Complete Solutions to Exercises 1.7

1. (a) The augmented matrix is

$$\begin{array}{c} R_1 \\ R_2 \\ R_3 \end{array} \left(\begin{array}{ccc|c} 1 & 3 & 2 & 5 \\ 2 & -1 & -1 & 1 \\ -1 & 2 & 1 & 3 \end{array}\right)$$

Carrying out the following row operations:

$$\begin{array}{c} R_1 \\ r_2 = R_2 - 2R_1 \\ r_3 = R_3 + R_1 \end{array} \left(\begin{array}{ccc|c} 1 & 3 & 2 & 5 \\ 2-2(1) & -1-2(3) & -1-2(2) & 1-2(5) \\ -1+1 & 2+3 & 1+2 & 3+5 \end{array}\right)$$

Simplifying the entries gives

$$\begin{array}{c} R_1 \\ r_2 \\ r_3 \end{array} \left(\begin{array}{ccc|c} 1 & 3 & 2 & 5 \\ 0 & -7 & -5 & -9 \\ 0 & 5 & 3 & 8 \end{array}\right)$$

We need to get 0 in place of the 5 in the bottom row. *How?*
By executing the row operation $7r_3 + 5r_2$:

$$\begin{array}{c} R_1 \\ r_2 \\ r_3^* = 7r_3 + 5r_2 \end{array} \left(\begin{array}{ccc|c} 1 & 3 & 2 & 5 \\ 0 & -7 & -5 & -9 \\ 0 & 7(5)+5(-7) & 7(3)+5(-5) & 7(8)+5(-9) \end{array}\right)$$

Simplifying the entries in the bottom row gives

$$\begin{array}{c} R_1 \\ r_2 \\ r_3^* \end{array} \left(\begin{array}{ccc|c} 1 & 3 & 2 & 5 \\ 0 & -7 & -5 & -9 \\ 0 & 0 & -4 & 11 \end{array}\right)$$

Dividing the second row by -7 and the last row by -4 gives

$$\begin{array}{c} \\ R_1 \\ r_2^* = r_2/(-7) \\ r_3^{**} = r_3^*/(-4) \end{array} \begin{array}{c} \begin{array}{ccc} x & y & z \end{array} \\ \left(\begin{array}{ccc|c} 1 & 3 & 2 & 5 \\ 0 & 1 & 5/7 & 9/7 \\ 0 & 0 & 1 & -11/4 \end{array}\right) \end{array}$$

This matrix is now in row echelon form. From the last row we have $z = -\frac{11}{4}$. Substituting this $z = -\frac{11}{4}$ into the second row gives

$$y + \frac{5}{7}z = y + \frac{5}{7}\left(-\frac{11}{4}\right) = \frac{9}{7} \quad \text{gives} \quad y = \frac{9}{7} + \frac{5}{7}\left(\frac{11}{4}\right) = \frac{13}{4}$$

Substituting the values of y and z into the first row gives

$$x + 3\left(\frac{13}{4}\right) + 2\left(-\frac{11}{4}\right) = 5 \quad \text{gives} \quad x = 5 - 3\left(\frac{13}{4}\right) + 2\left(\frac{11}{4}\right) = \frac{3}{4}$$

Hence $x = \frac{3}{4}$, $y = \frac{13}{4}$ and $z = -\frac{11}{4}$.

(b) The augmented matrix is

$$\begin{array}{c} R_1 \\ R_2 \\ R_3 \end{array}\left(\begin{array}{ccc|c} -1 & 1 & 1 & 0 \\ 3 & -2 & 5 & 0 \\ 4 & -1 & -2 & 0 \end{array}\right)$$

To get 0's in place of 3 and 4 we carry out the following row operations:

$$\begin{array}{c} R_1 \\ r_2 = R_2 + 3R_1 \\ r_3 = R_3 + 4R_3 \end{array}\left(\begin{array}{ccc|c} -1 & 1 & 1 & 0 \\ 3+3(-1) & -2+3(1) & 5+3(1) & 0+3(0) \\ 4+4(-1) & -1+4(1) & -2+4(1) & 0+4(0) \end{array}\right)$$

Simplifying the entries gives

$$\begin{array}{c} R_1 \\ r_2 \\ r_3 \end{array}\left(\begin{array}{ccc|c} -1 & 1 & 1 & 0 \\ 0 & 1 & 8 & 0 \\ 0 & 3 & 2 & 0 \end{array}\right)$$

How do we get 0 in place of the 3 in the bottom row?

$$\begin{array}{c} R_1 \\ r_2 \\ r_3^* = r_3 - 3r_2 \end{array}\left(\begin{array}{ccc|c} -1 & 1 & 1 & 0 \\ 0 & 1 & 8 & 0 \\ 0-3(0) & 3-3(1) & 2-3(8) & 0-3(0) \end{array}\right)$$

Simplifying the arithmetic in the bottom row:

$$\begin{array}{cc} & \begin{array}{ccc} x & y & z \end{array} \\ \begin{array}{c} R_1 \\ r_2 \\ r_3^* \end{array} & \left(\begin{array}{ccc|c} -1 & 1 & 1 & 0 \\ 0 & 1 & 8 & 0 \\ 0 & 0 & -22 & 0 \end{array}\right) \end{array} \qquad (*)$$

From the last row we have $-22z = 0$ which gives $z = 0$. Substituting $z = 0$ gives

$$y + 8(0) = 0 \text{ gives } y = 0$$

Similarly by the first row we have $x = 0$. We only have a trivial solution for the given linear system, $x = 0,\ y = 0$ and $z = 0$. We can also place the augmented matrix (*) in row echelon form which will give **no** zero rows, so we have $n = r = 3$ which means we **only** have the trivial solution $x = y = z = 0$.

(c) We have the augmented matrix given by

$$\begin{array}{c} R_1 \\ R_2 \\ R_3 \end{array}\left(\begin{array}{ccc|c} -1 & 1 & 1 & 2 \\ 2 & 2 & 3 & 5 \\ 6 & 6 & 9 & 7 \end{array}\right)$$

Note that the coefficients of the unknowns in the last two rows are multiples of each other. Execute the row operation $R_3 - 3R_2$:

$$\begin{array}{c} R_1 \\ R_2 \\ r_3 = R_3 - 3R_2 \end{array}\left(\begin{array}{ccc|c} -1 & 1 & 1 & 2 \\ 2 & 2 & 3 & 5 \\ 6-3(2) & 6-3(2) & 9-3(3) & 7-3(5) \end{array}\right)$$

Simplifying the entries in the last row we have

$$\begin{array}{c} \\ \mathrm{R}_1 \\ \mathrm{R}_2 \\ \mathrm{r}_3 \end{array}\begin{array}{c} \begin{array}{ccc} x & y & z \end{array} \\ \left(\begin{array}{ccc|c} -1 & 1 & 1 & 2 \\ 2 & 2 & 3 & 5 \\ 0 & 0 & 0 & -8 \end{array}\right) \end{array}$$

From the last row we have $0x+0y+0z=-8$ which means that $0=-8$. Clearly 0 **cannot equal** -8 therefore the system is **inconsistent**.

