Complete Solutions to **Miscellaneous Exercise 3**

1. *How do we find the null space of the given matrix?*

The solution space N of $\mathbf{Ax} = \mathbf{O}$ is the null space of matrix **A**. We examine

$$\mathbf{Ax} = \begin{bmatrix} 1 & -2 & 1 & 1 \\ -1 & 2 & 0 & 1 \\ 2 & -4 & 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Executing the following elementary row operations on matrix **A** we have

$$\begin{array}{c} R_1 \\ R_2 \\ R_3 \end{array} \begin{bmatrix} 1 & -2 & 1 & 1 \\ -1 & 2 & 0 & 1 \\ 2 & -4 & 1 & 0 \end{bmatrix} \Rightarrow \begin{array}{c} R_1 \\ R_2{*} = R_2 + R_1 \\ R_3{*} = R_3 - 2R_1 \end{array} \begin{bmatrix} 1 & -2 & 1 & 1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & -1 & -2 \end{bmatrix}$$

Carrying out the row operation $R_3{*} + R_2{*}$:

$$\begin{array}{c} R_1 \\ R_2{*} \\ R_3{**} = R_3{*} + R_2{*} \end{array} \begin{bmatrix} 1 & -2 & 1 & 1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Carrying out the row operation $R_1 - R_2{*}$:

$$\begin{array}{c} R_1{*} = R_1 - R_2{*} \\ R_2{*} \\ R_3{**} \end{array} \begin{bmatrix} 1 & -2 & 0 & -1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \mathbf{R}$$

This is now in reduced row echelon form **R**. We consider the equivalent linear system $\mathbf{Rx} = \mathbf{O}$:

$$\begin{bmatrix} 1 & -2 & 0 & -1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

From the middle row we have

$$z + 2w = 0 \text{ which gives } z = -2w$$

Let $w = t$ then $z = -2w = -2t$ where t is any real number. From the top row we have

$$x - 2y - w = 0 \text{ implies } x = 2y + w$$

Let $y = s$ where s is any real number. Therefore $x = 2y + w = 2s + t$.

The null space of the matrix **A** is the vector **x** whose entries are given by $x = 2s + t$, $y = s$, $z = -2t$ and $w = t$:

$$\mathbf{x} = \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} 2s + t \\ s \\ -2t \\ t \end{bmatrix} = s\begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t\begin{bmatrix} 1 \\ 0 \\ -2 \\ 1 \end{bmatrix}$$

Thus a basis for the null space is $N = \left\{ \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -2 \\ 1 \end{bmatrix} \right\}$.

2. By Proposition (3-26) part (a) we have:

$$rank(\mathbf{A}) = rank(\mathbf{A} \mid \mathbf{b}) = n \quad \Leftrightarrow \text{ the linear system has a } \textbf{unique} \text{ solution.}$$

Since there are 5 columns in a 7×5 matrix therefore $rank(\mathbf{A}) = 5$.

By Proposition (3-23) the solution of $\mathbf{AX} = \mathbf{B}$ is of the form $\mathbf{x}_H + \mathbf{x}_P$ where $\mathbf{x}_H$ is the homogeneous solution, that is the solution to $\mathbf{AX} = \mathbf{O}$, and $\mathbf{x}_P$ is the particular solution of $\mathbf{AX} = \mathbf{B}$. We are given that $\mathbf{x}_H$ is unique and $\mathbf{x}_P$ being a particular solution we have $\mathbf{x}_H + \mathbf{x}_P$ is unique.

Summarising the above, $rank(\mathbf{A}) = 5$ and $\mathbf{AX} = \mathbf{B}$ is uniquely solvable, so selection ② is correct option.

3. To find a basis for the column space of $\mathbf{A}$ we transpose the given matrix:

$$\mathbf{A}^T = \begin{bmatrix} 1 & 3 & -2 & 0 & 2 \\ 2 & 6 & -5 & -2 & 4 \\ 0 & 0 & 5 & 10 & 0 \\ 2 & 6 & 0 & 8 & 4 \end{bmatrix}^T = \begin{bmatrix} 1 & 2 & 0 & 2 \\ 3 & 6 & 0 & 6 \\ -2 & -5 & 5 & 0 \\ 0 & -2 & 10 & 8 \\ 2 & 4 & 0 & 4 \end{bmatrix}$$

Applying elementary row operations we have

$$\begin{array}{c} R_1 \\ R_2 \\ R_3 \\ R_4 \\ R_5 \end{array} \begin{bmatrix} 1 & 2 & 0 & 2 \\ 3 & 6 & 0 & 6 \\ -2 & -5 & 5 & 0 \\ 0 & -2 & 10 & 8 \\ 2 & 4 & 0 & 4 \end{bmatrix} \Rightarrow \begin{array}{c} R_1 \\ R_2{*} = R_2 - 3R_1 \\ R_3{*} = R_3 + 2R_1 \\ R_4 \\ R_5{*} = R_5 - 2R_1 \end{array} \begin{bmatrix} 1 & 2 & 0 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 5 & 4 \\ 0 & -2 & 10 & 8 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Executing the elementary row operation $2R_3{*} - R_4$ we have

$$\begin{array}{c} R_1 \\ R_2{*} \\ R_3{**} = 2R_3{*} - R_4 \\ R_4 \\ R_5{*} \end{array} \begin{bmatrix} 1 & 2 & 0 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & -2 & 10 & 8 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Interchanging second and fourth rows gives

$$\begin{bmatrix} 1 & 2 & 0 & 2 \\ 0 & -2 & 10 & 8 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The non-zero rows are a basis for the column space:

$$\left\{ \begin{pmatrix} 1 \\ 2 \\ 0 \\ 2 \end{pmatrix}, \begin{pmatrix} 0 \\ -2 \\ 10 \\ 8 \end{pmatrix} \right\}$$

These are columns 1 and 4 of the given matrix, therefore $C_1,\ C_4$ are a basis for the column space of **A**. Hence correct selection is ⑥.

4. (a) *Proof.*
We have $\mathbf{Av}=\mathbf{b}$ and $\mathbf{Aw}=\mathbf{O}$. Using these gives

$$\begin{aligned}\mathbf{A}(\mathbf{v}+\mathbf{w})&=\mathbf{Av}+\mathbf{Aw}\\&=\mathbf{b}+\mathbf{O}=\mathbf{b}\end{aligned}$$

Hence $\mathbf{x}=\mathbf{v}+\mathbf{w}$ is a solution to $\mathbf{Ax}=\mathbf{b}$.

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(b) *Proof.*
We have $\mathbf{Au}=\mathbf{b}$ and $\mathbf{Ax}=\mathbf{b}$

$$\begin{aligned}\mathbf{A}(\mathbf{x}-\mathbf{u})&=\mathbf{Ax}-\mathbf{Au}\\&=\mathbf{b}-\mathbf{b}=\mathbf{O}\end{aligned}$$

Hence $\mathbf{w}=\mathbf{x}-\mathbf{u}$ is a solution to $\mathbf{Aw}=\mathbf{O}$.

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5. If $\mathbf{Ax}=\mathbf{b}$ has exactly one solution then
   i. Matrix **A** is invertible (non-singular).
   ii. $\det(\mathbf{A})\neq 0$
   iii. $rank(\mathbf{A})=n$

6. (a) This part **cannot** be true because a homogeneous linear system

$$\mathbf{Ax}=\mathbf{O}$$

has a solution such as the trivial solution $\mathbf{x}=\mathbf{O}$.
(b) For part b let **A** be a m by n matrix and **B** be a q by p matrix:

$$\mathbf{A}=\begin{pmatrix}a_{11}&\cdots&a_{1n}\\\vdots&\ddots&\vdots\\a_{m1}&\cdots&a_{mn}\end{pmatrix}\text{ and }\mathbf{B}=\begin{pmatrix}b_{11}&\cdots&b_{1p}\\\vdots&\ddots&\vdots\\b_{q1}&\cdots&b_{qp}\end{pmatrix}$$

To carry out the matrix multiplication $\mathbf{AB}$ the number of columns of matrix **A** which is n must equal the number of rows of matrix **B** which is q. We have $n=q$.
Similarly for the matrix multiplication **BA** we must have the number of columns of matrix **B** which is p must equal the number of rows of matrix **A** which is m. Hence we have $p=m$. Thus we can have different size matrices for **A** $(m\times n)$ and **B** $(n\times m)$.
Take **B** to be the n by m zero matrix so we have

$$\mathbf{AB}=\mathbf{BA}=\mathbf{O}$$

Hence statement b is possible.
(c) *What is the dimension of $\mathbb{R}^4$?*
4. By Theorem (3-13) (b) which says:

Any spanning set of n vectors forms a basis for V.

Hence a set of four vectors that spans $\mathbb{R}^4$ is a basis for $\mathbb{R}^4$. Thus statement c is false.
(d) The set of 2×2 matrices has dimension 4 but we can form a subspace of dimension 3 with the following 2×2 matrices:

$$\begin{pmatrix}1&0\\0&0\end{pmatrix},\ \begin{pmatrix}0&1\\0&0\end{pmatrix}\text{ and }\begin{pmatrix}0&0\\1&0\end{pmatrix}$$

This means that it is possible to form a three dimensional subspace by the set of 2×2 matrices. Hence statement d is correct.

7. Part (a) is true because any *n* linearly independent vectors in a *n* dimensional vector space is a basis for that space therefore the vectors must span the space.
Checking Statement (b):
Remember P_2 is the vector space of quadratic polynomials

$$P_2=\left\{ax^2+bx+c \mid a\in\mathbb{R},\ b\in\mathbb{R},\ c\in\mathbb{R}\right\}$$

Let $p\in P_2$ such that $p(0)=2$, that is

$$p(0)=a\left(0^2\right)+b(0)+c=2 \text{ which gives } c=2$$

The dimension of P_2 is 3 so we only need 3 linearly independent vectors to be a basis for P_2. Consider the 3 vectors $\left\{2,\ x+2,\ x^2+2\right\}$, these are linearly independent and satisfy $p(0)=2$. Hence they are a basis for P_2.
Hence statement (b) is true.
Checking Statement (c):
Since we have the fundamental trigonometric identity

$$\sin^2(x)+\cos^2(x)=1$$

therefore $\left\{1,\ \sin^2(x),\ \cos^2(x)\right\}$ is **not** linearly independent. Statement (c) is false.
Only statements (a) and (b) are correct.

