Complete Solutions to Exercises 4.4

1. (a) We are given $\mathbf{Q} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$. Taking the transpose of this matrix and evaluating $\mathbf{Q}^T\mathbf{Q}$ gives:

$$\mathbf{Q}^T\mathbf{Q} = \frac{1}{\sqrt{2}}\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} = \mathbf{I}$$

(b) Similarly we have

$$\mathbf{Q}^T\mathbf{Q} = \frac{1}{5}\frac{1}{5}\begin{pmatrix} 3 & 4 \\ 4 & -3 \end{pmatrix}\begin{pmatrix} 3 & 4 \\ 4 & -3 \end{pmatrix} = \frac{1}{25}\begin{pmatrix} 25 & 0 \\ 0 & 25 \end{pmatrix} = \mathbf{I}$$

(c) We have

$$\mathbf{Q}^T\mathbf{Q} = \begin{pmatrix} \cos(\theta) & \sin(\theta) \\ \sin(\theta) & -\cos(\theta) \end{pmatrix}\begin{pmatrix} \cos(\theta) & \sin(\theta) \\ \sin(\theta) & -\cos(\theta) \end{pmatrix}$$

$$= \begin{pmatrix} \cos^2(\theta)+\sin^2(\theta) & 0 \\ 0 & \sin^2(\theta)+\cos^2(\theta) \end{pmatrix} \underbrace{=}_{\substack{\text{Because} \\ \cos^2(\theta)+\sin^2(\theta)=1}} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \mathbf{I}$$

2. By Proposition (4-13):

If $\mathbf{Q}$ is an orthogonal matrix then $\mathbf{Q}^{-1} = \mathbf{Q}^T$.

All the matrices in question 1 are symmetrical orthogonal matrices which means $\mathbf{Q}^T = \mathbf{Q}$. Hence in each case we have $\mathbf{Q}^{-1} = \mathbf{Q}^T = \mathbf{Q}$ which means the inverse is the same as the given matrix.

3. (a) $\mathbf{A} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}$ is not an orthogonal matrix because the second and third column vectors, $\begin{pmatrix} 0 & 2 & 0 \end{pmatrix}^T$ and $\begin{pmatrix} 0 & 0 & 3 \end{pmatrix}^T$, of matrix **A** are not normalized.

(b) We are given the matrix $\mathbf{B} = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \\ 1 & -1 \\ 1 & -1 \end{pmatrix}$. Since matrix **B** is not square so not orthogonal.

(c) With all the square roots in the matrix it looks as if it is orthogonal but we need to check. Evaluating the dot product between each of the column vectors of **C**:

$$\begin{pmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{pmatrix} = \left(\frac{1}{\sqrt{2}} \times \frac{1}{\sqrt{3}}\right) + \left(-\frac{1}{\sqrt{2}} \times \frac{1}{\sqrt{3}}\right) = 0$$

$$\begin{pmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 1/\sqrt{6} \\ 1/\sqrt{6} \\ -2/\sqrt{6} \end{pmatrix} = \left(\frac{1}{\sqrt{2}} \times \frac{1}{\sqrt{6}}\right) + \left(-\frac{1}{\sqrt{2}} \times \frac{1}{\sqrt{6}}\right) = 0$$

$$\begin{pmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{pmatrix} \cdot \begin{pmatrix} 1/\sqrt{6} \\ 1/\sqrt{6} \\ -2/\sqrt{6} \end{pmatrix} = \left(\frac{1}{\sqrt{3}} \times \frac{1}{\sqrt{6}}\right) + \left(\frac{1}{\sqrt{3}} \times \frac{1}{\sqrt{6}}\right) + \left(\frac{1}{\sqrt{3}} \times \frac{(-2)}{\sqrt{6}}\right) = 0$$

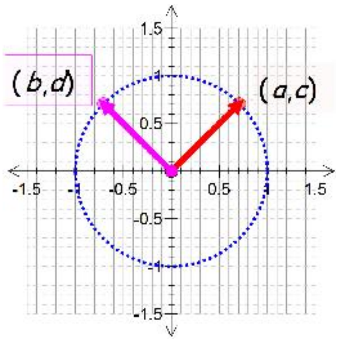
All 3 column vectors of matrix **C** are orthogonal. It is straightforward to check that each of these column vectors have a length of 1 so they are also normalized. Hence matrix **C** is orthogonal.