(d) The augmented matrix is

$$\begin{array}{c} \mathrm{R}_1 \\ \mathrm{R}_2 \\ \mathrm{R}_3 \end{array}\left(\begin{array}{ccc|c} 1 & 1 & -1 & 2 \\ 1 & 2 & 1 & 4 \\ 3 & 3 & -3 & 6 \end{array}\right)$$

Note that the last row is 3 times the first row, so we carry out the row operation $\mathrm{R}_3-3\mathrm{R}_1$:

$$\begin{array}{c} \mathrm{R}_1 \\ \mathrm{R}_2 \\ \mathrm{r}_3=\mathrm{R}_3-3\mathrm{R}_1 \end{array}\left(\begin{array}{ccc|c} 1 & 1 & -1 & 2 \\ 1 & 2 & 1 & 4 \\ 3-3(1) & 3-3(1) & -3-3(-1) & 6-3(2) \end{array}\right)$$

Simplifying the entries in the last row we have

$$\begin{array}{c} \mathrm{R}_1 \\ \mathrm{R}_2 \\ \mathrm{r}_3 \end{array}\left(\begin{array}{ccc|c} 1 & 1 & -1 & 2 \\ 1 & 2 & 1 & 4 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

Subtract the first two rows:

$$\begin{array}{c} \mathrm{R}_1 \\ \mathrm{r}_2=\mathrm{R}_2-\mathrm{R}_1 \\ \mathrm{r}_3 \end{array}\left(\begin{array}{ccc|c} 1 & 1 & -1 & 2 \\ 1-1 & 2-1 & 1-(-1) & 4-2 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

We have

$$\begin{array}{c} \\ \mathrm{R}_1 \\ \mathrm{r}_2 \\ \mathrm{r}_3 \end{array}\begin{array}{c} \begin{array}{ccc} x & y & z \end{array} \\ \left(\begin{array}{ccc|c} \boxed{1} & 1 & -1 & 2 \\ 0 & \boxed{1} & 2 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right) \end{array}$$

From the middle row we have $y+2z=2$ which gives $y=2-2z$. Since the z column has **no** leading 1 therefore z is our free variable. Let $z=t$ where t is any real number. Then $y=2-2t$. Substituting these, $z=t$ and $y=2-2t$, into the first row we have

$$x+y-z=2$$
$$x+(2-2t)-t=2$$
$$x+2-3t=2 \quad \text{gives} \quad x=3t$$

Hence we have an infinite number of solutions given by $x=3t$, $y=2-2t$ and $z=t$ where t is any real number.

(e) The augmented matrix is given by

$$\begin{array}{c} R_1 \\ R_2 \\ R_3 \\ R_4 \end{array} \left(\begin{array}{cccc|c} 3 & -3 & -1 & 2 & 0 \\ 6 & -7 & 1 & 1 & 0 \\ 1 & -1 & -2 & -1 & 0 \\ 2 & -2 & 6 & 8 & 0 \end{array}\right)$$

Interchanging the first and third rows gives

$$\begin{array}{c} r_1 = R_3 \\ R_2 \\ r_3 = R_1 \\ R_4 \end{array} \left(\begin{array}{cccc|c} 1 & -1 & -2 & -1 & 0 \\ 6 & -7 & 1 & 1 & 0 \\ 3 & -3 & -1 & 2 & 0 \\ 2 & -2 & 6 & 8 & 0 \end{array}\right)$$

Executing the following row operations we have

$$\begin{array}{c} r_1 \\ r_2 = R_2 - 6r_1 \\ r_3^* = r_3 - 3r_1 \\ r_4 = R_4 - 2r_1 \end{array} \left(\begin{array}{cccc|c} 1 & -1 & -2 & -1 & 0 \\ 6-6(1) & -7-6(-1) & 1-6(-2) & 1-6(-1) & 0 \\ 3-3(1) & -3-3(-1) & -1-3(-2) & 2-3(-1) & 0 \\ 2-2(1) & -2-2(-1) & 6-2(-2) & 8-2(-1) & 0 \end{array}\right)$$

Simplifying the entries gives

$$\begin{array}{c} r_1 \\ r_2 \\ r_3^* \\ r_4 \end{array} \left(\begin{array}{cccc|c} 1 & -1 & -2 & -1 & 0 \\ 0 & -1 & 13 & 7 & 0 \\ 0 & 0 & 5 & 5 & 0 \\ 0 & 0 & 10 & 10 & 0 \end{array}\right)$$

$$\begin{array}{c} r_1^* = r_1 - r_2 \\ r_2 \\ r_3^{**} = r_3^*/5 \\ r_4^* = r_4/10 \end{array} \left(\begin{array}{cccc|c} 1 & 0 & -15 & -8 & 0 \\ 0 & -1 & 13 & 7 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 \end{array}\right)$$

Subtracting the last two rows and multiplying the second row by -1 gives

$$\begin{array}{cc} & \begin{array}{cccc} x & y & z & w \end{array} \\ \begin{array}{c} r_1^* \\ -r_2 \\ r_3^{**} \\ r_4^{**} = r_4^* - r_3^{**} \end{array} & \left(\begin{array}{cccc|c} \boxed{1} & 0 & -15 & -8 & 0 \\ 0 & \boxed{1} & -13 & -7 & 0 \\ 0 & 0 & \boxed{1} & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right) \end{array}$$

The matrix is now in row echelon form. From the third row we have

$$z + w = 0 \text{ which gives } z = -w$$

Since none of our equations begin with w (or the w column does **not** have a leading 1) therefore w is a free variable and so we let $w = t$ where t is any real number. Hence $z = -w = -t$. From the second row we have

$$y - 13z - 7w = 0$$

Substituting $z = -t$ and $w = t$ into this:

$$y - 13(-t) - 7t = 0 \text{ gives } y = -6t$$

From the first row we have

$$x-15z-8w=0$$
$$x-15(-t)-8t=0 \qquad [\text{Substituting } z=-t \text{ and } w=t]$$
$$x+15t-8t=0 \quad \text{gives } x=-7t$$

Hence we have an infinite number of solutions given by $x=-7t,\ y=-6t,\ z=-t$ and $w=t$ where t is any real number.

(f) The augmented matrix is

$$\begin{array}{c} R_1 \\ R_2 \\ R_3 \\ R_4 \end{array}\left(\begin{array}{cccc|c} 2 & 3 & 5 & 2 & 6 \\ 2 & 3 & 2 & 2 & 7 \\ 8 & 12 & 20 & 8 & 24 \\ 1 & 2 & 4 & 5 & 6 \end{array}\right)$$

Notice that the bottom row is 4 times the first row. Executing the row operations $R_3-4R_1,\ R_2-R_1$ and $2R_4-R_1$ gives:

$$\begin{array}{c} R_1 \\ r_2=R_2-R_1 \\ r_3=R_3-4R_1 \\ r_4=2R_4-R_1 \end{array}\left(\begin{array}{cccc|c} 2 & 3 & 5 & 2 & 6 \\ 2-2 & 3-3 & 2-5 & 2-2 & 7-6 \\ 8-4(2) & 12-4(3) & 20-4(5) & 8-4(2) & 24-4(6) \\ 2(1)-2 & 2(2)-3 & 2(4)-5 & 2(5)-2 & 2(6)-6 \end{array}\right)$$

Simplifying the entries gives

$$\begin{array}{c} R_1 \\ r_2 \\ r_3 \\ r_4 \end{array}\left(\begin{array}{cccc|c} 2 & 3 & 5 & 2 & 6 \\ 0 & 0 & -3 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 3 & 8 & 6 \end{array}\right)$$

Add the bottom row to the second row to convert the 3 into 0 and simplify:

$$\begin{array}{cc} & \begin{array}{cccc} x & y & z & w \end{array} \\ \begin{array}{c} R_1 \\ r_2 \\ r_3 \\ r_4 \end{array} & \left(\begin{array}{cccc|c} 2 & 3 & 5 & 2 & 6 \\ 0 & 0 & -3 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 8 & 7 \end{array}\right) \end{array}$$

Expanding the bottom row, we have $y+8w=7$ which gives $y=7-8w$. Since none of the equations begin with w therefore w is the free variable and we assign the parameter t to it, that is $w=t$ where t is any real number. Hence $y=7-8w=7-8t$.