8. Statement (a) is true because if $\{\mathbf{u},\ \mathbf{v}\}$ is linearly dependent then

$$\mathbf{u}=m\mathbf{v} \text{ where } m \text{ is a scalar}$$

This means that $\{\mathbf{u},\ \mathbf{v},\ \mathbf{w}\}=\{\mathbf{u},\ m\mathbf{u},\ \mathbf{w}\}$ is linearly dependent because

$$\mathbf{u}-m\mathbf{u}+0\mathbf{w}=\mathbf{O}$$

But we are given that $\{\mathbf{u},\ \mathbf{v},\ \mathbf{w}\}$ is linearly independent. Hence our supposition $\{\mathbf{u},\ \mathbf{v}\}$ is linearly dependent must be wrong so $\{\mathbf{u},\ \mathbf{v}\}$ is linearly independent.
Statement (b):
This statement is also true because linearly independent vectors in any spanning set of V form a basis for *V*.
Statement (c):
This is also true because of Theorem (3-13) part (a) which says:

> Any linearly independent set of *n* vectors in *V* forms a basis for *V* where $\dim(V)=n$.

All three statements are true.

9. We need to find the following:
(a) The rank of **A** – this is the number of non-zero rows of the reduced row echelon form matrix **B** which is 3. Thus $rank(\mathbf{A})=3$.
(b) The nullity of **A** can be found by applying Theorem (3-22) which is

$$nullity(\mathbf{A})+rank(\mathbf{A})=n$$

where *n* is the number of columns of matrix **A**. Since for the given matrix **A** we have 5 columns therefore $n=5$ and substituting $rank(\mathbf{A})=3$ into this

$$nullity(\mathbf{A})+3=5 \text{ gives } nullity(\mathbf{A})=2$$

(c) *How do we find a basis for the column space of* ***A****?*

Transpose matrix **A** and then place it into reduced row echelon form:

$$\mathbf{A}^T=\begin{bmatrix}1 & 0 & -2 & 1 & 3\\ -1 & 1 & 5 & -1 & -3\\ 0 & 2 & 6 & 0 & 1\\ 1 & 1 & 1 & 1 & 4\end{bmatrix}^T=\begin{bmatrix}1 & -1 & 0 & 1\\ 0 & 1 & 2 & 1\\ -2 & 5 & 6 & 1\\ 1 & -1 & 0 & 1\\ 3 & -3 & 1 & 4\end{bmatrix}$$

Applying elementary row operations:

$$\begin{array}{c}R_1\\R_2\\R_3\\R_4\\R_5\end{array}\begin{bmatrix}1 & -1 & 0 & 1\\ 0 & 1 & 2 & 1\\ -2 & 5 & 6 & 1\\ 1 & -1 & 0 & 1\\ 3 & -3 & 1 & 4\end{bmatrix} \Rightarrow \begin{array}{c}R_1\\R_2\\R_3{*}=R_3+2R_1\\R_4{*}=R_4-R_1\\R_5{*}=R_5-3R_1\end{array}\begin{bmatrix}1 & -1 & 0 & 1\\ 0 & 1 & 2 & 1\\ 0 & 3 & 6 & 3\\ 0 & 0 & 0 & 0\\ 0 & 0 & 1 & 1\end{bmatrix}$$

Carrying out the row operation $R_3{*}-3R_2$ gives:

$$\begin{array}{c}R_1\\R_2\\R_3{**}=R_3{*}-3R_2\\R_4{*}\\R_5{*}\end{array}\begin{bmatrix}1 & -1 & 0 & 1\\ 0 & 1 & 2 & 1\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 1 & 1\end{bmatrix}$$

Executing the row operations $R_1{*}=R_1+R_2$ and $R_1{*}-R_5{*}$, $R_2-2R_5{*}$ gives:

$$\begin{array}{c}R_1{*}=R_1+R_2\\R_2\\R_3{**}\\R_4{*}\\R_5{*}\end{array}\begin{bmatrix}1 & 0 & 2 & 2\\ 0 & 1 & 2 & 1\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 1 & 1\end{bmatrix} \Rightarrow \begin{array}{c}R_1{**}=R_1{*}-2R_5{*}\\R_2-2R_5{*}\\R_3{**}\\R_4{*}\\R_5{*}\end{array}\begin{bmatrix}1 & 0 & 0 & 0\\ 0 & 1 & 0 & -1\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 1 & 1\end{bmatrix}$$

Interchanging rows gives us our reduced row echelon form:

$$\begin{bmatrix}1 & 0 & 0 & 0\\ 0 & 1 & 0 & -1\\ 0 & 0 & 1 & 1\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\end{bmatrix}=\mathbf{R}$$

A basis for the column space of matrix **A** is $\left\{\begin{pmatrix}1\\0\\0\\0\end{pmatrix},\begin{pmatrix}0\\1\\0\\-1\end{pmatrix},\begin{pmatrix}0\\0\\1\\1\end{pmatrix}\right\}$.

(d) A basis for the row space of **A** are the non-zero rows of the reduced row echelon form matrix $\mathbf{B}=\begin{bmatrix}1&0&-2&1&0\\0&1&3&0&0\\0&0&0&0&1\\0&0&0&0&0\end{bmatrix}$. Hence $\left\{\begin{pmatrix}1\\0\\-2\\1\\0\end{pmatrix},\begin{pmatrix}0\\1\\3\\0\\0\end{pmatrix},\begin{pmatrix}0\\0\\0\\0\\1\end{pmatrix}\right\}$ is a basis for the row space of **A**.

(e) A basis for the null space of **A** is given by the solution space **x** such that $\mathbf{Ax}=\mathbf{O}$. We solve the equivalent system $\mathbf{Bx}=\mathbf{O}$.

$$\begin{bmatrix}1&0&-2&1&0\\0&1&3&0&0\\0&0&0&0&1\\0&0&0&0&0\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\\x_4\\x_5\end{bmatrix}=\begin{bmatrix}0\\0\\0\\0\end{bmatrix}$$

From the third row we have $x_5=0$. From the second row we have

$$x_2+3x_3=0 \text{ which gives } x_2=-3x_3$$

Let $x_3=s$ where s is any real number. Thus $x_2=-3x_3=-3s$. From the top row we have

$$x_1-2x_3+x_4=0 \text{ implies } x_1=2x_3-x_4$$

Let $x_4=t$ where t is any real number. Hence $x_1=2s-t$.

Substituting $x_1=2s-t$, $x_2=-3s$, $x_3=s$, $x_4=t$ and $x_5=0$ into the vector **x**:

$$\mathbf{x}=\begin{bmatrix}x_1\\x_2\\x_3\\x_4\\x_5\end{bmatrix}=\begin{bmatrix}2s-t\\-3s\\s\\t\\0\end{bmatrix}=s\begin{bmatrix}2\\-3\\1\\0\\0\end{bmatrix}+t\begin{bmatrix}-1\\0\\0\\1\\0\end{bmatrix}$$

A basis for the null space is $\left\{\begin{bmatrix}2\\-3\\1\\0\\0\end{bmatrix},\begin{bmatrix}-1\\0\\0\\1\\0\end{bmatrix}\right\}$.

10. (a) (i) The vectors $\mathbf{u}_1,\ldots,\ \mathbf{u}_k$ are linearly independent if

$$m_1\mathbf{u}_1+\ldots\ +m_k\mathbf{u}_k=\mathbf{O}\quad\Rightarrow\quad m_1=m_2=\cdots=m_k=0$$

where $m_1,\ m_2,\cdots,\ m_k$ are scalars.

(ii) See (3-16) which is equivalent to:

If **every** vector in U can be produced by a linear combination of these vectors $\mathbf{u}_1,\ldots,\ \mathbf{u}_k$ then we say these vectors **span** or **generate** the subspace U.

(b) Applying elementary row operations to the given matrix we have

$$\begin{array}{c} R_1 \\ R_2 \\ R_3 \end{array}\begin{pmatrix} 1 & 3 & -4 \\ 2 & -1 & -1 \\ -2 & -6 & 8 \end{pmatrix} \Rightarrow \begin{array}{c} R_1 \\ R_2{*} = R_2 - 2R_1 \\ R_3{*} = R_3 + 2R_1 \end{array}\begin{pmatrix} 1 & 3 & -4 \\ 0 & -7 & 7 \\ 0 & 0 & 0 \end{pmatrix}$$

Dividing the middle row by -7 gives

$$\begin{array}{c} R_1 \\ R_2{**} \\ R_3{*} \end{array}\begin{pmatrix} 1 & 3 & -4 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix}$$

Carrying out the row operation $R_1 - 3R_2{**}$ gives

$$\begin{array}{c} R_1{*} = R_1 - 3R_2{**} \\ R_2{**} \\ R_3{*} \end{array}\begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix} = \mathbf{R}$$

The last matrix **R** is the reduced row echelon form of the given matrix **A**. We can find the null space, which is the solution space of $\mathbf{Ax} = \mathbf{O}$, by solving the equivalent system $\mathbf{Rx} = \mathbf{O}$. We have

$$\mathbf{Rx} = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

From the middle row we have $y - z = 0$ which gives $y = z$.

Let $z = s$ where *s* is any real number then $y = z = s$. From the first row we have

$$x - z = 0 \Rightarrow x = z = s$$

Thus our null space of **A** is the solution space **x** which has the entries $x = z = y = s$,

$$\mathbf{x} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} s \\ s \\ s \end{pmatrix} = s\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \text{ where } s \in \mathbb{R}$$

The null space of the matrix **A** is given by $\left\{ s\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \middle| \; s \in \mathbb{R} \right\}$.

(c) Using the definition given in part (a) (i) with m_1, m_2 and m_3 as scalars we have:

$$m_1\begin{bmatrix} -5 \\ 4 \\ -6 \end{bmatrix} + m_2\begin{bmatrix} -1 \\ 2 \\ -1 \end{bmatrix} + m_3\begin{bmatrix} 0 \\ -2 \\ 3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

If we expand this and solve for m_1, m_2 and m_3 we get

$$m_1 = m_2 = m_3 = 0$$

Thus the given vectors $\mathbf{v}_1$, $\mathbf{v}_2$ and $\mathbf{v}_3$ are linearly independent.

11. (a) *V* is a subspace of $\mathbb{R}^n$ if and only if

(i) $\mathbf{O} \in V$

(ii) If $\mathbf{u}, \mathbf{v} \in V$ then the linear combination $k\mathbf{u} + c\mathbf{v}$ is also in *V*.