The inverse $\mathbf{C}^{-1} = \mathbf{C}^T = \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{3} & 1/\sqrt{6} \\ -1/\sqrt{2} & 1/\sqrt{3} & 1/\sqrt{6} \\ 0 & 1/\sqrt{3} & -2/\sqrt{6} \end{pmatrix}^T = \begin{pmatrix} 1/\sqrt{2} & -1/\sqrt{2} & 0 \\ 1/\sqrt{3} & 1/\sqrt{3} & 1/\sqrt{3} \\ 1/\sqrt{6} & 1/\sqrt{6} & -2/\sqrt{6} \end{pmatrix}$.

4. We are given that matrix **A** is orthogonal where $\mathbf{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. We have

$$\begin{pmatrix} a \\ c \end{pmatrix} \cdot \begin{pmatrix} b \\ d \end{pmatrix} = ab + cd = 0 \quad \Leftrightarrow \quad \begin{pmatrix} a \\ c \end{pmatrix} \text{ and } \begin{pmatrix} b \\ d \end{pmatrix} \text{ are orthogonal}$$

Also $\begin{pmatrix} a \\ c \end{pmatrix} \cdot \begin{pmatrix} a \\ c \end{pmatrix} = a^2 + c^2 = 1$ and $\begin{pmatrix} b \\ d \end{pmatrix} \cdot \begin{pmatrix} b \\ b \end{pmatrix} = b^2 + d^2 = 1$. This means these vectors $\begin{pmatrix} a \\ c \end{pmatrix}$ and $\begin{pmatrix} b \\ d \end{pmatrix}$ lie on the unit circle and are perpendicular to each other.



5. Let $\begin{pmatrix} a & b & c \end{pmatrix}^T$ be the third column. Since we want to create an orthogonal matrix, the entries *a*, *b* and *c* must satisfy:

$$\frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \cdot \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \frac{1}{\sqrt{2}}(a - c) = 0 \quad \Rightarrow \quad a = c$$

$$\frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \frac{1}{\sqrt{3}}[a + b + c] \underbrace{=}_{\text{Substituting } a=c} \frac{1}{\sqrt{3}}[2a + b] = 0 \quad \Rightarrow \quad b = -2a$$

We have $a,\ b = -2a$ and $c = a$. We also need the norm of $\begin{pmatrix} a & b & c \end{pmatrix}^T$ to equal 1:

$$a^2+b^2+c^2=1 \;\Rightarrow\; a^2+(-2a)^2+a^2=6a^2=1$$

$$\Rightarrow\; a^2=\frac{1}{6} \;\Rightarrow\; a=\pm\frac{1}{\sqrt{6}}$$

We have two vectors $a=\frac{1}{\sqrt{6}},\ b=-\frac{2}{\sqrt{6}}$ and $c=\frac{1}{\sqrt{6}}$ and

$$a=-\frac{1}{\sqrt{6}},\ b=\frac{2}{\sqrt{6}} \text{ and } c=-\frac{1}{\sqrt{6}}$$

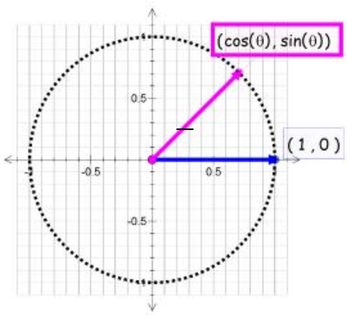
The two possible orthogonal matrices are

$$\begin{pmatrix} 1/\sqrt{3} & 1/\sqrt{2} & 1/\sqrt{6} \\ 1/\sqrt{3} & 0 & -2/\sqrt{6} \\ 1/\sqrt{3} & -1/\sqrt{2} & 1/\sqrt{6} \end{pmatrix} \text{ or } \begin{pmatrix} 1/\sqrt{3} & 1/\sqrt{2} & -1/\sqrt{6} \\ 1/\sqrt{3} & 0 & 2/\sqrt{6} \\ 1/\sqrt{3} & -1/\sqrt{2} & -1/\sqrt{6} \end{pmatrix}$$

6. Applying the matrix $\mathbf{Q}=\begin{pmatrix} \cos(\theta) & \sin(\theta) \\ \sin(\theta) & -\cos(\theta) \end{pmatrix}$ to the vector $\mathbf{u}=\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ gives

$$\mathbf{Qu}=\begin{pmatrix} \cos(\theta) & \sin(\theta) \\ \sin(\theta) & -\cos(\theta) \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix}=\begin{pmatrix} \cos(\theta) \\ \sin(\theta) \end{pmatrix}$$

Plotting these in $\mathbb{R}^2$ gives:



The orthogonal matrix $\mathbf{Q}$ rotates the vector $\mathbf{u}$ by an angle θ counter-clockwise.