By the second row we have $-3z=1$ gives $z=-\dfrac{1}{3}$. Need to find the last unknown x by using the first row and substituting the unknowns we already have, $y=7-8t$, $z=-\dfrac{1}{3}$ and $w=t$:

$$2x+3y+5z+2w=6$$
$$2x+3(7-8t)+5\left(-\frac{1}{3}\right)+2t=6$$
$$2x-24t+2t+21-\frac{5}{3}=6$$
$$2x=6+\frac{5}{3}-21+22t=-\frac{40}{3}+22t$$
$$x=11t-\frac{20}{3} \qquad [\text{Dividing by } 2]$$

Our infinite number of solutions are given by $x=11t-\frac{20}{3}$, $y=7-8t$, $z=-\frac{1}{3}$ and $w=t$.

(g) The augmented matrix is

$$\begin{array}{c} R_1 \\ R_2 \\ R_3 \end{array}\left(\begin{array}{cccc|c} 0 & -10 & 38 & 0 & 6 \\ 5 & 6 & -8 & 4 & 3 \\ 10 & 7 & 3 & 8 & 9 \end{array}\right)$$

Executing the following row operation:

$$\begin{array}{c} R_1 \\ R_2 \\ r_3=R_3-2R_2 \end{array}\left(\begin{array}{cccc|c} 0 & -10 & 38 & 0 & 6 \\ 5 & 6 & -8 & 4 & 3 \\ 10-2(5) & 7-2(6) & 3-2(-8) & 8-2(4) & 9-2(3) \end{array}\right)$$

Simplifying the entries in the bottom row we have

$$\begin{array}{c} R_1 \\ R_2 \\ r_3 \end{array}\left(\begin{array}{cccc|c} 0 & -10 & 38 & 0 & 6 \\ 5 & 6 & -8 & 4 & 3 \\ 0 & -5 & 19 & 0 & 3 \end{array}\right)$$

Note that the first row is twice the last row. Carrying out the row operation $2r_3-R_1$ gives:

$$\begin{array}{c} R_1 \\ R_2 \\ r_3'=2r_3-R_1 \end{array}\left(\begin{array}{cccc|c} 0 & -10 & 38 & 0 & 6 \\ 5 & 6 & -8 & 4 & 3 \\ 0 & 2(-5)-(-10) & 2(19)-38 & 0 & 2(3)-6 \end{array}\right)$$

Simplifying the entries in the bottom row we have

$$\begin{array}{c} R_1 \\ R_2 \\ r_3' \end{array}\left(\begin{array}{cccc|c} 0 & -10 & 38 & 0 & 6 \\ 5 & 6 & -8 & 4 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

Dividing the first two rows by -10 and 4 respectively gives

$$\begin{array}{c} R_1/(-10) \\ R_2/4 \\ r_3' \end{array}\left(\begin{array}{cccc|c} 0 & 1 & 38/(-10) & 0 & 6/(-10) \\ 5/4 & 6/4 & -2 & 1 & 3/4 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

Simplifying these entries gives

$$\begin{array}{c} \\ R_1/(-10) \\ R_2/4 \\ r_3{}' \end{array}\begin{array}{c} \begin{array}{cccc} x & y & z & w \end{array} \\ \left(\begin{array}{cccc|c} 0 & 1 & -19/5 & 0 & -3/5 \\ 5/4 & 3/2 & -2 & 1 & 3/4 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right) \end{array}$$

Since none of the equations begin with z and w therefore these are our free variables and by expanding the top row we have

$$y-\frac{19}{5}z=-\frac{3}{5} \quad \text{gives} \quad y=\frac{1}{5}(19z-3)$$

We assign parameter t to the free variable w therefore $z=t$ and so $y=\frac{1}{5}(19t-3)$.

Using the middle row we have

$$\frac{5}{4}x+\frac{3}{2}y-2z+w=\frac{3}{4}$$

Since w is the other free variable we assign the parameter s to it, that is $w=s$. Substituting $w=s$, $z=t$ and $y=\frac{1}{5}(19t-3)$ we have

$$\frac{5}{4}x+\frac{3}{2}\left[\frac{1}{5}(19t-3)\right]-2t+s=\frac{3}{4}$$

$$\frac{5}{4}x+\frac{3}{10}\left[(19t-3)\right]-2t+s=\frac{3}{4}$$

Multiplying this by 20 gives

$$25x+6\left[(19t-3)\right]-40t+20s=15$$

$$25x+114t-18-40t+20s=15$$

$$25x+74t-18+20s=15$$

$$25x=15+18-74t-20s=33-74t-20s$$

$$x=\frac{1}{25}(33-74t-20s)$$

Our infinite number of solutions are given by $x=\frac{1}{25}(33-74t-20s)$, $y=\frac{1}{5}(19t-3)$, $z=t$ and $w=s$.

(h) The augmented matrix is

$$\begin{array}{c} R_1 \\ R_2 \\ R_3 \\ R_4 \end{array}\left(\begin{array}{ccccc|c} 0 & -1 & 0 & 0 & 5 & 3 \\ 3 & -4 & 1 & 6 & 7 & 5 \\ 15 & -20 & 2 & 30 & 3 & -1 \\ 12 & -16 & 7 & 24 & 60 & 10 \end{array}\right)$$

Executing the following row operations:

$$\begin{array}{c} R_1 \\ R_2 \\ r_3=R_3-5R_2 \\ r_4=R_4-4R_2 \end{array}\left(\begin{array}{ccccc|c} 0 & -1 & 0 & 0 & 5 & 3 \\ 3 & -4 & 1 & 6 & 7 & 5 \\ 15-5(3) & -20-5(-4) & 2-5(1) & 30-5(6) & 3-5(7) & -1-5(5) \\ 12-4(3) & -16-4(-4) & 7-4(1) & 24-4(6) & 60-4(7) & 10-4(5) \end{array}\right)$$

Simplifying the entries gives

$$\begin{array}{c} R_1 \\ R_2 \\ r_3 \\ r_4 \end{array} \left(\begin{array}{ccccc|c} 0 & -1 & 0 & 0 & 5 & 3 \\ 3 & -4 & 1 & 6 & 7 & 5 \\ 0 & 0 & -3 & 0 & -32 & -26 \\ 0 & 0 & 3 & 0 & 32 & -10 \end{array}\right)$$

Adding the last two rows gives

$$\begin{array}{c} R_1 \\ R_2 \\ r_3 \\ r_4 + r_3 \end{array} \left(\begin{array}{ccccc|c} 0 & -1 & 0 & 0 & 5 & 3 \\ 3 & -4 & 1 & 6 & 7 & 5 \\ 0 & 0 & -3 & 0 & -32 & -26 \\ 0 & 0 & 0 & 0 & 0 & -36 \end{array}\right)$$

From the bottom row we have $0 = -36$ which is clearly **inconsistent**. Hence the given linear system has **no** solutions.

(i) The given linear system is very similar to part (h). The only difference is that the last constant is 46 rather than 10. If we carry out the above operations and simplify we have

$$\begin{array}{c} R_1 \\ R_2 \\ r_3 \\ r_4 \end{array} \left(\begin{array}{ccccc|c} 0 & -1 & 0 & 0 & 5 & 3 \\ 3 & -4 & 1 & 6 & 7 & 5 \\ 0 & 0 & -3 & 0 & -32 & -26 \\ 0 & 0 & 3 & 0 & 32 & 26 \end{array}\right)$$

Adding the last two rows gives

$$\begin{array}{c} R_1 \\ R_2 \\ r_3 \\ r_4^* = r_4 + r_3 \end{array} \left(\begin{array}{ccccc|c} 0 & -1 & 0 & 0 & 5 & 3 \\ 3 & -4 & 1 & 6 & 7 & 5 \\ 0 & 0 & -3 & 0 & -32 & -26 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

Multiplying the first row by -1, second row by 1/3 and third row by $-1/3$ we have

$$\begin{array}{cc} & \begin{array}{ccccc} x & y & z & w & u \end{array} \\ \begin{array}{c} -R_1 \\ R_2/3 \\ -r_3/3 \\ r_4^* = r_4 + r_3 \end{array} & \left(\begin{array}{ccccc|c} 0 & 1 & 0 & 0 & -5 & -3 \\ 1 & -4/3 & 1/3 & 2 & 7/3 & 5/3 \\ 0 & 0 & 1 & 0 & 32/3 & 26/3 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right) \end{array}$$

From the top row we have

$$y - 5u = -3 \text{ implies that } y = 5u - 3$$

Note that u is a free variable so we assign $u = t$ where t is any real number. Hence $y = 5t - 3$.