Subspaces of $\mathbb{R}^n$ are the zero vector space $\{\mathbf{O}\}$ and the space $\mathbb{R}^n$ itself.

(b) See solution to question 10(a) above.
(c) Let $k_1,\ k_2,\ k_3$ be scalars such that

$$k_1\begin{pmatrix}1\\1\\1\end{pmatrix}+k_2\begin{pmatrix}2\\3\\-2\end{pmatrix}+k_3\begin{pmatrix}-2\\-5\\10\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix}$$

From these we have the simultaneous equations

$$\left.\begin{array}{rcrcrcr} k_1 & + & 2k_2 & - & 2k_3 & = & 0\\ k_1 & + & 3k_2 & - & 5k_3 & = & 0\\ k_1 & - & 2k_2 & + & 10k_3 & = & 0\end{array}\right\} \text{ gives } k_1=-4,\ k_2=3 \text{ and } k_3=1$$

This means that we have $-4\mathbf{v}_1+3\mathbf{v}_2+\mathbf{v}_3=\mathbf{O}$ so we conclude that $\mathbf{v}_1,\ \mathbf{v}_2$ and $\mathbf{v}_3$ are linearly dependent.
Since the given space is a subspace of $\mathbb{R}^3$ therefore the dimension can only be 0, 1, 2 and 3. Since the subspace spanned by $\mathbf{v}_1,\ \mathbf{v}_2$ and $\mathbf{v}_3$ is linearly dependent therefore it **cannot** be 3. Similarly we do **not** have the zero vector space so the dimension **cannot** be 0. Additionally any two vectors amongst $\mathbf{v}_1,\ \mathbf{v}_2$ and $\mathbf{v}_3$ are linearly independent so dimension **cannot** be 1. Thus dimension is equal to 2.

12. (a) Writing out the augmented matrix and carrying out elementary row operations we have

$$\begin{array}{c} R_1\\ R_2\\ R_3\end{array}\left[\begin{array}{ccc|c} 2 & 6 & -4 & 0\\ -2 & -5 & 4 & 1\\ 4 & 11 & -8 & -1\end{array}\right] \Rightarrow \begin{array}{c} R_1\\ R_2^*=R_2+R_1\\ R_3^*=R_3-2R_1\end{array}\left[\begin{array}{ccc|c} 2 & 6 & -4 & 0\\ 0 & 1 & 0 & 1\\ 0 & -1 & 0 & -1\end{array}\right]$$

Carrying out the elementary row operations $R_1-6R_2^*$ and $R_3^*+R_2^*$:

$$\begin{array}{c} R_1^*=R_1-6R_2^*\\ R_2^*\\ R_3^{**}=R_3^*+R_2^*\end{array}\left[\begin{array}{ccc|c} 2 & 0 & -4 & -6\\ 0 & 1 & 0 & 1\\ 0 & 0 & 0 & 0\end{array}\right]$$

Dividing the top row by 2 gives the reduced row echelon form:

$$\left[\begin{array}{ccc|c} 1 & 0 & -2 & -3\\ 0 & 1 & 0 & 1\\ 0 & 0 & 0 & 0\end{array}\right]=\mathbf{R}$$

(b) Clearly this system has a solution so $rank(\mathbf{A})=rank\left(\left[\mathbf{A}\quad \vec{b}\right]\right)$ otherwise we would have an inconsistent system with zeros on the Left Hand of the vertical bar equal to a non-zero to the Right Hand side of the vertical bar which is impossible in this case.
(c) We solve the equivalent system $\mathbf{R}\vec{x}=\vec{b}$. The solution to this system is the homogeneous solution $\mathbf{x}_H$ plus the particular solution $\mathbf{x}_P$, that is $\vec{x}=\mathbf{x}_H+\mathbf{x}_P$.
Consider the reduced row echelon form augmented matrix $\mathbf{R}$:

$$\left[\begin{array}{ccc|c} 1 & 0 & -2 & -3\\ 0 & 1 & 0 & 1\\ 0 & 0 & 0 & 0\end{array}\right]$$

From the middle row we have $y=1$ and the top row yields

$$x - 2z = -3 \text{ which implies } x = 2z - 3$$

Let $z = s$ where s is any real number. Thus we have $x = 2z - 3 = 2s - 3$. The solution $\vec{x}$ is given by substituting $x = 2s - 3$, $y = 1$ and $z = s$ into $\vec{x}$:

$$\vec{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2s-3 \\ 1 \\ s \end{bmatrix} = \underbrace{s\begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}}_{=\mathbf{x}_H} + \underbrace{\begin{bmatrix} -3 \\ 1 \\ 0 \end{bmatrix}}_{=\mathbf{x}_P}$$

13. Labelling the rows of the given matrix we have

$$\begin{array}{c} R_1 \\ R_2 \\ R_3 \end{array} \begin{pmatrix} 1 & 2 & 3 & 1 & 2 & 3 \\ 2 & 4 & 6 & 1 & 2 & 5 \\ 3 & 6 & 9 & 4 & 8 & 3 \end{pmatrix}$$

Carrying out the elementary row operations $R_2 - 2R_1$ and $R_3 - 3R_1$ gives

$$\begin{array}{c} R_1 \\ R_2{*} = R_2 - 2R_1 \\ R_3{*} = R_3 - 3R_1 \end{array} \begin{pmatrix} 1 & 2 & 3 & 1 & 2 & 3 \\ 0 & 0 & 0 & -1 & -2 & -1 \\ 0 & 0 & 0 & 1 & 2 & -6 \end{pmatrix}$$

Executing $R_1 + R_2{*}$ and $R_3{*} + R_2{*}$ gives

$$\begin{array}{c} R_1{*} = R_1 + R_2{*} \\ R_2{*} \\ R_3{**} = R_3{*} + R_2{*} \end{array} \begin{pmatrix} 1 & 2 & 3 & 0 & 0 & 2 \\ 0 & 0 & 0 & -1 & -2 & -1 \\ 0 & 0 & 0 & 0 & 0 & -7 \end{pmatrix}$$

Multiplying the middle row by -1 and bottom row by $-1/7$ gives us

$$\begin{array}{c} R_1{*} \\ R_2{**} \\ R_3{***} \end{array} \begin{pmatrix} 1 & 2 & 3 & 0 & 0 & 2 \\ 0 & 0 & 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

Executing $R_1{*} - 2R_3{***}$ and $R_2{**} - R_3{***}$ gives us the reduced row echelon form matrix **R**:

$$\begin{array}{c} R_1{**} = R_1{*} - 2R_3{***} \\ R_2{***} = R_2{**} - R_3{***} \\ R_3{***} \end{array} \begin{pmatrix} 1 & 2 & 3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} = \mathbf{R}$$

The null space is the solution space of $\mathbf{Ax} = \mathbf{O}$ which is equivalent to $\mathbf{Rx} = \mathbf{O}$.

$$\mathbf{Rx} = \begin{pmatrix} 1 & 2 & 3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

From the bottom row we have $x_6 = 0$. From the middle row we have

$$x_4 + 2x_5 = 0 \text{ which gives } x_4 = -2x_5$$

Let $x_5 = s$ where s is any real number. Thus $x_4 = -2x_5 = -2s$.

Expanding the top row gives

$$x_1 + 2x_2 + 3x_3 = 0 \text{ yields } x_1 = -2x_2 - 3x_3$$

Let $x_2 = t$ and $x_3 = r$ where *r* and *t* are real numbers. Substituting this, $x_2 = t$ and $x_3 = r$, into $x_1 = -2x_2 - 3x_3$ gives

$$x_1 = -2x_2 - 3x_3 = -2t - 3r$$

Substituting $x_1 = -2t - 3r$, $x_2 = t$, $x_3 = r$, $x_4 = -2s$, $x_5 = s$ and $x_6 = 0$ into the vector **x** gives:

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{pmatrix} = \begin{pmatrix} -2t-3r \\ t \\ r \\ -2s \\ s \\ 0 \end{pmatrix} = s\begin{pmatrix} 0 \\ 0 \\ 0 \\ -2 \\ 1 \\ 0 \end{pmatrix} + t\begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} + r\begin{pmatrix} -3 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

A basis for the null space *N(A)* of the given matrix is

$$\left\{ \begin{pmatrix} 0 \\ 0 \\ 0 \\ -2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -3 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \right\}$$

14. (a) $\mathbf{A} \in M_{m\times n}(\mathbb{R})$ means that the matrix **A** is a $m \times n$ real matrix. The solution space **x** of the homogeneous system $\mathbf{Ax} = \mathbf{O}$ is called the null space, *N(A)*, of matrix **A**. The column space *C(A)* of the matrix **A** is the space spanned by the column vectors of matrix **A**.

The rank of a matrix **A** is dimension of the column space *C(A)*.

The nullity of **A** is the dimension of the null space *N(A)*.

(b) We need to apply elementary row operations to the given matrix:

$$\begin{array}{c} R_1 \\ R_2 \\ R_3 \end{array} \begin{bmatrix} 1 & -1 & 2 & 0 \\ 2 & -2 & 1 & 3 \\ -2 & 2 & -7 & 3 \end{bmatrix} \Rightarrow \begin{array}{r} R_1 \\ R_2{*} = R_2 - 2R_1 \\ R_3{*} = R_3 + 2R_1 \end{array} \begin{bmatrix} 1 & -1 & 2 & 0 \\ 0 & 0 & -3 & 3 \\ 0 & 0 & -3 & 3 \end{bmatrix}$$

Executing the row operations $R_3{**} = R_3{*} - R_2{*}$ and $R_2{**} = -R_2{*}/3$:

$$\begin{array}{r} R_1 \\ R_2{**} \\ R_3{**} \end{array} \begin{bmatrix} 1 & -1 & 2 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Carrying out the row operation $R_1 - 2R_2{**}$ gives the reduced row echelon form of the given matrix **A**:

$$\begin{bmatrix} 1 & -1 & 0 & 2 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \mathbf{R}$$

R is the reduced row echelon form matrix. We can determine the null space of matrix **A** be solving the equivalent $\mathbf{Rx} = \mathbf{O}$:

$$\begin{bmatrix} 1 & -1 & 0 & 2 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

By expanding the middle row we have

$$z - w = 0 \text{ which gives } z = w$$

Let $w = t$ where t is any real number. Hence $z = w = t$. From the top row we have

$$x - y + 2w = 0 \text{ implies } x = y - 2w$$

Let $y = s$ where s is any real number. We have $x = y - 2w = s - 2t$.
Substituting $x = s - 2t$, $y = s$, $z = w = t$ into **x** we have:

$$\mathbf{x} = \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} s-2t \\ s \\ t \\ t \end{bmatrix} = s\begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t\begin{bmatrix} -2 \\ 0 \\ 1 \\ 1 \end{bmatrix}$$

A basis for the null space of matrix **A**, *N(A)*, is $\left\{\begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ 1 \\ 1 \end{bmatrix}\right\}$.