7. (i) By the solution to question 4(b) of Exercises 4.3 we have

$$\mathbf{u}_1=\frac{1}{\sqrt{3}}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix},\ \mathbf{u}_2=\frac{1}{\sqrt{6}}\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} \text{ and } \mathbf{u}_3=\frac{1}{\sqrt{2}}\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$$

Hence $\mathbf{Q}=(\mathbf{u}_1 \quad \mathbf{u}_2 \quad \mathbf{u}_3)=\begin{pmatrix} 1/\sqrt{3} & 1/\sqrt{6} & -1/\sqrt{2} \\ 1/\sqrt{3} & -2/\sqrt{6} & 0 \\ 1/\sqrt{3} & 1/\sqrt{6} & 1/\sqrt{2} \end{pmatrix}$. Simplifying these entries by using:

$$\frac{1}{\sqrt{3}}=\frac{\sqrt{2}}{\sqrt{2}\sqrt{3}}=\frac{\sqrt{2}}{\sqrt{6}} \qquad \begin{bmatrix} \text{Multiplying numerator} \\ \text{and denominator by } \sqrt{2} \end{bmatrix}$$

$$\frac{1}{\sqrt{2}}=\frac{\sqrt{3}}{\sqrt{3}\sqrt{2}}=\frac{\sqrt{3}}{\sqrt{6}} \qquad \begin{bmatrix} \text{Multiplying numerator} \\ \text{and denominator by } \sqrt{3} \end{bmatrix}$$

Gives

$$\mathbf{Q}=\begin{pmatrix}\sqrt{2}/\sqrt{6} & 1/\sqrt{6} & -\sqrt{3}/\sqrt{6}\\ \sqrt{2}/\sqrt{6} & -2/\sqrt{6} & 0\\ \sqrt{2}/\sqrt{6} & 1/\sqrt{6} & \sqrt{3}/\sqrt{6}\end{pmatrix}=\frac{1}{\sqrt{6}}\begin{pmatrix}\sqrt{2} & 1 & -\sqrt{3}\\ \sqrt{2} & -2 & 0\\ \sqrt{2} & 1 & \sqrt{3}\end{pmatrix}$$

Transposing this gives

$$\mathbf{Q}^T=\frac{1}{\sqrt{6}}\begin{pmatrix}\sqrt{2} & \sqrt{2} & \sqrt{2}\\ 1 & -2 & 1\\ -\sqrt{3} & 0 & \sqrt{3}\end{pmatrix}$$

We have

$$\mathbf{R}=\mathbf{Q}^T\mathbf{A}=\frac{1}{\sqrt{6}}\begin{pmatrix}\sqrt{2} & \sqrt{2} & \sqrt{2}\\ 1 & -2 & 1\\ -\sqrt{3} & 0 & \sqrt{3}\end{pmatrix}\begin{pmatrix}2 & -1 & -1\\ 2 & 0 & 2\\ 2 & -1 & 3\end{pmatrix}$$

$$=\frac{1}{\sqrt{6}}\begin{pmatrix}6\sqrt{2} & -2\sqrt{2} & 4\sqrt{2}\\ 0 & -2 & -2\\ 0 & 0 & 4\sqrt{3}\end{pmatrix}$$

The QR factorization of matrix **A** is

$$\mathbf{A}=\mathbf{QR}=\frac{1}{\sqrt{6}}\frac{1}{\sqrt{6}}\begin{pmatrix}\sqrt{2} & 1 & -\sqrt{3}\\ \sqrt{2} & -2 & 0\\ \sqrt{2} & 1 & \sqrt{3}\end{pmatrix}\begin{pmatrix}6\sqrt{2} & -2\sqrt{2} & 4\sqrt{2}\\ 0 & -2 & -2\\ 0 & 0 & 4\sqrt{3}\end{pmatrix}$$