By expanding the third row we have

$$z + \frac{32}{3}u = \frac{26}{3}$$

$$z = \frac{1}{3}(26 - 32t) \quad [\text{Because } u = t]$$

By the second row we have

$$x - \frac{4}{3}y + \frac{1}{3}z + 2w + \frac{7}{3}u = \frac{5}{3} \qquad (*)$$

We have the unknowns y, z and u from above and also w is the other free variable. Hence let $w = s$ where s is any real number. Substituting these, $y = 5t - 3$, $z = \frac{1}{3}(26 - 32t)$, $w = s$ and $u = t$, into (*) gives

$$x - \frac{4}{3}(5t-3) + \frac{1}{3}\left[\frac{1}{3}(26-32t)\right] + 2s + \frac{7}{3}t = \frac{5}{3}$$

Multiplying through by 9 gives

$$\begin{aligned} 9x - 12(5t-3) + (26-32t) + 18s + 21t &= 15 \\ 9x - 60t + 36 + 26 - 32t + 18s + 21t &= 15 \\ 9x - 71t + 18s + 62 &= 15 \\ x &= \frac{1}{9}(71t - 18s - 47) \end{aligned}$$

Our solution is $x = \frac{1}{9}(71t - 18s - 47)$, $y = 5t - 3$, $z = \frac{1}{3}(26 - 32t)$, $w = s$ and $u = t$.

(j) The augmented matrix is

$$\begin{array}{c} \mathrm{R}_1 \\ \mathrm{R}_2 \\ \mathrm{R}_3 \\ \mathrm{R}_4 \end{array} \left(\begin{array}{ccccc|c} 2 & -1 & 3 & 5 & 5 & 1 \\ 1 & 7 & 6 & -11 & 7 & -8 \\ 1 & -2 & 6 & 4 & 1 & 5 \\ 2 & -6 & 0 & 14 & 2 & 20/3 \end{array}\right)$$

We execute the following row operations:

$$\begin{array}{c} \mathrm{R}_1 \\ \mathrm{R}_2 \\ \mathrm{r}_3 = \mathrm{R}_3 - \mathrm{R}_2 \\ \mathrm{r}_4 = \mathrm{R}_4 - \mathrm{R}_1 \end{array} \left(\begin{array}{ccccc|c} 2 & -1 & 3 & 5 & 5 & 1 \\ 1 & 7 & 6 & -11 & 7 & -8 \\ 1-1 & -2-7 & 6-6 & 4-(-11) & 1-7 & 5-(-8) \\ 2-2 & -6-(-1) & 0-3 & 14-5 & 2-5 & 20/3-1 \end{array}\right)$$

Simplifying gives

$$\begin{array}{c} \mathrm{R}_1 \\ \mathrm{R}_2 \\ \mathrm{r}_3 \\ \mathrm{r}_4 \end{array} \left(\begin{array}{ccccc|c} 2 & -1 & 3 & 5 & 5 & 1 \\ 1 & 7 & 6 & -11 & 7 & -8 \\ 0 & -9 & 0 & 15 & -6 & 13 \\ 0 & -5 & -3 & 9 & -3 & 17/3 \end{array}\right)$$

Carrying out the row operation $2\mathrm{R}_2 - \mathrm{R}_1$ gives

$$\begin{array}{c} \mathrm{R}_1 \\ \mathrm{r}_2 = 2\mathrm{R}_2 - \mathrm{R}_1 \\ \mathrm{r}_3 \\ \mathrm{r}_4 \end{array} \left(\begin{array}{ccccc|c} 2 & -1 & 3 & 5 & 5 & 1 \\ 2(1)-2 & 2(7)-(-1) & 2(6)-3 & 2(-11)-5 & 2(7)-5 & 2(-8)-1 \\ 0 & -9 & 0 & 15 & -6 & 13 \\ 0 & -5 & -3 & 9 & -3 & 17/3 \end{array}\right)$$

Simplifying the entries in the second row gives

$$\begin{array}{c} \mathrm{R}_1 \\ \mathrm{r}_2 \\ \mathrm{r}_3 \\ \mathrm{r}_4 \end{array} \left(\begin{array}{ccccc|c} 2 & -1 & 3 & 5 & 5 & 1 \\ 0 & 15 & 9 & -27 & 9 & -17 \\ 0 & -9 & 0 & 15 & -6 & 13 \\ 0 & -5 & -3 & 9 & -3 & 17/3 \end{array}\right)$$

Note that the last row is -3 times the second row so we execute the row operation $3\mathrm{r}_4 + \mathrm{r}_2$:

$$\begin{array}{r}\mathrm{R}_1\\ \mathrm{r}_2\\ \mathrm{r}_3\\ \mathrm{r}_4^*=3\mathrm{r}_4+\mathrm{r}_2\end{array}\left(\begin{array}{cccccc|c}2&-1&3&5&5&1\\0&15&9&-27&9&-17\\0&-9&0&15&-6&13\\0&3(-5)+15&3(-3)+9&3(9)-27&3(-3)+9&3(17/3)-17\end{array}\right)$$

Simplifying the entries in the last row gives

$$\begin{array}{r}\mathrm{R}_1\\ \mathrm{r}_2\\ \mathrm{r}_3\\ \mathrm{r}_4^*\end{array}\left(\begin{array}{ccccc|c}2&-1&3&5&5&1\\0&15&9&-27&9&-17\\0&-9&0&15&-6&13\\0&0&0&0&0&0\end{array}\right)$$

Carrying out the row operation $5\mathrm{r}_3+3\mathrm{r}_2$:

$$\begin{array}{r}\mathrm{R}_1\\ \mathrm{r}_2\\ \mathrm{r}_3^*=5\mathrm{r}_3+3\mathrm{r}_2\\ \mathrm{r}_4^*\end{array}\left(\begin{array}{ccccc|c}2&-1&3&5&5&1\\0&15&9&-27&9&-17\\0&5(-9)+3(15)&27&5(15)+3(-27)&5(-6)+3(9)&5(13)+3(-17)\\0&0&0&0&0&0\end{array}\right)$$

Simplifying the entries in the third row:

$$\begin{array}{rc}& \begin{array}{ccccc}x&y&z&w&u\end{array}\\ \begin{array}{r}\mathrm{R}_1\\ \mathrm{r}_2\\ \mathrm{r}_3^*=5\mathrm{r}_3+3\mathrm{r}_2\\ \mathrm{r}_4^*\end{array} & \left(\begin{array}{ccccc|c}2&-1&3&5&5&1\\0&15&9&-27&9&-17\\0&0&27&-6&-3&14\\0&0&0&0&0&0\end{array}\right)\end{array}\qquad (\dagger)$$

We have 3 non-zero rows (equations) and 5 unknowns therefore there are $5-3=2$ free variables. *Which of the unknowns are the free variables?*

Since none of the rows (equations) start with u and w therefore these are the free variables. Let $u=s$ and $w=t$ where s and t are any real numbers.