Nullity of matrix **A** is 2 because we have 2 basis vectors for the null space. The rank of **A** is also 2 because we have 2 non-zero rows in the reduced row echelon form matrix **R**.

Let $\mathbf{u} = \begin{pmatrix} 1, & 5, & 1 \end{pmatrix}^T = \begin{pmatrix} 1 \\ 5 \\ 1 \end{pmatrix}$ and $\mathbf{v} = \begin{pmatrix} 0, & 3, & 3 \end{pmatrix}^T = \begin{pmatrix} 0 \\ 3 \\ 3 \end{pmatrix}$.

From above we have that the dimension of the column space is 2 because rank of **A** is 2. We only need 2 linearly independent vectors and vectors **u** and **v** are linearly independent and they are the last column of **A** and addition of the last and first columns of A. Hence they form a basis for the column space of matrix **A**.

15. (a) (i) The given subset $U = \left\{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M(2,\ 2) \;\middle|\; a^2 = d^2\right\}$ is **not** a subspace of $M(2,\ 2)$ because if we consider the matrices $\mathbf{A} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ then both of these are members of the set U but

$$\begin{aligned} k\mathbf{A} + c\mathbf{B} &= k\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + c\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad \text{where } k \text{ and } c \text{ are scalars} \\ &= \begin{pmatrix} k+c & 0 \\ 0 & k-c \end{pmatrix} \end{aligned}$$

is **not** a member of *U*. *Why not?*

Because $(k+c)^2 \neq (k-c)^2$ [Not Equal].

(ii) *What does the notation* P_n *mean?*

P_n is the set of nth degree polynomials, that is polynomials of the type

$$p(x) = c_n x^n + c_{n-1}x^{n-1} + \cdots + c_1 x + c_0$$

What does $p(3) = 0$ *mean?*

$$p(3) = 3^n c_n + 3^{n-1}c_{n-1} + \cdots + 3c_1 + c_0 = 0$$

Let $q(3) \in V$ where

$$q(3) = 3^n a_n + 3^{n-1}a_{n-1} + \cdots + 3a_1 + a_0 = 0$$

Let k_1 and k_2 be scalars. Need to check that $k_1 p(3) + k_2 q(3)$ is also in *V*.

$$\begin{aligned} k_1 p(3) + k_2 q(3) &= k_1\left(3^n c_n + \cdots + 3c_1 + c_0\right) + k_2\left(3^n a_n + \cdots + 3a_1 + a_0\right) \\ &= k_1(0) + k_2(0) = 0 \end{aligned}$$

Hence $k_1 p(3) + k_2 q(3)$ is in *V* and the zero polynomial (zero vector) is also in *V* therefore *V* is a subspace of P_n

(iii) *Is* $W = \{(x,\ y,\ z,\ t) \in \mathbb{R}^4 \mid y = z + t\}$ *a subspace of* $\mathbb{R}^4$ *?*

Need to check two things:

1. The zero vector is in *W*.
2. If **u** and **v** are vectors in *W* then $k\mathbf{u} + c\mathbf{v}$ (*k* and *c* are scalars) are also in *W*.

Clearly the zero vector $(0,\ 0,\ 0,\ 0)$ is in *W* because $x = y = z = t = 0$ and we have $y = z + t = 0$.

Let $\mathbf{u} = \begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix}$ and $\mathbf{w} = \begin{pmatrix} x' \\ y' \\ z' \\ t' \end{pmatrix}$ be members of given set *W*. This means we have $y = z + t$ and $y' = z' + t'$. Consider $k\mathbf{u} + c\mathbf{v}$:

$$k\mathbf{u} + c\mathbf{v} = k\begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix} + c\begin{pmatrix} x' \\ y' \\ z' \\ t' \end{pmatrix} = \begin{pmatrix} kx + cx' \\ ky + cy' \\ kz + cz' \\ kt + ct' \end{pmatrix}$$

What do we need to show?

Required to show that

$$ky + cy' = kz + cz' + kt + ct'$$

$k\mathbf{u} + c\mathbf{v}$ is a member of *W* because

$$\begin{aligned} ky + cy' &= k(z + t) + c(z' + t') \\ &= kz + kt + cz' + ct' \\ &= kz + cz' + kt + ct' \end{aligned}$$

Hence the given set $W = \{(x,\ y,\ z,\ t) \in \mathbb{R}^4 \mid y = z + t\}$ is a subspace of $\mathbb{R}^4$.

(b) The vectors $\begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\end{pmatrix}, \begin{pmatrix}3\\5\\-4\end{pmatrix}$ is a basis for $\mathbb{R}^3$.

(c) (i) $\{(1, \ 0, \ 1), (1, \ 1, \ 0), (0, \ 1, \ 1), (1 \ 1 \ 1)\}$ is **not** a basis for $\mathbb{R}^3$ because this set contains 4 vectors whilst dimension of $\mathbb{R}^3$ is 3 so a basis for $\mathbb{R}^3$ consists of exactly 3 vectors.

They are **not** linearly independent because

$$\frac{1}{2}\begin{pmatrix}1\\0\\1\end{pmatrix}+\frac{1}{2}\begin{pmatrix}1\\1\\0\end{pmatrix}+\frac{1}{2}\begin{pmatrix}0\\1\\1\end{pmatrix}=\begin{pmatrix}1\\1\\1\end{pmatrix}$$

Since the last vector $\begin{pmatrix}1\\1\\1\end{pmatrix}$ is a linear combination of $\begin{pmatrix}1\\0\\1\end{pmatrix}, \begin{pmatrix}1\\1\\0\end{pmatrix}, \begin{pmatrix}0\\1\\1\end{pmatrix}$ therefore the given vectors are linearly dependent.

The given vectors do span $\mathbb{R}^3$ because

$$k_1\begin{pmatrix}1\\0\\1\end{pmatrix}+k_2\begin{pmatrix}1\\1\\0\end{pmatrix}+k_3\begin{pmatrix}0\\1\\1\end{pmatrix}+k_4\begin{pmatrix}1\\1\\1\end{pmatrix}=\begin{pmatrix}a\\b\\c\end{pmatrix}$$

gives the equations

$$\left.\begin{array}{ccccccccc} k_1 & + & k_2 & & & + & k_4 & = & a \\ & & k_2 & + & k_3 & + & k_4 & = & b \\ k_1 & & & + & k_3 & + & k_4 & = & c \end{array}\right\} \text{ which yields}$$

$$k_1=\frac{a+c-b}{2}, \ k_2=\frac{a-c+b}{2}, \ k_3=\frac{c-a+b}{2}, \ k_4=0$$

Since we can generate the arbitrary vector $\begin{pmatrix}a\\b\\c\end{pmatrix}$ with $\begin{pmatrix}1\\0\\1\end{pmatrix}, \begin{pmatrix}1\\1\\0\end{pmatrix}, \begin{pmatrix}0\\1\\1\end{pmatrix}, \begin{pmatrix}1\\1\\1\end{pmatrix}$ therefore these vectors span $\mathbb{R}^3$.

(ii) The dimension of P_2 is 3 therefore it is possible that the given vectors $\{5, \ 2+x-3x^2, \ 4x-1\}$ are a basis for P_2. We need to check two things:

<u>Linear independence</u>:

Let k_1, k_2 and k_3 be scalars such that

$$k_1(5)+k_2(2+x-3x^2)+k_3(4x-1)=0$$

For linear independence what do we need to show?

All the scalars are zero. Expanding the above equation gives

$$-3x^2k_2+(k_2+4k_3)x+(5k_1+2k_2-k_3)=0 \qquad (*)$$

Equating coefficients of (*) gives

x^2: $\quad -3k_2=0$ implies $k_2=0$

x: $\quad k_2+4k_3=0$ implies $k_3=0$ because $k_2=0$

constant: $5k_1 + 2k_2 - k_3 = 0$ implies $k_1 = 0$ because $k_2 = 0$, $k_3 = 0$

Hence $\{5,\ 2 + x - 3x^2,\ 4x - 1\}$ is a linearly independent set of vectors.

These vectors form a basis for P_2 because we have 3 linearly independent vectors in P_2 and dimension of P_2 is 3. This is because of:

> Theorem (3-13). Let V be a finite n-dimensional vector space. We have the following:
> Any linearly independent set of n vectors forms a basis for V.

16. We need to prove that $V + W$ is a subspace of $\mathbb{R}^n$ given that both V and W are subspaces of $\mathbb{R}^n$. *How?*

Show the two conditions of subspaces, these are

i) The zero vector $\mathbf{O}$ is in $V + W$.

ii) If $\mathbf{u}_1$ and $\mathbf{u}_2$ are in $V + W$ then the linear combination $k\mathbf{u}_1 + c\mathbf{u}_2$ is also in $V + W$.

Proof.

Zero Vector:

We are given that $V + W = \{\mathbf{u} \in \mathbb{R}^n \mid \mathbf{u} = \mathbf{v} + \mathbf{w} \text{ for some } \mathbf{v} \in V \text{ and some } \mathbf{w} \in W\}$.

Since V is a subspace of $\mathbb{R}^n$ therefore the zero vector, $\mathbf{O}_V$, is in V. Similarly $\mathbf{O}_W \in W$.
We have

$$\mathbf{O}_V + \mathbf{O}_W = \mathbf{O} \in V + W$$

Thus the zero vector $\mathbf{O}$ is in $V + W$.