(ii) We need to solve $\mathbf{Ax}=\mathbf{b}$ where $\mathbf{b}=\begin{pmatrix}-3 & 8 & 9\end{pmatrix}^T$. Equivalently we solve

$$\mathbf{Rx}=\mathbf{c} \text{ where } \mathbf{Q}^T\mathbf{b}=\mathbf{c}$$

Evaluating $\mathbf{Q}^T\mathbf{b}=\mathbf{c}$:

$$\mathbf{Q}^T\mathbf{b}=\frac{1}{\sqrt{6}}\begin{pmatrix}\sqrt{2} & \sqrt{2} & \sqrt{2}\\ 1 & -2 & 1\\ -\sqrt{3} & 0 & \sqrt{3}\end{pmatrix}\begin{pmatrix}-3\\ 8\\ 9\end{pmatrix}=\frac{1}{\sqrt{6}}\begin{pmatrix}14\sqrt{2}\\ -10\\ 12\sqrt{3}\end{pmatrix}$$

Equating this to $\mathbf{Rx}$ gives

$$\mathbf{Rx}=\frac{1}{\sqrt{6}}\begin{pmatrix}6\sqrt{2} & -2\sqrt{2} & 4\sqrt{2}\\ 0 & -2 & -2\\ 0 & 0 & 4\sqrt{3}\end{pmatrix}\begin{pmatrix}x\\ y\\ z\end{pmatrix}=\frac{1}{\sqrt{6}}\begin{pmatrix}14\sqrt{2}\\ -10\\ 12\sqrt{3}\end{pmatrix}$$

From the bottom row we have

$$4\sqrt{3}z=12\sqrt{3} \quad\Rightarrow\quad z=3$$

Expanding the middle row we have

$$-2y-2z=-10 \quad\Rightarrow\quad -2y-6=-10 \quad\Rightarrow\quad y=2$$

By the top row we have

$$6\sqrt{2}x-2\sqrt{2}y+4\sqrt{2}z=14\sqrt{2}$$

$$6x-2y+4z=14 \quad\Rightarrow\quad 6x-2(2)+4(3)=14 \quad\Rightarrow\quad x=1$$

Our solution is $x=1$, $y=2$ and $z=3$.

(iii) We need to solve $\mathbf{Ax}=\mathbf{b}$ where $\mathbf{b}=\begin{pmatrix}-5 & 12 & 11\end{pmatrix}^T$. Equivalently we solve

$$\mathbf{Rx}=\mathbf{c} \text{ where } \mathbf{Q}^T\mathbf{b}=\mathbf{c}$$

Evaluating $\mathbf{Q}^T\mathbf{b}=\mathbf{c}$:

$$\mathbf{Q}^T\mathbf{b}=\frac{1}{\sqrt{6}}\begin{pmatrix}\sqrt{2} & \sqrt{2} & \sqrt{2}\\ 1 & -2 & 1\\ -\sqrt{3} & 0 & \sqrt{3}\end{pmatrix}\begin{pmatrix}-5\\ 12\\ 11\end{pmatrix}=\frac{1}{\sqrt{6}}\begin{pmatrix}18\sqrt{2}\\ -18\\ 16\sqrt{3}\end{pmatrix}$$

Equating this to $\mathbf{Rx}$ and multiplying through by $\sqrt{6}$ gives

$$\begin{pmatrix}6\sqrt{2} & -2\sqrt{2} & 4\sqrt{2}\\ 0 & -2 & -2\\ 0 & 0 & 4\sqrt{3}\end{pmatrix}\begin{pmatrix}x\\ y\\ z\end{pmatrix}=\begin{pmatrix}18\sqrt{2}\\ -18\\ 16\sqrt{3}\end{pmatrix}$$

From the bottom row we have $4z=16 \quad\Rightarrow\quad z=4$.
Expanding the middle row we have

$$-2y-2z=-18 \;\Rightarrow\; -2y-2(4)=-18 \;\Rightarrow\; y=5$$

Using the top row we have

$$6x-2y+4z=18 \;\Rightarrow\; 6x-2(5)+4(4)=18 \;\Rightarrow\; x=2$$

Our solution is $x=2,\ y=5$ and $z=4$.