By expanding the third row of $(\dagger)$ we have

$$\begin{aligned}27z-6w-3u&=14\\27z&=14+6w+3u\\z&=\frac{1}{27}(14+6t+3s)\qquad [\text{Substituting } u=s \text{ and } w=t]\end{aligned}$$

By the second row we have

$$15y+9z-27w+9u=-17$$
$$15y=-17-9z+27w-9u$$
$$y=\frac{1}{15}\left(-17-9\frac{1}{27}(14+6t+3s)+27t-9s\right)$$
$$=\frac{1}{15}\left(-17-\frac{1}{3}(14+6t+3s)+27t-9s\right)$$
$$=\frac{1}{45}\left(-51-(14+6t+3s)+81t-27s\right)$$
$$=\frac{1}{45}\left(-51-14-6t-3s+81t-27s\right)$$
$$=\frac{1}{45}(75t-30s-65)=\frac{1}{9}(15t-6s-13)$$

We have $y=\frac{1}{9}(15t-6s-13)$. By expanding the top row of $(\dagger)$ we have

$$2x-y+3z+5w+5u=1$$
$$2x=1+y-3z-5w-5u$$

Substituting $y=\frac{1}{9}(15t-6s-13)$, $z=\frac{1}{27}(14+6t+3s)$, $w=t$ and $u=s$ into the above:

$$2x=1+\frac{1}{9}(15t-6s-13)-3\frac{1}{27}(14+6t+3s)-5t-5s$$
$$=1+\frac{1}{9}(15t-6s-13)-\frac{1}{9}(14+6t+3s)-5t-5s$$
$$=\frac{1}{9}\left(9+(15t-6s-13)-(14+6t+3s)-45t-45s\right)$$
$$=\frac{1}{9}\left(9+15t-6s-13-14-6t-3s-45t-45s\right)$$
$$=\frac{1}{9}(-18-36t-54s)$$
$$=\frac{-9}{9}(2+4t+6s)=-(2+4t+6s)$$

Dividing both sides by 2 gives $x=-(1+2t+3s)$.

Our solution is $x=-(1+2t+3s)$, $y=\frac{1}{9}(15t-6s-13)$, $z=\frac{1}{27}(14+6t+3s)$, $w=t$ and $u=s$.

2. (a) We are given the reduced row echelon form is

$$\begin{array}{cccc}x & y & z & w\end{array}$$
$$\left(\begin{array}{cccc|c}1 & 1 & 0 & -10 & -9\\ 0 & 0 & 1 & -7 & -7\\ 0 & 0 & 0 & 0 & 0\end{array}\right)$$

There are 2 non-zero equations and 4 unknowns so we have $4-2=2$ free variables. Since non of the equations start with *y* and *w* so these are our free variables. Let $y=s$ and $w=t$.

Expanding the middle row we have

$$z-7w=-7 \quad\Rightarrow\quad z=-7+7w=-7+7t$$

By the top row we have

$$x+y-10w=-9 \quad\Rightarrow\quad x=-9+10w-y=-9+10t-s$$

We have $x=-9+10t-s$, $y=s$, $z=-7+7t$ and $w=t$. The vector of unknowns **x** is given by:

$$\mathbf{x}=\begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix}=\begin{pmatrix} -9+10t-s \\ s \\ -7+7t \\ t \end{pmatrix}=\begin{pmatrix} -9 \\ 0 \\ -7 \\ 0 \end{pmatrix}+s\begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \end{pmatrix}+t\begin{pmatrix} 10 \\ 0 \\ 7 \\ 1 \end{pmatrix}$$

(b) The reduced row echelon form is:

$$\begin{array}{cccccc} x_1 & x_2 & x_3 & x_4 & x_5 & x_6 \end{array}$$
$$\left(\begin{array}{cccccc|c} 1 & 1 & 1 & 1 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

We have 6 unknowns and 2 non-zero equations which means that there are $6-2=4$ free variables. None of the equations start with $x_2,\ x_3,\ x_4$ and x_6 so these are our free variables. Let $x_2=p,\ x_3=q,\ x_4=r$ and $x_6=s$.

Expanding the middle row we have $x_5=0$. Using the top row we have

$$\begin{aligned} x_1+x_2+x_3+x_4+2x_5=0 \quad\Rightarrow\quad x_1 &= -x_2-x_3-x_4-2x_5 \\ &= -p-q-r-2(0)=-p-q-r \end{aligned}$$

Our general solution is $x_1=-p-q-r$, $x_2=p,\ x_3=q,\ x_4=r,\ x_5=0$ and $x_6=s$. We can write this in vector form as

$$\mathbf{x}=\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{pmatrix}=\begin{pmatrix} -p-q-r \\ p \\ q \\ r \\ 0 \\ s \end{pmatrix}=p\begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}+q\begin{pmatrix} -1 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}+r\begin{pmatrix} -1 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}+s\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

(c) We are given the reduced row echelon form as

$$\begin{array}{cccccc} x_1 & x_2 & x_3 & x_4 & x_5 & x_6 \end{array}$$
$$\left(\begin{array}{cccccc|c} 1 & 3 & 0 & 0 & 6 & 2 & 2 \\ 0 & 0 & 1 & 0 & 3 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

There are $6-3=3$ free variables. None of the equations start with $x_2,\ x_5,\ x_6$ so these are our free variables. Let $x_2=r,\ x_5=s,\ x_6=t$.

By the third row we have $x_4=0$. Expanding the second row we have

$$x_3+3x_5=1 \Rightarrow x_3=1-3x_5=1-3s$$

Using the top row we have

$$x_1 + 3x_2 + 6x_5 + 2x_6 = 2 \quad \Rightarrow \quad x_1 = 2 - 3x_2 - 6x_5 - 2x_6 = 2 - 3r - 6s - 2t$$

Hence we have the solution $x_1 = 2 - 3r - 6s - 2t$, $x_2 = r$, $x_3 = 1 - 3s$, $x_4 = 0$, $x_5 = s$, $x_6 = t$.
In vector form the solution is

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{pmatrix} = \begin{pmatrix} 2-3r-6s-2t \\ r \\ 1-3s \\ 0 \\ s \\ t \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + r\begin{pmatrix} -3 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} + s\begin{pmatrix} -6 \\ 0 \\ -3 \\ 0 \\ 1 \\ 0 \end{pmatrix} + t\begin{pmatrix} -2 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

3. The question should say that $z \neq 0$. Writing the augmented matrix we have

$$\begin{array}{c} \mathrm{R}_1 \\ \mathrm{R}_2 \\ \mathrm{R}_3 \end{array} \left(\begin{array}{ccc|c} 2 & -1 & -4 & k \\ -1 & 1 & 2 & k \\ -1 & 1 & k & k \end{array}\right)$$

Carrying out the row operation $\mathrm{R}_3 - \mathrm{R}_2$:

$$\begin{array}{c} \mathrm{R}_1 \\ \mathrm{R}_2 \\ \mathrm{R}_3{*} = \mathrm{R}_3 - \mathrm{R}_2 \end{array} \left(\begin{array}{ccc|c} 2 & -1 & -4 & k \\ -1 & 1 & 2 & k \\ 0 & 0 & k-2 & 0 \end{array}\right)$$

By expanding the bottom row we notice that the system is consistent if $k - 2 = 0 \quad \Rightarrow \quad k = 2$.
Substituting $k = 2$ into the above we have

$$\begin{array}{c} \mathrm{R}_1 \\ \mathrm{R}_2 \\ \mathrm{R}_3{*} = \mathrm{R}_3 - \mathrm{R}_2 \end{array} \left(\begin{array}{ccc|c} 2 & -1 & -4 & 2 \\ -1 & 1 & 2 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

Executing the row operation $2\mathrm{R}_2 + \mathrm{R}_1$ gives

$$\begin{array}{cc} & \begin{array}{ccc} x & y & z \end{array} \\ \begin{array}{c} \mathrm{R}_1 \\ \mathrm{R}_2{*} = 2\mathrm{R}_2 + \mathrm{R}_1 \\ \mathrm{R}_3{*} \end{array} & \left(\begin{array}{ccc|c} 2 & -1 & -4 & 2 \\ 0 & 1 & 0 & 6 \\ 0 & 0 & 0 & 0 \end{array}\right) \end{array}$$

None of the equations begin with *z* so this is our free variable, let $z = s$ where *s* is any real number.
Expanding the middle row we have $y = 6$
Examining the top row we have

$$2x - y - 4z = 2 \quad \Rightarrow \quad 2x = 2 + y + 4z$$

$$x = 1 + \frac{1}{2}y + 2z$$

Putting $y = 6$ and $z = s$ into this gives

$$x = 1 + 3 + 2s = 4 + 2s$$

Our solution is $x = 4 + 2s$, $y = 6$ and $z = s$ where *s* is any real number.