Linear Combination:

Let $\mathbf{u}_1$ and $\mathbf{u}_2$ be in $V + W$ then

$$\mathbf{u}_1 = \mathbf{v}_1 + \mathbf{w}_1 \text{ and } \mathbf{u}_2 = \mathbf{v}_2 + \mathbf{w}_2 \text{ where } \mathbf{v}_1, \mathbf{v}_2 \in V \text{ and } \mathbf{w}_1, \mathbf{w}_2 \in W$$

Let k and c be scalars. Consider the linear combination

$$\begin{aligned} k\mathbf{u}_1 + c\mathbf{u}_2 &= k(\mathbf{v}_1 + \mathbf{w}_1) + c(\mathbf{v}_2 + \mathbf{w}_2) \\ &= k\mathbf{v}_1 + k\mathbf{w}_1 + c\mathbf{v}_2 + c\mathbf{w}_2 \\ &= k\mathbf{v}_1 + c\mathbf{v}_2 + k\mathbf{w}_1 + c\mathbf{w}_2 \end{aligned}$$

Since $\mathbf{v}_1, \mathbf{v}_2 \in V$ and V is a subspace therefore $k\mathbf{v}_1 + c\mathbf{v}_2$ is also in V. Similarly because $\mathbf{w}_1, \mathbf{w}_2 \in W$ therefore $k\mathbf{w}_1 + c\mathbf{w}_2$ is also in W. Hence $k\mathbf{u}_1 + c\mathbf{u}_2$ is in $V + W$.

Both the above conditions are satisfied so we conclude that $V + W$ is a subspace of $\mathbb{R}^n$.

■

17. (a) Let k_1, k_2, k_3 and k_4 be scalars such that

$$k_1\begin{bmatrix} 1 & 0 \\ -1 & 0 \end{bmatrix} + k_2\begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} + k_3\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} + k_4\begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 1 \\ -1 & 3 \end{bmatrix}$$

$$\begin{bmatrix} k_1 & 0 \\ -k_1 & 0 \end{bmatrix} + \begin{bmatrix} 0 & k_2 \\ 0 & k_2 \end{bmatrix} + \begin{bmatrix} k_3 & k_3 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ k_4 & k_4 \end{bmatrix} = \begin{bmatrix} 3 & 1 \\ -1 & 3 \end{bmatrix}$$

$$\begin{bmatrix} k_1 + k_3 & k_2 + k_3 \\ -k_1 + k_4 & k_2 + k_4 \end{bmatrix} = \begin{bmatrix} 3 & 1 \\ -1 & 3 \end{bmatrix}$$

Equating corresponding entries gives

$$\begin{array}{rcrcrcrcrl} k_1 & & & + & k_3 & & & = & 3 & (*) \\ & & k_2 & + & k_3 & & & = & 1 & (**) \\ -k_1 & & & & & + & k_4 & = & -1 & (***) \\ & & k_2 & & & + & k_4 & = & 3 & (****) \end{array}$$

(*) – (**) gives

$$k_1 - k_2 = 2$$

(***) – (****) gives

$$-k_1 - k_2 = -4$$

Adding these two equations yields $-2k_2 = -2$ which means $k_2 = 1$ and from

$$k_1 - k_2 = 2 \text{ we have } k_1 = 3$$

Substituting $k_1 = 3$ into (***) we have

$$-3 + k_4 = -1 \text{ implies that } k_4 = 2$$

Substituting $k_2 = 1$ into (**) gives

$$1 + k_3 = 1 \text{ which implies that } k_3 = 0$$

Hence the scalars $k_1 = 3$, $k_2 = 1$, $k_3 = 0$ and $k_4 = 2$ generate the given vector **v**. This means that **v** is in the span of *S*.

(b) Again let k_1, k_2, k_3 and k_4 be scalars such that

$$k_1(1+x) + k_2(x + x^2) + k_3(x + x^3) + k_4(1 + x + x^2 + x^3) = 2 - 3x + 4x^2 + x^3$$

Expanding and collecting like terms in the above

$$k_1 + k_1x + k_2x + k_2x^2 + k_3x + k_3x^3 + k_4 + k_4x + k_4x^2 + k_4x^3$$

$$= (k_1 + k_4) + (k_1 + k_2 + k_3 + k_4)x + (k_2 + k_4)x^2 + (k_3 + k_4)x^3 = 2 - 3x + 4x^2 + x^3 \quad (\dagger)$$

Equating coefficients of

$$\begin{array}{lrcrcrcrcr} x^3: & & & & & k_3 & + & k_4 & = & 1 \\ x^2: & & & k_2 & & & + & k_4 & = & 4 \\ x: & k_1 & + & k_2 & + & k_3 & + & k_4 & = & -3 \\ \textit{constant}: & k_1 & & & & & + & k_4 & = & 2 \end{array}$$

Solving these equations gives $k_1 = -3$, $k_2 = -1$, $k_3 = -4$ and $k_4 = 5$. This means that **v** is in the span of *S*.

18. Applying elementary row operations to the given matrix:

$$\begin{array}{c} R_1 \\ R_2 \\ R_3 \end{array} \begin{bmatrix} 1 & 2 & 2 & 8 \\ 2 & 4 & 4 & 13 \\ 1 & 1 & 1 & 5 \end{bmatrix} \Rightarrow \begin{array}{r} R_1 \\ R_2{*} = R_2 - 2R_1 \\ R_3{*} = R_3 - R_1 \end{array} \begin{bmatrix} 1 & 2 & 2 & 8 \\ 0 & 0 & 0 & -3 \\ 0 & -1 & -1 & -3 \end{bmatrix}$$

Carrying out the elementary row operation $R_1 + 2R_3{*}$ gives

$$\begin{array}{r} R_1{*} = R_1 + 2R_3{*} \\ R_2{*} \\ R_3{*} \end{array} \begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 0 & 0 & -3 \\ 0 & -1 & -1 & -3 \end{bmatrix}$$

Executing the row operations $R_2{*}/(-3)$ and $-R_3{*}$ gives

$$\begin{matrix} R_1* \\ R_2**=R_2*/(-3) \\ R_3**=-R_3* \end{matrix}\begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 3 \end{bmatrix}$$

Carrying out R_1*-2R_2** and R_3**-3R_2** yields

$$\begin{matrix} R_1**=R_1*-2R_2** \\ R_2** \\ R_3***=R_3**-3R_2** \end{matrix}\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}$$

Interchanging the middle and bottom rows gives the reduced row echelon form **R**:

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}=\mathbf{R}$$

(a) A basis for the null space is given by solution space **x** which satisfies $\mathbf{Rx}=\mathbf{O}$:

$$\mathbf{Rx}=\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

From the bottom row we have $w=0$. From the top row we have $x=0$. From the middle row we have

$$y+z=0 \text{ which gives } y=-z$$

Let $z=s$ where s is any real number, then $y=-s$ and

$$\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix}=\begin{bmatrix} 0 \\ -s \\ s \\ 0 \end{bmatrix}=s\begin{bmatrix} 0 \\ -1 \\ 1 \\ 0 \end{bmatrix}$$

Thus the vector $\begin{bmatrix} 0 \\ -1 \\ 1 \\ 0 \end{bmatrix}$ is a basis for null space of the given matrix **B**.

(b) Solution 1: The dimension of the column space is 3 because the rank of the matrix is 3 since we have 3 non-zero rows in reduced row echelon form matrix **R**. Thus the column space is $\mathbb{R}^3$ and a basis for this space is $\left\{\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}\right\}$.

Solution 2: Alternatively we can find a basis through the normal procedure which is outlined below.

To find a basis for the column space of **B** we need to transpose the matrix **B**:

$$\mathbf{B}^T=\begin{bmatrix} 1 & 2 & 2 & 8 \\ 2 & 4 & 4 & 13 \\ 1 & 1 & 1 & 5 \end{bmatrix}^T=\begin{bmatrix} 1 & 2 & 1 \\ 2 & 4 & 1 \\ 2 & 4 & 1 \\ 8 & 13 & 5 \end{bmatrix}$$

Applying the following row operations to $\mathbf{B}^T$ we have

$$\begin{array}{c} R_1 \\ R_2 \\ R_3 \\ R_4 \end{array}\begin{bmatrix} 1 & 2 & 1 \\ 2 & 4 & 1 \\ 2 & 4 & 1 \\ 8 & 13 & 5 \end{bmatrix} \Rightarrow \begin{array}{c} R_1 \\ R_2{*} = R_2 - 2R_1 \\ R_3{*} = R_3 - R_2 \\ R_4{*} = R_4 - 8R_1 \end{array}\begin{bmatrix} 1 & 2 & 1 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \\ 0 & -3 & -3 \end{bmatrix}$$

Executing the row operation $R_4 {*}/(-3)$ gives

$$\begin{array}{c} R_1 \\ R_2{*} \\ R_3{*} \\ R_4{**} = R_4{*}/(-3) \end{array}\begin{bmatrix} 1 & 2 & 1 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$

Carrying out the row operations $R_1 - 2R_4{**}$ and $R_4{**} + R_2{*}$

$$\begin{array}{c} R_1{*} = R_1 - 2R_4{**} \\ R_2{*} \\ R_3{*} \\ R_4{***} = R_4{**} + R_2{*} \end{array}\begin{bmatrix} 1 & 0 & -1 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

Carrying out the operation $R_1{*} - R_2{*}$ gives

$$\begin{array}{c} R_1{**} = R_1{*} - R_2{*} \\ R_2{*} \\ R_3{*} \\ R_4{***} \end{array}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

Interchanging rows and multiplying the second row by -1 gives the reduced row echelon form matrix **R**:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} = \mathbf{R}$$

A basis for the column space of **B** is $\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}$.

(c) The nullity of **B** is 1 because we only have one vector as a basis for the null space.
(d) The rank of matrix **B** is the dimension of the column space which is 3 because we only need 3 vectors in a basis for the column space.

19. Let M_{nn} be the vector space of *n* by *n* matrices and

$$S = \left\{ \mathbf{A} \;\middle|\; \mathbf{A}^T = -\mathbf{A} \text{ where } \mathbf{A} \in M_{nn} \right\}$$

How do we show S is a subspace of M_{nn}*?*

Clearly *S* is a subset of M_{nn} because $\mathbf{A} \in M_{nn}$. We also need to show that

i. The zero matrix, **O**, is in *S*.
ii. If matrices **A** and **B** are in *S* then the linear combination $k\mathbf{A} + c\mathbf{B}$ is also in *S*.

Proof.
Showing i:
Since $\mathbf{O}^T = \mathbf{O} = -\mathbf{O}$ therefore the zero matrix **O** is in *S*.
Showing ii:
Let **A** and **B** be in *S* and $k,\ c$ be scalars. Consider the linear combination $k\mathbf{A}+c\mathbf{B}$:

$$\begin{aligned}(k\mathbf{A}+c\mathbf{B})^T &= (k\mathbf{A})^T + (c\mathbf{B})^T \\ &= k\mathbf{A}^T + c\mathbf{B}^T \\ &= -k\mathbf{A} - c\mathbf{B} = -(k\mathbf{A}+c\mathbf{B})\end{aligned}$$

Hence the linear combination $k\mathbf{A}+c\mathbf{B}$ is in *S*.
Since both conditions are satisfied therefore *S* is a subspace of M_{nn} which means that skew-symmetric matrices are a subspace of M_{nn}.