8. We need to prove that $\mathbf{Q}^T\mathbf{Q}=\mathbf{I} \;\Rightarrow\; \mathbf{Q}$ is an orthogonal matrix.
Proof.
Let $\mathbf{Q}=\begin{pmatrix}\mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3 & \cdots & \mathbf{v}_n\end{pmatrix}$. We assume $\mathbf{Q}^T\mathbf{Q}=\mathbf{I}$ and writing this out:

$$\mathbf{Q}^T\mathbf{Q}=\begin{pmatrix}\mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3 & \cdots & \mathbf{v}_n\end{pmatrix}^T\begin{pmatrix}\mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3 & \cdots & \mathbf{v}_n\end{pmatrix}$$

$$=\begin{pmatrix}\mathbf{v}_1^T\\ \mathbf{v}_2^T\\ \vdots\\ \mathbf{v}_n^T\end{pmatrix}\begin{pmatrix}\mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3 & \cdots & \mathbf{v}_n\end{pmatrix}$$

$$=\begin{pmatrix}\mathbf{v}_1^T\mathbf{v}_1 & \mathbf{v}_1^T\mathbf{v}_2 & \cdots & \mathbf{v}_1^T\mathbf{v}_n\\ \mathbf{v}_2^T\mathbf{v}_1 & \mathbf{v}_2^T\mathbf{v}_2 & \cdots & \mathbf{v}_2^T\mathbf{v}_n\\ \vdots & \vdots & & \vdots\\ \mathbf{v}_n^T\mathbf{v}_1 & \mathbf{v}_n^T\mathbf{v}_2 & \cdots & \mathbf{v}_n^T\mathbf{v}_n\end{pmatrix}=\begin{pmatrix}1 & 0 & \cdots & 0\\ 0 & 1 & 0 & \vdots\\ 0 & \cdots & \ddots & \vdots\\ 0 & \cdots & \cdots & 1\end{pmatrix}$$

Equating the leading diagonal entries in the last 2 matrices we have

$$\mathbf{v}_i^T\mathbf{v}_i=1 \text{ for } i=1 \text{ to } n$$

Remember this equates to $\mathbf{v}_i^T\mathbf{v}_i=\mathbf{v}_i\cdot\mathbf{v}_i=\left\|\mathbf{v}_i\right\|^2=1$. Hence for $i=1$ to n the vector $\mathbf{v}_i$ is normalized which means each of the vectors in $\mathbf{Q}=\begin{pmatrix}\mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3 & \cdots & \mathbf{v}_n\end{pmatrix}$ are normalized.
Equating the remaining entries in the above we have

$$\mathbf{v}_i^T\mathbf{v}_j=0 \text{ for } i \text{ not equal to } j$$

Remember this equates to $\mathbf{v}_i^T\mathbf{v}_j = \mathbf{v}_i \cdot \mathbf{v}_j = 0$. Hence for each i not equal to j the vectors are orthogonal to each other. This means all the distinct vectors in $\mathbf{Q} = \begin{pmatrix}\mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3 & \cdots & \mathbf{v}_n\end{pmatrix}$ are orthogonal. By

Definition $(4\text{-}5)$.
A square matrix $\mathbf{Q} = \begin{pmatrix}\mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3 & \cdots & \mathbf{v}_n\end{pmatrix}$ whose columns $\mathbf{v}_1, \quad \mathbf{v}_2, \quad \mathbf{v}_3, \quad \cdots \quad , \mathbf{v}_n$ are orthonormal (perpendicular unit) vectors is called an orthogonal matrix.

We have **Q** is an orthogonal matrix.

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9. We have to prove the following result:

Let **Q** be an *n* by *n* matrix and **u** be a vector in $\mathbb{R}^n$. We have

$$\mathbf{Q} \text{ is an orthogonal matrix} \Leftrightarrow \|\mathbf{Qu}\| = \|\mathbf{u}\|$$

Proof.

$(\Rightarrow)$. This follows directly by applying Proposition(4-14):

Proposition (4-14). Let **Q** be an *n* by *n* matrix and **u** and **v** be vectors in $\mathbb{R}^n$. We have

$$\mathbf{Q} \text{ is an orthogonal matrix} \Leftrightarrow \mathbf{Qu}\cdot\mathbf{Qw} = \mathbf{u}\cdot\mathbf{w}$$

Because $\mathbf{Qu}\cdot\mathbf{Qu} = \|\mathbf{Qu}\|^2$ and $\mathbf{u}\cdot\mathbf{u} = \|\mathbf{u}\|^2$. Hence we have

$$\mathbf{Q} \text{ is an orthogonal matrix} \Rightarrow \|\mathbf{Qu}\|^2 = \|\mathbf{u}\|^2 \Rightarrow \|\mathbf{Qu}\| = \|\mathbf{u}\|$$

$(\Leftarrow)$**.** Assume $\|\mathbf{Qu}\| = \|\mathbf{u}\|$ and we need to show that $\mathbf{Qu}\cdot\mathbf{Qw} = \mathbf{u}\cdot\mathbf{w}$ for any **u** and **w** because then by Proposition (4-14) we have that **Q** is an orthogonal matrix.