4. Need to prove $k\mathbf{u}+c\mathbf{v}$ is a solution of $\mathbf{Ax}=\mathbf{O}$ provided **u** and **v** are distinct solutions.
Proof.
Since **u** and **v** are solutions of $\mathbf{Ax}=\mathbf{O}$ so

$$\mathbf{Au}=\mathbf{O} \text{ and } \mathbf{Av}=\mathbf{O}$$

Consider

$$\begin{aligned}\mathbf{A}(k\mathbf{u}+c\mathbf{v}) &= \mathbf{A}(k\mathbf{u})+\mathbf{A}(c\mathbf{v})\\ &= k(\mathbf{Au})+c(\mathbf{Av})=k(\mathbf{O})+c(\mathbf{O})=\mathbf{O}\end{aligned}$$

We have $\mathbf{A}(k\mathbf{u}+c\mathbf{v})=\mathbf{O}$ therefore $k\mathbf{u}+c\mathbf{v}$ is a solution of $\mathbf{Ax}=\mathbf{O}$. Since *k* and *c* are any scalars so $k\mathbf{u}+c\mathbf{v}$ covers an infinite number of solutions.

■

5. We need to prove the following:
If $\mathbf{x}_p$ is a particular solution to this $\mathbf{Ax}=\mathbf{b}$ and $\mathbf{x}_h$ is the solution to the associated homogeneous system $\mathbf{Ax}=\mathbf{O}$ then $\mathbf{x}_p+\mathbf{x}_h$ is a solution to $\mathbf{Ax}=\mathbf{b}$.
Proof.
We are given that

$$\mathbf{Ax}_h=\mathbf{O} \text{ and } \mathbf{Ax}_P=\mathbf{b}$$

Adding these two results gives

$$\mathbf{Ax}_h+\mathbf{Ax}_P=\mathbf{O}+\mathbf{b}=\mathbf{b}$$
$$\mathbf{A}(\mathbf{x}_h+\mathbf{x}_P)=\mathbf{b}$$

Hence $\mathbf{x}=\mathbf{x}_p+\mathbf{x}_h$ is a solution to $\mathbf{Ax}=\mathbf{b}$. This is our required result.

■

6. *Proof*
(Very similar to the proof of Proposition (1-12).)
We are given that the number of non-zero rows (equations) *r* is less than the number of unknowns *n*.
Consider the reduced row echelon form **R** with *r* non-zero rows. This means there are exactly *r* leading 1's. *Why?*
Because **R** is in reduced row echelon form which means any non-zero row has a leading 1.
Every column of **R** cannot have a leading 1 because you would need *n* leading 1's but there are only *r* which is less than *n*.
Therefore there must be a column call it x_j which has **no** leading one. We have

$$\begin{array}{c|cccccc|c|}
 & x_1 & \cdots & x_j & x_{j+1} & \cdots & x_n & \text{jth Column} \\
\hline
 & a_{11} & \cdots & & & & & b_1 \\
 & \vdots & & & & & & \vdots \\
jth\ \text{Row} & 0 & \cdots & 0 & a_{j(j+1)} & & a_{jn} & b_j \\
 & & & & & & & b_{j+1}
\end{array}$$

If the jth (x_j) column **only** has zero entries then we can let $x_j=t$ where *t* is any real number.
The solutions are

$$x_1=s_1,\ x_2=s_2,\ \cdots, x_{j-1}=s_{j-1},\ x_{j+1}=s_{j+1},\ \cdots,\ x_n=s_n \text{ and } x_j=t \text{ where } t\in\mathbb{R}$$

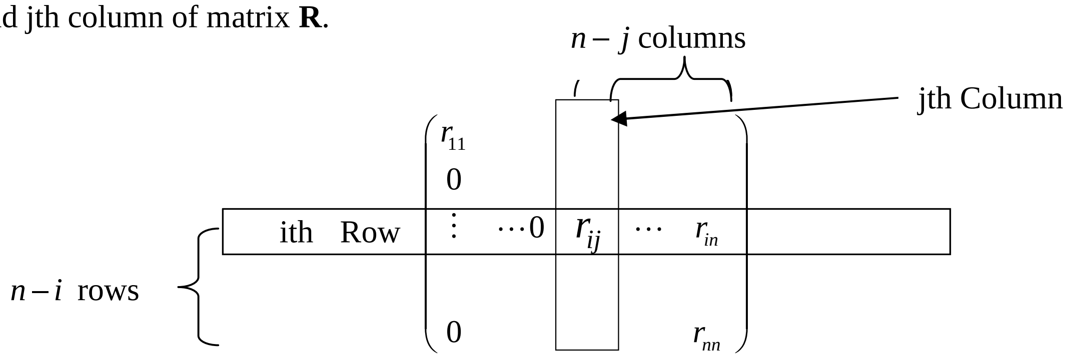
Hence our infinite solution set is

$$\left\{s_1,\ s_2,\ \cdots,\ s_{j-1},\ t,\ s_{j+1},\ \cdots,\ x_n=s_n \ \middle|\ t\in\mathbb{R}\right\}$$

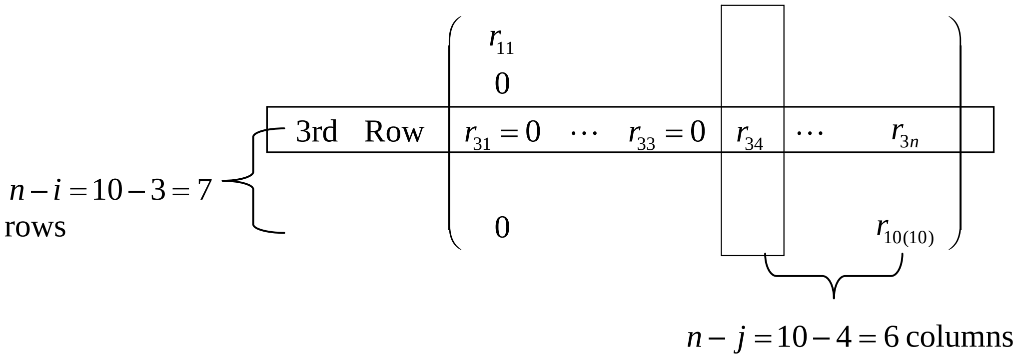
We have an infinite number of solutions to $\mathbf{Ax} = \mathbf{b}$.
If there is a non-zero entry in the jth column then at least one of unknowns must be written in terms of x_j. Let the unknown x_j be *t* where *t* is any real number. This means that at least one of the other unknowns can be written in terms of *t*. Thus we have infinite number of solutions.

■

7. To prove that the reduced row echelon form of a $n \times n$ matrix $\mathbf{A}$ is either the identity matrix or it has at least one row of zeros.
Proof.
If $\mathbf{A} = \mathbf{O}$ then we have a row of zeros and our proof is complete. Suppose $\mathbf{A} \neq \mathbf{O}$.
Let **R** be the reduced row echelon form of this matrix **A**. Let r_{ij} be a non-zero entry in the ith row and jth column of matrix **R**.