■

20. (a) We can use the result of question 13 of Exercise 4(e) which says:
Let **A** be any matrix whose rows are given by the set of **distinct** vectors

$$S = \{\mathbf{v}_1,\ \mathbf{v}_2,\ \mathbf{v}_3,\ \cdots,\ \mathbf{v}_k\}$$

$rank(\mathbf{A}) = k \iff S$ is a set of linearly independent vectors.

Let $\mathbf{A} = \begin{pmatrix}\mathbf{v}_1\\ \mathbf{v}_2\\ \mathbf{v}_3\end{pmatrix} = \begin{pmatrix}2&4&1&1\\4&7&2&2\\6&8&7&5\end{pmatrix}$ and using elementary row operations we have

$$\begin{array}{c}R_1\\R_2\\R_3\end{array}\begin{pmatrix}2&4&1&1\\4&7&2&2\\6&8&7&5\end{pmatrix} \Rightarrow \begin{array}{c}R_1\\R_2{*}=R_2-2R_1\\R_3{*}=R_3-3R_1\end{array}\begin{pmatrix}2&4&1&1\\0&-1&0&0\\0&-4&4&2\end{pmatrix}$$

Carrying out the row operation $R_1 + R_3*$ gives

$$\begin{array}{c}R_1{*}=R_1+R_3{*}\\R_2{*}\\R_3{*}\end{array}\begin{pmatrix}2&0&5&3\\0&-1&0&0\\0&-4&4&2\end{pmatrix}$$

Executing $R_3{*}-4R_2{*}$ gives

$$\begin{array}{c}R_1{*}\\R_2{*}\\R_3{**}=R_3{*}-4R_2{*}\end{array}\begin{pmatrix}2&0&5&3\\0&-1&0&0\\0&0&4&2\end{pmatrix}$$

Multiplying the middle row by -1 and the bottom row by 2 gives

$$\begin{array}{c}R_1{*}\\R_2{**}=-R_2{*}\\R_3{***}=R_3{**}/2\end{array}\begin{pmatrix}2&0&5&3\\0&1&0&0\\0&0&2&1\end{pmatrix}$$

Clearly there are going to be **no** zero rows therefore $rank(\mathbf{A}) = 3$ which means that the given vectors $\mathbf{v}_1 = (2\ \ 4\ \ 1\ \ 1)$, $\mathbf{v}_2 = (4\ \ 7\ \ 2\ \ 2)$ and $\mathbf{v}_3 = (6\ \ 8\ \ 7\ \ 5)$ are linearly independent.
(b) Since $rank(\mathbf{A}) = 3$ therefore dimension of *V* is 3. Because the given vectors or the vectors obtained in part (a) are linearly independent so they form a basis for *V*. Hence

$$\mathbf{u}_1 = \begin{pmatrix}2 & 0 & 5 & 3\end{pmatrix},\ \mathbf{u}_2 = \begin{pmatrix}0 & 1 & 0 & 0\end{pmatrix} \text{ and } \mathbf{u}_3 = \begin{pmatrix}0 & 0 & 2 & 1\end{pmatrix}$$

is a basis for *V*.

(c) For $\mathbf{v} = \begin{bmatrix}2 & 1 & 1 & 1\end{bmatrix}$ to be a member of *V* means it must be a linear combination of the **u** vectors stated in part (b). Let k_1, k_2 and k_3 be scalars such that

$$k_1\begin{bmatrix}2\\0\\5\\3\end{bmatrix} + k_2\begin{bmatrix}0\\1\\0\\0\end{bmatrix} + k_3\begin{bmatrix}0\\0\\2\\1\end{bmatrix} = \begin{bmatrix}2\\1\\1\\1\end{bmatrix}$$

Expanding these we have

$$\begin{aligned} 2k_1 &= 2 \quad \text{gives } k_1 = 1 \\ k_2 &= 1 \\ 5k_1 + 2k_3 &= 1 \\ 3k_1 + k_3 &= 1 \end{aligned}$$

From the first 2 equations we have $k_1 = 1,\ k_2 = 1$. Substituting $k_1 = 1$ into the third equations gives

$$5 + 2k_3 = 1 \ \Rightarrow\ k_3 = -2$$

Also $k_3 = -2$ satisfies the bottom equation.

Since we have values for scalars therefore the given vector $\mathbf{v} = \begin{bmatrix}2 & 1 & 1 & 1\end{bmatrix}$ is in *V*.

(d) The dimension of the null space *W* is $nullity(\mathbf{A})$.

The matrix **A** is the one given in part (a), that is $\mathbf{A} = \begin{pmatrix}\mathbf{v}_1\\ \mathbf{v}_2\\ \mathbf{v}_3\end{pmatrix} = \begin{pmatrix}2 & 4 & 1 & 1\\ 4 & 7 & 2 & 2\\ 6 & 8 & 7 & 5\end{pmatrix}$.

Since $rank(\mathbf{A}) = 3$ and the number of columns of matrix **A** is 4 therefore

$$nullity(\mathbf{A}) = 4 - 3 = 1$$

(e) From part (a) we have

$$\begin{array}{r} R_1{*} \\ R_2{**} \\ R_3{***} \end{array} \begin{pmatrix}2 & 0 & 5 & 3\\ 0 & 1 & 0 & 0\\ 0 & 0 & 2 & 1\end{pmatrix}$$

However this is not in reduced row echelon form. We need to carry further row operations. Carrying out the row operation $R_1{*} - \frac{5}{2}R_3{***}$ gives

$$\begin{array}{r} R_1{***} = R_1{*} - 5R_3{***}/2 \\ R_2{**} \\ R_3{***} \end{array} \begin{pmatrix}2 & 0 & 0 & 0.5\\ 0 & 1 & 0 & 0\\ 0 & 0 & 2 & 1\end{pmatrix}$$

Dividing the top and bottom row by 2 gives the reduced row echelon form matrix **R**:

$$\begin{pmatrix}1 & 0 & 0 & 0.25\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0.5\end{pmatrix} = \mathbf{R}$$

The null space is the solution space **x** of $\mathbf{Rx} = \mathbf{O}$. Thus

$$\mathbf{Rx}=\begin{pmatrix}1&0&0&0.25\\0&1&0&0\\0&0&1&0.5\end{pmatrix}\begin{pmatrix}x\\y\\z\\w\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix}$$

From the middle row we have $y=0$. From the bottom row we have

$$z+0.5w=0 \text{ gives } z=-0.5w$$

Let $w=s$ where s is any real number. This means that $z=-0.5w=-0.5s$.
From the top row we have

$$x+0.25w=0 \text{ implies that } x=-0.25w=-0.25s$$

Substituting $x=-0.25s$, $y=0$, $z=-0.5s$ and $w=s$ into $\mathbf{x}$ gives the null space:

$$\mathbf{x}=\begin{pmatrix}x\\y\\z\\w\end{pmatrix}=\begin{pmatrix}-0.25s\\0\\-0.5s\\s\end{pmatrix}=s\begin{pmatrix}-0.25\\0\\-0.5\\1\end{pmatrix}$$

A basis for the null space is $\begin{pmatrix}-0.25\\0\\-0.5\\1\end{pmatrix}$.

21. We need to prove that if $\mathbf{A}$ is a 5×3 matrix then the rows of $\mathbf{A}$ are linearly dependent.
Proof.
By Proposition (3-22) we have

$$nullity(\mathbf{A})+rank(\mathbf{A})=n$$

where n is the number of columns of matrix $\mathbf{A}$. Since matrix $\mathbf{A}$ has 3 columns therefore $n=3$. Remember $nullity(\mathbf{A})$ is the dimension of the null space of the matrix $\mathbf{A}$ so $nullity(\mathbf{A})\geq 0$. This implies that $rank(\mathbf{A})\leq 3$. Since $\mathbf{A}$ is a 5×3 which means it has 5 rows therefore the reduced row echelon form of $\mathbf{A}$ has at least 2 zero rows because $rank(\mathbf{A})\leq 3$.

By question 16 of Exercise 4(e) which says:

> If $\mathbf{A}$ is $m\times n$ matrix and the reduced row echelon form matrix $\mathbf{R}$ of $\mathbf{A}$ contains zero rows then the rows of $\mathbf{A}$ are linearly dependent.

Since the reduced row echelon form of $\mathbf{A}$ contains zero rows therefore the rows of $\mathbf{A}$ are linearly dependent.

■

22. A matrix of the given size 4×10 means that it has 4 rows and 10 columns. By Proposition (3-22) we have

$$nullity(\mathbf{A})+rank(\mathbf{A})=n$$

where n is the number of columns of the matrix which is 10, thus $n=10$. Since $\mathbf{A}$ has 4 rows therefore $r=rank(\mathbf{A})\leq 4$. Using the above formula we have

$$nullity(\mathbf{A})=n-rank(\mathbf{A})=10-r\geq 6$$

This says that $nullity(\mathbf{A}) \geq 6$ which means that the dimension of the null space **cannot** be less than 6. It is impossible for the dimension of the null space to be 2.

