathbf{v}_j\rangle + \underbrace{\cdots + 0}_{=0} && \text{[Because the vectors are orthogonal]} \\
&= k_j\|\mathbf{v}_j\|^2 = k_j(1) = k_j && \text{[Because the vectors are normalized]}
\end{aligned}$$

Hence we have $\langle \mathbf{u},\ \mathbf{v}_j\rangle = k_j$ which is our required result.

■