The matrix **R** is in reduced row echelon form therefore the non-zero entries go **strictly** from top left to bottom right as you move down the matrix **R**.
Consider the first non-zero entry r_{ij} in the ith row.
This means that *j* must be greater than or equal to *i* because r_{ij} is non-zero so it is strictly to the right of r_{ii} or is the r_{ii} entry itself. This means we have $j \geq i$.
If $j > i$ then $r_{ii} = 0$ and every **non-zero** entry below ith row is **strictly** to the right of r_{ij}. There are **only** $n - j$ columns to the right of r_{ij} but $n - i$ rows below r_{ij}. We have $n - j < n - i$ because $j > i$ therefore we conclude that the $i + (n - j) + 1$ row is a row of zeros. *Why?*
As an example let $n = 10,\ i = 3,\ j = 4$ say. Then $r_{ii} = r_{33} = 0$ and every non-zero entry below the third row is strictly to the right of $r_{ij} = r_{34}$. There are only 6 columns to the right of r_{34}.This means as we work across the columns and down the rows the zeros must move to the right and down. However we have 7 rows below r_{34} which suggests that last row must be a row of zeros.



If $j = i$ then $r_{ij} \neq 0$ (because r_{ij} is non-zero entry that we are considering) for $i = j$. Now **R** is in reduced row echelon form therefore $r_{ij} = 1$ for $i = j$. Again by the procedure of rref $r_{ij} = 0$ for $i \neq j$. Hence $\mathbf{R} = \mathbf{I}$ where **I** is the identity matrix.
This completes our proof.

■

8. We are given the following:

x_1	x_2	x_3	6
x_4	x_5	x_6	15
5	7	9	

Forming the equations by summing up the rows and columns we have

$$\begin{array}{ccccccccccccl} x_1 & + & x_2 & + & x_3 & & & & & & = & 6 & \text{First Row} \\ & & & & & & x_4 & + & x_5 & + \; x_6 & = & 15 & \text{Second Row} \\ x_1 & & & & & + & x_4 & & & & = & 5 & \text{First Column} \\ & & x_2 & & & & & + & x_5 & & = & 7 & \text{Second Column} \\ & & & & x_3 & & & & & + \; x_6 & = & 9 & \text{Third Column} \end{array}$$

As long as this system is consistent we can be sure that we have an infinite number of solutions because we have more unknowns (6) than equations (5).
The augment matrix and reduced row echelon form which is evaluated by using mathematical software such as Maple or MATLAB is given by:

$$\left(\begin{array}{cccccc|c} 1 & 1 & 1 & 0 & 0 & 0 & 6 \\ 0 & 0 & 0 & 1 & 1 & 1 & 15 \\ 1 & 0 & 0 & 1 & 0 & 0 & 5 \\ 0 & 1 & 0 & 0 & 1 & 0 & 7 \\ 0 & 0 & 1 & 0 & 0 & 1 & 9 \end{array}\right) \Rightarrow \begin{array}{c} \\ \text{row 1} \\ \text{row 2} \\ \text{row 3} \\ \text{row 4} \\ \text{row 5} \end{array} \begin{array}{c} \begin{array}{cccccc} x_1 & x_2 & x_3 & x_4 & x_5 & x_6 \end{array} \\ \left(\begin{array}{cccccc|c} 1 & 0 & 0 & 0 & -1 & -1 & -10 \\ 0 & 1 & 0 & 0 & 1 & 0 & 7 \\ 0 & 0 & 1 & 0 & 0 & 1 & 9 \\ 0 & 0 & 0 & 1 & 1 & 1 & 15 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right) \end{array}$$

In row echelon form we have 4 non-zero equations and 6 unknowns therefore there are $6 - 4 = 2$ free variables. None of the equations start with x_5 and x_6 so these are our free variables. Let $x_5 = s$ and $x_6 = t$ where *s* and *t* are any real numbers. Expanding row 4 we have

$$x_4 + x_5 + x_6 = 15 \text{ implies } x_4 = 15 - x_5 - x_6 = 15 - s - t$$

Expanding row 3 we have $x_3 + x_6 = 9$ implies $x_3 = 9 - x_6 = 9 - t$.
Expanding row 2 we have $x_2 + x_5 = 7$ implies $x_2 = 7 - x_5 = 7 - s$.
Expanding row 1 we have $x_1 - x_5 - x_6 = -10$ implies $x_1 = -10 + x_5 + x_6 = -10 + s + t$.
Our general solution is

$$x_1 = -10 + s + t, \; x_2 = 7 - s, \; x_3 = 9 - t, \; x_4 = 15 - s - t, \; x_5 = s \text{ and } x_6 = t \qquad (*)$$

where *s* and *t* are any real numbers. This is the general solution of the given puzzle which we can write in vector form as

$$\mathbf{x}=\begin{pmatrix}x_1\\x_2\\x_3\\x_4\\x_5\\x_6\end{pmatrix}=\begin{pmatrix}-10+s+t\\7-s\\9-t\\15-s-t\\s\\t\end{pmatrix}=s\begin{pmatrix}1\\-1\\0\\-1\\1\\0\end{pmatrix}+t\begin{pmatrix}1\\0\\-1\\-1\\0\\1\end{pmatrix}+\begin{pmatrix}-10\\7\\9\\15\\0\\0\end{pmatrix}$$

A particular solution can be found by letting s and t take on particular values. For instance let $s=1,\ t=2$ then substituting this into (*) gives

$$x_1=-7\,,\ x_2=6\,,\ x_3=7\,,\ x_4=12\,,\ x_5=1\ \text{and}\ x_6=2$$

Check that this is actually a solution to the above puzzle.
If you want your solution so that the solution is whole numbers less than 10 then try $s=5,\ t=6$.

9. We are given the following puzzle:

x_1	x_2	x_3	16
x_4	x_5	x_6	21
x_7	x_8	x_9	8
17	15	13	

Formulating the equations we have

$$\begin{array}{ccccccccccccccccccc}
x_1 & + & x_2 & + & x_3 & & & & & & & & & & & & & = & 16\\
& & & & & & x_4 & + & x_5 & + & x_6 & & & & & & & = & 21\\
& & & & & & & & & & & & x_7 & + & x_8 & + & x_9 & = & 8\\
x_1 & & & & & + & x_4 & & & & & + & x_7 & & & & & = & 17\\
& & x_2 & & & & & + & x_5 & & & & & + & x_8 & & & = & 15\\
& & & & x_3 & & & & & + & x_6 & & & & & + & x_9 & = & 13
\end{array}$$

As long as this system is consistent we can be sure that we have an infinite number of solutions because we have more unknowns (6) than equations (5).
The augment matrix and reduced row echelon form which is evaluated by using mathematical software such as Maple or MATLAB is given by:

$$\left(\begin{array}{ccccccccc|c}
1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 16\\
0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 21\\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 8\\
1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 17\\
0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 15\\
0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 13
\end{array}\right)\Rightarrow$$

$$\begin{array}{c} \begin{array}{ccccccccc} x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & x_7 & x_8 & x_9 \end{array} \\ \left(\begin{array}{ccccccccc|c} 1 & 0 & 0 & 0 & -1 & -1 & 0 & -1 & -1 & -12 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 15 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 13 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 21 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 8 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right) \end{array}$$

In reduced row echelon form we have 5 non-zero equations and 9 unknowns therefore there are $9-5=4$ free variables. None of the equations start with x_5, x_6 x_8 and x_9 so these are our free variables. Let $x_5 = p$, $x_6 = q$, $x_8 = s$ and $x_9 = t$ where p, q, s and t are any real numbers. Expanding row 5 we have

$$x_7 + x_8 + x_9 = 8 \text{ implies } x_7 = 8 - x_8 - x_9 = 8 - s - t$$

Expanding row 4 we have $x_4 + x_5 + x_6 = 21$ implies $x_4 = 21 - x_5 - x_6 = 21 - p - q$.

Expanding row 3 we have $x_3 + x_6 + x_9 = 13$ implies $x_3 = 13 - x_6 - x_9 = 13 - q - t$.

Expanding row 2 we have $x_2 + x_5 + x_8 = 15$ implies $x_2 = 15 - x_5 - x_8 = 15 - p - s$.