23. (a) We apply elementary row operations to the given matrix:

$$\begin{array}{c} R_1 \\ R_2 \end{array}\begin{bmatrix} 2 & 3 & 1 & -1 \\ 6 & 9 & 3 & -2 \end{bmatrix} \Longrightarrow \begin{array}{c} R_1 \\ R_2{*} = R_2 - 3R_1 \end{array}\begin{bmatrix} 2 & 3 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Carrying out the row operation $R_1 + R_2{*}$ gives

$$\begin{array}{c} R_1{*} = R_1 + R_2{*} \\ R_2{*} \end{array}\begin{bmatrix} 2 & 3 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Dividing the top row by 2 gives the reduced row echelon form

$$\begin{array}{c} R_1{*}/2 \\ R_2{*} \end{array}\begin{bmatrix} 1 & 3/2 & 1/2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \mathbf{R}$$

This $\mathbf{R}$ is in reduced row echelon form and we can find the solution space $\mathbf{x}$ of $\mathbf{Ax} = \mathbf{O}$ by solving the equivalent system $\mathbf{Rx} = \mathbf{O}$.

$$\mathbf{Rx} = \begin{bmatrix} 1 & 3/2 & 1/2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Expanding the bottom row gives $x_4 = 0$. Opening the top row yields

$$x_1 + \frac{3}{2}x_2 + \frac{1}{2}x_3 = 0 \text{ gives } x_1 = -\frac{3}{2}x_2 - \frac{1}{2}x_3$$

Let $x_3 = 2s$ and $x_2 = 2t$ where $s,\ t \in \mathbb{R}$. Substituting these into $x_1 = -\frac{3}{2}x_2 - \frac{1}{2}x_3$ gives

$$x_1 = -\frac{3}{2}(2t) - \frac{1}{2}(2s) = -3t - s$$

We have $x_1 = -3t - s,\ x_2 = 2t,\ x_3 = 2s$ and $x_4 = 0$. Substituting these into the vector $\mathbf{x}$ gives us the null space

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -3t - s \\ 2t \\ 2s \\ 0 \end{bmatrix} = t\begin{bmatrix} -3 \\ 2 \\ 0 \\ 0 \end{bmatrix} + s\begin{bmatrix} -1 \\ 0 \\ 2 \\ 0 \end{bmatrix}$$

Thus the null space $N = \left\{ t\begin{bmatrix} -3 \\ 2 \\ 0 \\ 0 \end{bmatrix} + s\begin{bmatrix} -1 \\ 0 \\ 2 \\ 0 \end{bmatrix} \;\middle|\; s,\ t \in \mathbb{R} \right\}$. This is a subspace of $\mathbb{R}^4$.

(b) By Proposition (3-23) we have

all the solutions of $\mathbf{Ax} = \mathbf{b}$ are of the form $\mathbf{x}_P + \mathbf{x}_H$

where $\mathbf{x}_H$ is the homogeneous solution given in part (a) and $\mathbf{x}_P$ is the particular solution to the non-homogeneous system.

We can find $\mathbf{x}_P$ by solving the augmented matrix which is the same as part (a) but we have entries to the right hand side of the vertical bar. The row operations are identical to part (a).

$$\begin{array}{c} R_1 \\ R_2 \end{array}\left[\begin{array}{cccc|c} 2 & 3 & 1 & -1 & 1 \\ 6 & 9 & 3 & -2 & 2 \end{array}\right] \Rightarrow \begin{array}{c} R_1 \\ R_2{*} = R_2 - 3R_1 \end{array}\left[\begin{array}{cccc|c} 2 & 3 & 1 & -1 & 1 \\ 0 & 0 & 0 & 1 & -1 \end{array}\right]$$

$$\Rightarrow \begin{array}{c} R_1{*} = R_1 + R_2{*} \\ R_2{*} \end{array}\left[\begin{array}{cccc|c} 2 & 3 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 \end{array}\right]$$

$$\Rightarrow \begin{array}{c|c} & \begin{array}{ccccc} x_1 & x_2 & x_3 & x_4 & \end{array} \\ \begin{array}{c} R_1{*}/2 \\ R_2{*} \end{array} & \left[\begin{array}{cccc|c} 1 & 3/2 & 1/2 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 \end{array}\right] \end{array}$$

From the bottom row we have $x_4 = -1$. From the first row we have

$$x_1 + \frac{3}{2}x_2 + \frac{1}{2}x_3 = 0 \text{ gives } x_1 = x_2 = x_3 = 0$$

Our particular solution is $\mathbf{x}_P = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ -1 \end{bmatrix}$.

The general solution of $\mathbf{Ax} = \mathbf{b}$ is

$$\mathbf{x} = \mathbf{x}_P + \mathbf{x}_H = \begin{bmatrix} 0 \\ 0 \\ 0 \\ -1 \end{bmatrix} + \underbrace{t\begin{bmatrix} -3 \\ 2 \\ 0 \\ 0 \end{bmatrix} + s\begin{bmatrix} -1 \\ 0 \\ 2 \\ 0 \end{bmatrix}}_{\mathbf{x}_H \text{ solution of part (a)}}$$

(c) Proposition (3-22) we have

$$nullity(\mathbf{A}) + rank(\mathbf{A}) = n$$

where n is the number of columns of matrix **A**. We are given that $rank(\mathbf{A}) = m$ so we have

$$nullity(\mathbf{A}) = n - m$$

We must have $m \le n$ that is the number of rows of matrix **A** must be less than or equal to the number of columns of matrix **A**. If this is the case then $nullity(\mathbf{A}) = n - m$ and the number of special solutions to $\mathbf{Ax} = \mathbf{O}$ is $n - m$ because the dimension of the null space of **A** is $nullity(\mathbf{A}) = n - m$.

24. We are given the vectors

$$\begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix}, \begin{pmatrix} 2 \\ 3 \\ 5 \end{pmatrix}, \begin{pmatrix} 2 \\ 7 \\ c \end{pmatrix}$$

We want to find a value for c so that these vectors are linearly dependent.

$$k_1\begin{pmatrix}1\\2\\4\end{pmatrix}+k_2\begin{pmatrix}2\\3\\5\end{pmatrix}=\begin{pmatrix}2\\7\\c\end{pmatrix}$$

Writing out these equations we have

$$\begin{array}{rcrcl} k_1 & + & 2k_2 & = & 2 \qquad (\dagger)\\ 2k_1 & + & 3k_2 & = & 7 \qquad (\dagger\dagger)\\ 4k_1 & + & 5k_2 & = & c \qquad (\dagger\dagger\dagger)\end{array}$$

Multiplying (†) by 2 and subtracting $(\dagger\dagger)$ gives

$$\begin{array}{rcrcl} 2k_1 & + & 4k_2 & = & 4\\ -(2k_1 & + & 3k_2 & = & 7)\\ \hline 0 & + & k_2 & = & -3\end{array}$$

Substituting $k_2=-3$ into $(\dagger)$ gives

$$k_1 + 2(-3) = 2 \Rightarrow k_1=8$$

Substituting $k_1=8$ and $k_2=-3$ into $(\dagger\dagger\dagger)$ we have

$$4(8) + 5(-3) = c \Rightarrow c=17$$

Thus for $c=17$ the given vectors are linearly dependent which means that they **cannot** be a basis for $\mathbb{R}^3$.

25. Let **A** be the 3×3 matrix whose column space is the span of the vectors:

$$(4,\ 5,\ 6) \text{ and } (7,\ 8,\ 9)$$

This means the columns of **A** are given by $\begin{pmatrix}4\\5\\6\end{pmatrix}s$ and $\begin{pmatrix}7\\8\\9\end{pmatrix}t$ where $s,\ t\in\mathbb{R}$. Let $\begin{pmatrix}a\\b\\c\end{pmatrix}$ be last column of the matrix **A**:

$$\mathbf{A}=\begin{pmatrix}4s & 7t & a\\ 5s & 8t & b\\ 6s & 9t & c\end{pmatrix} \qquad (*)$$

Since the null space is the span of the vector $(1,\ 2,\ 3)$ we have

$$\mathbf{Ax}=\mathbf{O} \text{ where } \mathbf{x}=\begin{pmatrix}1\\2\\3\end{pmatrix}r \text{ for any } r\in\mathbb{R}$$

We can find the entries *a*, *b* and *c* in (*) by considering particular values of *r*, *s* and *t*.
Let $r=s=t=1$ then we have

$$\mathbf{Ax}=\begin{pmatrix}4 & 7 & a\\ 5 & 8 & b\\ 6 & 9 & c\end{pmatrix}\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix} \text{ has the solution } \mathbf{x}=\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}1\\2\\3\end{pmatrix}$$

Thus substituting **x** into $\mathbf{Ax}=\mathbf{O}$ gives

$$\begin{pmatrix} 4 & 7 & a \\ 5 & 8 & b \\ 6 & 9 & c \end{pmatrix}\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Expanding this out yields

$$4+14+3a=0 \Rightarrow a=-6$$
$$5+16+3b=0 \Rightarrow b=-7$$
$$6+18+3c=0 \Rightarrow c=-8$$

Hence a matrix **A** is given by $\mathbf{A}=\begin{pmatrix} 4 & 7 & -6 \\ 5 & 8 & -7 \\ 6 & 9 & -8 \end{pmatrix}$.

26. (a) We need to show that $\{\cos(x),\ \cos(2x),\ \cos(3x)\}$ is linearly independent.

Let $k_1,\ k_2$ and k_3 be scalars and consider the linear combination

$$k_1\cos(x)+k_2\cos(2x)+k_3\cos(3x)=0 \qquad (*)$$

From trigonometric identities we have

$$\cos(2x)=\cos^2(x)-\sin^2(x)$$
$$\cos(3x)=4\cos^3(x)-3\cos(x)$$

Substituting these into (*) gives

$$k_1\cos(x)+k_2\left[\cos^2(x)-\sin^2(x)\right]+k_3\left[4\cos^3(x)-3\cos(x)\right]=0 \qquad (**)$$

Substituting $x=\frac{\pi}{2}$ into (**) gives

$$k_1(0)+k_2\left[(0)-1\right]+k_3\left[4(0)-3(0)\right]=0 \text{ implies that } k_2=0$$

Substituting $x=0$ and $k_2=0$ into (**) yields

$$k_1+k_3\left[4-3\right]=0 \text{ implies } k_1=-k_3$$

Substituting $k_1=-k_3$ and $k_2=0$ into (**) we have

$$-k_3\cos(x)+k_3\left[4\cos^3(x)-3\cos(x)\right]=0$$
$$4k_3\left[\cos^3(x)-\cos(x)\right]=0$$

Since these results are identities therefore they are valid for **all** *x* and we have

$$4k_3\left[\cos^3(x)-\cos(x)\right]=0 \text{ implies } k_3=0$$

From $k_1=-k_3$ we have $k_1=0$. Since

$$k_1\cos(x)+k_2\cos(2x)+k_3\cos(3x)=0 \text{ gives } k_1=0,\ k_2=0 \text{ and } k_3=0$$

therefore $\{\cos(x),\ \cos(2x),\ \cos(3x)\}$ is linearly independent.

(b) Let $V=span\{1,\ x,\ x^2,\ x^3\}$. Clearly $\dim(V)=4$ because 4 linearly independent vectors span *V* and so $\{1,\ x,\ x^2,\ x^3\}$ is a basis for *V*.