Let $\mathbf{u} = \mathbf{v} - \mathbf{w}$ then

$$\begin{aligned}
\|\mathbf{Qu}\|^2 = \mathbf{Qu}\cdot\mathbf{Qu} &= \mathbf{Q}(\mathbf{v}-\mathbf{w})\cdot\mathbf{Q}(\mathbf{v}-\mathbf{w}) \\
&= \mathbf{Qv}\cdot\mathbf{Qv} - 2(\mathbf{Qv}\cdot\mathbf{Qw}) + \mathbf{Qw}\cdot\mathbf{Qw} \\
&= \|\mathbf{Qv}\|^2 - 2(\mathbf{Qv}\cdot\mathbf{Qw}) + \|\mathbf{Qw}\|^2 \\
&= \|\mathbf{v}\|^2 - 2(\mathbf{Qv}\cdot\mathbf{Qw}) + \|\mathbf{w}\|^2 \qquad (\dagger)
\end{aligned}$$

Squaring our assumption $\|\mathbf{Qu}\| = \|\mathbf{u}\|$ we have

$$\begin{aligned}
\|\mathbf{Qu}\|^2 = \|\mathbf{u}\|^2 &= (\mathbf{v}-\mathbf{w})\cdot(\mathbf{v}-\mathbf{w}) \\
&= \|\mathbf{v}\|^2 - 2(\mathbf{v}\cdot\mathbf{w}) + \|\mathbf{w}\|^2 \qquad (\dagger\dagger)
\end{aligned}$$

Equating $(\dagger)$ and $(\dagger\dagger)$ gives

$$\|\mathbf{v}\|^2 - 2(\mathbf{Qv}\cdot\mathbf{Qw}) + \|\mathbf{w}\|^2 = \|\mathbf{v}\|^2 - 2(\mathbf{v}\cdot\mathbf{w}) + \|\mathbf{w}\|^2 \quad \text{implies } \mathbf{Qv}\cdot\mathbf{Qw} = \mathbf{v}\cdot\mathbf{w}$$

Since we have $\mathbf{Qv}\cdot\mathbf{Qw} = \mathbf{v}\cdot\mathbf{w}$ so by (4-14) we conclude that **Q** is orthogonal.

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10. We are required to prove that if $\|\mathbf{Au}\| = 1$ for all unit vectors $\mathbf{u}$ then **A** is an orthogonal matrix. *How?*

We use the above Proposition (4-15) of the above question 9.

Proof.

We have been given that $\|\mathbf{Au}\| = 1$ for all unit vectors $\mathbf{u}$. We have

$$\|\mathbf{Au}\| = 1 = \|\mathbf{u}\| \quad [\text{Because } \mathbf{u} \text{ is a unit vector}]$$

By (4-15) we conclude that matrix **A** is orthogonal.

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11. We need to prove that if **Q** is orthogonal then $\mathbf{Q}^T$ is also orthogonal.

Proof.

Using

Proposition (4-13). **Q** is an orthogonal matrix $\Leftrightarrow$ $\mathbf{Q}^{-1} = \mathbf{Q}^T$.

Taking the transpose of both sides of $\mathbf{Q}^{-1} = \mathbf{Q}^T$ we have

$$\left(\mathbf{Q}^{-1}\right)^T = \left(\mathbf{Q}^T\right)^T = \mathbf{Q}$$

We can also interchange the inverse and transpose operations:

$$\left(\mathbf{Q}^T\right)^{-1} = \mathbf{Q}$$

Hence by (4-13) the matrix $\mathbf{Q}^T$ is orthogonal.

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12. We need to prove that the columns and rows of an orthogonal matrix **Q** are linearly independent.

Proof.

Let **Q** be an orthogonal matrix. Since the columns of an orthogonal matrix **Q** are orthogonal to each other so by:

Proposition (4-8). If is an *orthogonal* set of non-zero vectors in an inner product space then they are *linearly independent*.

The columns of the matrix **Q** are linearly independent.

By result of question 11 which says the transposed matrix $\mathbf{Q}^T$ is also orthogonal so the rows of the matrix **Q** are orthogonal because the rows of **Q** become columns of matrix $\mathbf{Q}^T$ which is orthogonal.

■