By row 1 we have

$$x_1 - x_5 - x_6 - x_8 - x_9 = -12 \quad \Rightarrow \quad x_1 = -12 + x_5 + x_6 + x_8 + x_9 = -12 + p + q + s + t$$

Our solution is

$$x_1 = -12 + p + q + s + t,\ x_2 = 15 - p - s,\ x_3 = 13 - q - t,\ x_4 = 21 - p - q,\ x_5 = p,\ x_6 = q,$$
$$x_7 = 8 - s - t,\ x_8 = s \text{ and } x_9 = t \qquad (*)$$

where p, q, s and t are any real numbers. This is the general solution of the given puzzle which we can write in vector form as

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \\ x_8 \\ x_9 \end{pmatrix} = \begin{pmatrix} -12 + p + q + s + t \\ 15 - p - s \\ 13 - q - t \\ 21 - p - q \\ p \\ q \\ 8 - s - t \\ s \\ t \end{pmatrix} = \begin{pmatrix} -12 \\ 15 \\ 13 \\ 21 \\ 0 \\ 0 \\ 8 \\ 0 \\ 0 \end{pmatrix} + p\begin{pmatrix} 1 \\ -1 \\ 0 \\ -1 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} + q\begin{pmatrix} 1 \\ 0 \\ -1 \\ -1 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + s\begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \\ 0 \\ 0 \\ -1 \\ 1 \\ 0 \end{pmatrix} + t\begin{pmatrix} 1 \\ 0 \\ -1 \\ 0 \\ 0 \\ 0 \\ -1 \\ 0 \\ 1 \end{pmatrix}$$

You can substitute particular values for p, q, s and t to find a particular solution.

Let $p = 1$, $q = 2$, $s = 3$, $t = 4$ then substituting these into (*) we have

$$x_1 = -12 + 1 + 2 + 3 + 4 = -2,\ x_2 = 15 - 1 - 3 = 11,\ x_3 = 13 - 2 - 4 = 7,\ x_4 = 21 - 1 - 2 = 18,$$
$$x_5 = 1,\ x_6 = 2,\ x_7 = 8 - 3 - 4 = 1,\ x_8 = 3 \text{ and } x_9 = 4$$

Check that the puzzle works for these numbers.

10. Formulating the equations in the puzzle:

$$\begin{array}{ccccccccccccccccccc}
x_1 & + & x_2 & & & + & x_4 & + & x_5 & & & & & & & & & = & 19 \\
 & & x_2 & + & x_3 & & & + & x_5 & + & x_6 & & & & & & & = & 18 \\
 & & & & & & x_4 & + & x_5 & & & + & x_7 & + & x_8 & & & = & 28 \\
 & & & & & & & & x_5 & + & x_6 & & & + & x_8 & + & x_9 & = & 25
\end{array}$$

Writing this as an augmented matrix and finding the reduced row echelon form by using Maple or MATLAB we have:

$$\left(\begin{array}{ccccccccc|c}
1 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 19 \\
0 & 1 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 18 \\
0 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 28 \\
0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 25
\end{array}\right)$$

$$\begin{array}{c}
\begin{array}{ccccccccc} x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & x_7 & x_8 & x_9 \end{array} \\
\left(\begin{array}{ccccccccc|c}
1 & 0 & -1 & 0 & 0 & 0 & -1 & 0 & 1 & -2 \\
0 & 1 & 1 & 0 & 0 & 0 & 0 & -1 & -1 & -7 \\
0 & 0 & 0 & 1 & 0 & -1 & 1 & 0 & -1 & 3 \\
0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 25
\end{array}\right)
\end{array}$$

We have 9 unknowns and 4 non-zero equations which means there are $9-4=5$ free variables. Since none of the equations begin with x_3, x_6, x_7, x_8 and x_9 so these are our 5 free variables. Let $x_3 = p$, $x_6 = q$, $x_7 = r$, $x_8 = s$ and $x_9 = t$.

Expanding the bottom row we have

$$x_5 + x_6 + x_8 + x_9 = 25 \Rightarrow x_5 = 25 - x_6 - x_8 - x_9 = 25 - q - s - t$$

Expanding the third row we have

$$x_4 - x_6 + x_7 - x_9 = 3 \Rightarrow x_4 = 3 + x_6 - x_7 + x_9 = 3 + q - r + t$$

Expanding the second row we have

$$x_2 + x_3 - x_8 - x_9 = -7 \quad \Rightarrow x_2 = -7 - x_3 + x_8 + x_9 = -7 - p + s + t$$

By the top row we have

$$x_1 - x_3 - x_7 + x_9 = -2 \Rightarrow x_1 = -2 + x_3 + x_7 - x_9 = -2 + p + r - t$$

The general solution is given by

$$x_1 = -2 + p + r - t,\ x_2 = -7 - p + s + t, x_3 = p, x_4 = 3 + q - r + t,$$

$$x_5 = 25 - q - s - t,\ x_6 = q,\ x_7 = r,\ x_8 = s,\ x_9 = t$$

The general solution in vector form is:

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \\ x_8 \\ x_9 \end{pmatrix} = \begin{pmatrix} -2+p+r-t \\ -7-p+s+t \\ p \\ 3+q-r+t \\ 25-q-s-t \\ q \\ r \\ s \\ t \end{pmatrix} = \begin{pmatrix} -2 \\ -7 \\ 0 \\ 3 \\ 25 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} + p\begin{pmatrix} 1 \\ -1 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} + q\begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ -1 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + r\begin{pmatrix} 1 \\ 0 \\ 0 \\ -1 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} + s\begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ -1 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} + t\begin{pmatrix} -1 \\ 1 \\ 0 \\ 1 \\ -1 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

Substituting in particular values for $x_3 = p$, $x_6 = q$, $x_7 = r$, $x_8 = s$ and $x_9 = t$ we can find a particular solution. The solution for $p = q = r = s = t = 1$ gives

$$x_1 = -2 + p + r - t = -2 + 1 + 1 - 1 = -1,$$
$$x_2 = -7 - p + s + t = -7 - 1 + 1 + 1 = -6,$$
$$x_4 = 3 + q - r + t = 3 + 1 - 1 + 1 = 4$$
$$x_5 = 25 - q - s - t = 25 - 1 - 1 - 1 = 22$$
$$x_3 = 1,\ x_6 = 1,\ x_7 = 1,\ x_8 = 1 \text{ and } x_9 = 1$$

Check that this solution

$$x_1 = -1,\ x_2 = -6,\ x_3 = 1,\ x_4 = 4,\ x_5 = 22,\ x_6 = 1,\ x_7 = 1,\ x_8 = 1 \text{ and } x_9 = 1$$

actually works by substituting these numbers into the puzzle.
Remember there are an infinite number of solutions to this puzzle.

11. Writing the given equations in an augmented matrix gives:

$$\left(\begin{array}{cccc|c} 2 & -4 & 4 & 0.077 & 3.86 \\ 0 & -2 & 2 & -0.056 & -3.47 \\ 2 & -2 & 0 & 0 & 0 \end{array}\right)$$

Using software to place this into reduced row echelon form yields:

$$\begin{array}{c} \begin{array}{cccc} x & y & z & t \end{array} \\ \left(\begin{array}{cccc|c} 1 & 0 & 0 & 0.095 & 5.41 \\ 0 & 1 & 0 & 0.095 & 5.41 \\ 0 & 0 & 1 & 0.067 & 3.67 \end{array}\right) \end{array}$$

Clearly t is our free variable. We have the following solutions:

$$x + 0.095t = 5.41 \quad \Rightarrow \quad x = 5.41 - 0.095t$$
$$y + 0.095t = 5.41 \quad \Rightarrow \quad y = 5.41 - 0.095t$$
$$z + 0.067t = 3.67 \quad \Rightarrow \quad z = 3.67 - 0.067t$$

Our general solution is $x = 5.41 - 0.095t$, $y = 5.41 - 0.095t$, $z = 3.67 - 0.067t$ and t is any real number.