Expanding the given vectors we have

$$(1+x)^3=1+3x+3x^2+x^3$$

$$(1-x)^3 = 1-3x+3x^2-x^3$$
$$(1+2x)^3 = 1+6x+12x^2+8x^3$$
$$(1-2x)^3 = 1-6x+12x^2-8x^3$$
$$(1+3x)^3 = 1+9x+27x^2+27x^3$$

We have all 5 vectors are members of *V*. Since $\dim(V)=4$ and we have 5 vectors therefore they must be linearly dependent because of:

Lemma (3-12). Let *V* be a finite *n*-dimensional vector space and $\{\mathbf{v}_1,\ \mathbf{v}_2,\ \mathbf{v}_3,\ \cdots,\ \mathbf{v}_n\}$ be a set of linearly independent vectors. Then $\{\mathbf{v}_1,\ \mathbf{v}_2,\ \cdots,\ \mathbf{v}_m\}$ where $m>n$ (*m* is greater than n) is linearly dependent.

27. (a) Writing the given vectors as row vectors of a matrix we have

$$\mathbf{A}=\begin{pmatrix}1 & 2 & -1 & -2\\ 2 & 1 & 2 & -1\end{pmatrix},\quad \mathbf{B}=\begin{pmatrix}-1 & 4 & -7 & -4\\ 8 & 7 & 4 & -7\end{pmatrix}$$

This means that the row space of **A** is equal to

$$U=Span\big((1,\ 2,\ -1,\ -2),\ (2,\ 1,\ 2,\ -1)\big)$$

Similarly the row space of matrix **B** is equal to

$$V=Span\big((-1,\ 4,\ -7,\ -4),\ (8,\ 7,\ 4,\ -7)\big)$$

If we can show that these row spaces have the same basis vectors then they are equal, that is $U=V$.

Applying elementary row operations to matrix **A** we have

$$\begin{array}{c} R_1 \\ R_2 \end{array}\begin{pmatrix}1 & 2 & -1 & -2\\ 2 & 1 & 2 & -1\end{pmatrix} \Rightarrow \begin{array}{c} R_1 \\ R_2^*=R_2-2R_1 \end{array}\begin{pmatrix}1 & 2 & -1 & -2\\ 0 & -3 & 4 & 3\end{pmatrix}$$

Dividing the bottom row by -3 gives

$$\begin{array}{c} R_1 \\ R_2^{**}=-R_2^*/3 \end{array}\begin{pmatrix}1 & 2 & -1 & -2\\ 0 & 1 & -4/3 & -1\end{pmatrix}$$

Carrying out the row operation $R_1-2R_2^{**}$ gives the reduced row echelon form matrix **R**:

$$\begin{array}{c} R_1^*=R_1-2R_2^{**} \\ R_2^{**} \end{array}\begin{pmatrix}1 & 0 & 5/3 & 0\\ 0 & 1 & -4/3 & -1\end{pmatrix}=\mathbf{R}$$

Similarly apply elementary row operations to matrix **B** we obtain the same reduced row echelon form matrix **R**, that is $\mathbf{R}=\begin{pmatrix}1 & 0 & 5/3 & 0\\ 0 & 1 & -4/3 & -1\end{pmatrix}$.

This means that $U=Span\big((1,\ 2,\ -1,\ -2),\ (2,\ 1,\ 2,\ -1)\big)$ has the same basis vectors as $V=Span\big((-1,\ 4,\ -7,\ -4),\ (8,\ 7,\ 4,\ -7)\big)$ which means they are equal, that is $U=V$.

(b) We need to find values of *k* for which

$$c_1(k\mathbf{u}+\mathbf{v})+c_2(\mathbf{v}+k\mathbf{w})+c_3(\mathbf{w}+k\mathbf{u})=\mathbf{O} \text{ gives } c_1=c_2=c_3=0$$

Expanding the Left Hand Side and collecting like terms:

$$c_1k\mathbf{u}+c_1\mathbf{v}+c_2\mathbf{v}+c_2k\mathbf{w}+c_3\mathbf{w}+c_3k\mathbf{u}=(c_1k+c_3k)\mathbf{u}+(c_1+c_2)\mathbf{v}+(c_2k+c_3)\mathbf{w}=\mathbf{O}$$

Since **u**, **v** and **w** are linearly independent therefore

$$c_1k+c_3k=0,\ c_1+c_2=0,\ c_2k+c_3=0$$

From the middle equation we have $c_1 = -c_2$. Substituting this, $c_1 = -c_2$, into the first equation we have $-c_2k + c_3k = 0 \Rightarrow c_2 = c_3$ or $k = 0$. Putting $c_2 = c_3$ into the last equation $c_2k + c_3 = 0$ we have

$$c_3k + c_3 = c_3(k+1) = 0 \Rightarrow c_3 = 0 \text{ or } k = -1$$

Thus the values of k for which the vectors $k\mathbf{u}+\mathbf{v},\ \mathbf{v}+k\mathbf{w},\ \mathbf{w}+k\mathbf{u}$ are linearly independent are $k = 0,\quad k = -1$.

28. (a) We are given that $\mathbf{Ax} = \mathbf{y}$ has more than one solution therefore by the result of question 9 (b) of Exercise 4(e) which says

Let $\mathbf{A}$ be a square n by n matrix then

$\mathbf{A}$ has rank $n \Leftrightarrow$ the linear system $\mathbf{Ax} = \mathbf{b}$ has a unique solution.

we can say $rank(\mathbf{A}) = r \neq n$. Actually $r < n$ because rank is the number of non-zero rows in reduced row echelon form which cannot be greater than n because we only have n rows.

We have $rank(\mathbf{A}) = r$ is the dimension of the column space (or the column rank) so the columns of **A cannot** span $\mathbb{R}^n$ because we need n basis vectors for $\mathbb{R}^n$.

This is a false result.

(a) For $H = \left\{(x,\quad y) \in \mathbb{R}^2 \;\middle|\; y = x^2\right\}$ to be in the subspace of $\mathbb{R}^2$ we need to show that (i) **O** (zero vector is in H) and (ii) if **u** and **v** are in H then $k_1\mathbf{u}+k_2\mathbf{v}$ is also in H.

Clearly **O** is in H because we have $\mathbf{O} = \left\{(0,\quad 0) \in \mathbb{R}^2 \;\middle|\; 0 = 0^2\right\}$.

Let $\mathbf{u} = \begin{pmatrix} a \\ a^2 \end{pmatrix} \in H$ and $\mathbf{v} = \begin{pmatrix} b \\ b^2 \end{pmatrix} \in H$. Consider the linear combination

$$k_1\mathbf{u}+k_2\mathbf{v} \text{ where } k_1,\ k_2 \text{ are scalars}$$

We have

$$k_1\mathbf{u}+k_2\mathbf{v} = k_1\begin{pmatrix} a \\ a^2 \end{pmatrix} + k_2\begin{pmatrix} b \\ b^2 \end{pmatrix}$$
$$= \begin{pmatrix} k_1a + k_2b \\ k_1a^2 + k_2b^2 \end{pmatrix} \notin H$$

This is **not** a member of H because $(k_1a+k_2b)^2 \neq k_1a^2 + k_2b^2$ [Not Equal].

Thus the given subspace H is **not** a subspace of $\mathbb{R}^2$.

(c) Since **A** is a 5×6 therefore we have 6 columns and so by Proposition (3-20). If **A** is a matrix of size m by n then the null space N of **A** is a subspace of $\mathbb{R}^n$.

The null space of **A** is a subspace of $\mathbb{R}^6$ **not** $\mathbb{R}^5$. Hence the given statement is false.

All three statements (a), (b) and (c) are false.

29. We need to prove that if $\mathbf{Av}_1, \ldots, \mathbf{Av}_n$ form a basis of $\mathbb{R}^n$ then $\mathbf{v}_1, \ldots, \mathbf{v}_n$ form a basis of $\mathbb{R}^n$ and that **A** is invertible.

Proof.

Suppose $\mathbf{v}_1, \ldots, \mathbf{v}_n$ are linearly dependent. By Proposition (3-9) one of these vectors, say $\mathbf{v}_k$, can be written as a linear combination of the preceding vectors, that is

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3 + \cdots + c_{k-1}\mathbf{v}_{k-1} = \mathbf{v}_k \text{ where } 1 \le k \le n$$

Multiplying this by the matrix **A** gives

$$c_1(\mathbf{Av}_1) + c_2(\mathbf{Av}_2) + c_3(\mathbf{Av}_3) + \cdots + c_{k-1}(\mathbf{Av}_{k-1}) = \mathbf{Av}_k$$

This implies that $\mathbf{Av}_1, \ldots, \mathbf{Av}_k$ are linearly dependent. However this is impossible because $\mathbf{Av}_1, \ldots, \mathbf{Av}_k, \cdots, \mathbf{Av}_n$ is a basis for $\mathbb{R}^n$ so it must be linearly independent which means that $\mathbf{Av}_1, \ldots, \mathbf{Av}_k$ is linearly independent. Hence we have a contradiction so our supposition $\mathbf{v}_1, \ldots, \mathbf{v}_n$ are linearly dependent must be false which means that $\mathbf{v}_1, \ldots, \mathbf{v}_n$ are linearly independent. Since there are *n* of these and dimension of $\mathbb{R}^n$ is *n* therefore they $\mathbf{v}_1, \ldots, \mathbf{v}_n$ form a basis for $\mathbb{R}^n$.

We also need to prove that **A** is invertible.

Let $\mathbf{u}$ be an arbitrary vector in $\mathbb{R}^n$. Since we are given that $\mathbf{Av}_1, \ldots, \mathbf{Av}_n$ forms a basis for $\mathbb{R}^n$ therefore we can write the vector **u** uniquely as the following linear combination

$$c_1\mathbf{Av}_1 + c_2\mathbf{Av}_2 + \ldots + c_n\mathbf{Av}_n = \mathbf{u} \qquad (*)$$

$$\mathbf{A}(c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \ldots + c_n\mathbf{v}_n) = \mathbf{u}$$

Let $\mathbf{x} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \ldots + c_n\mathbf{v}_n$ then we have

$$\mathbf{Ax} = \mathbf{u} \qquad (**)$$

Since the representation in (*) is unique therefore the **x** in (**) is unique. This means that the matrix **A** has rank *n* which means that it is invertible.

■

Proposition (3-9). The vectors in the set $S = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \cdots, \mathbf{v}_n\}$ are linearly dependent if and only if one of these vectors, say $\mathbf{v}_k$, is a linear combination of the preceding vectors, that is $\mathbf{v}_k = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3 + \cdots + c_{k-1}\mathbf{v}_{k-1}$.