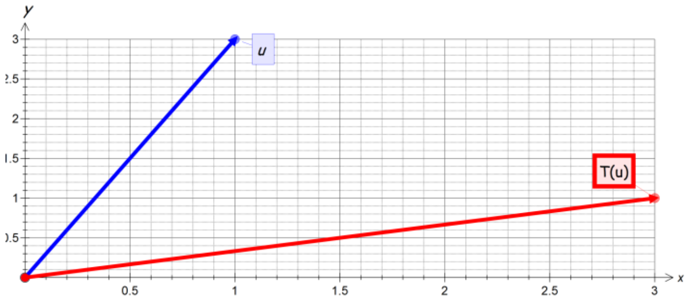
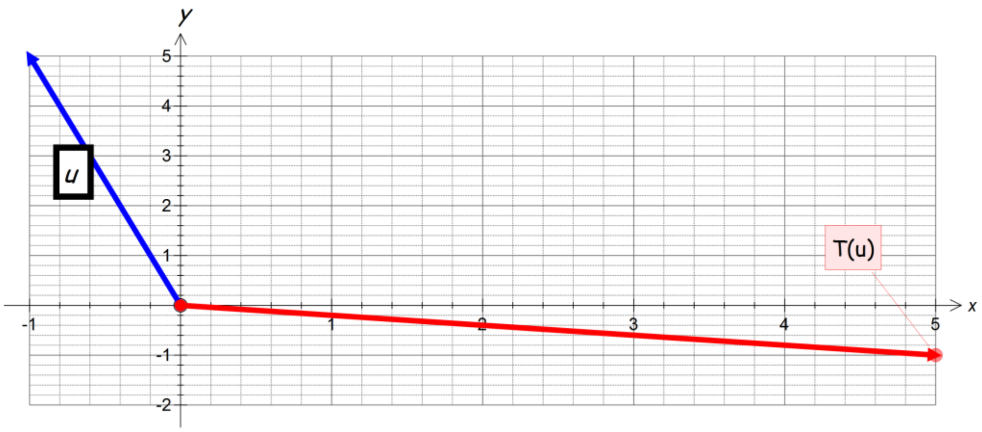
Complete Solutions to Exercises 5.1

1. Let $\mathbf{u}=\begin{pmatrix}x\\y\end{pmatrix}$ then $T\left(\begin{bmatrix}x\\y\end{bmatrix}\right)=\begin{pmatrix}0&1\\1&0\end{pmatrix}\begin{bmatrix}x\\y\end{bmatrix}=\begin{pmatrix}y\\x\end{pmatrix}$. This transformation interchanges the x and y values.

(a) We have $T\left(\begin{bmatrix}1\\3\end{bmatrix}\right)=\begin{pmatrix}3\\1\end{pmatrix}$. Plotting these on $\mathbb{R}^2$ gives:



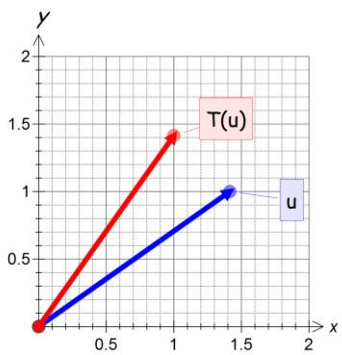
(b) Similarly we have $T\left(\begin{pmatrix}-1\\5\end{pmatrix}\right)=\begin{pmatrix}5\\-1\end{pmatrix}$ and the graph of these is:



(c) We apply the transformation to the vector $\mathbf{u}=\begin{pmatrix}\sqrt{2}\\1\end{pmatrix}$. Thus we have

$$T\left(\begin{bmatrix}\sqrt{2}\\1\end{bmatrix}\right)=\begin{pmatrix}1\\\sqrt{2}\end{pmatrix}$$

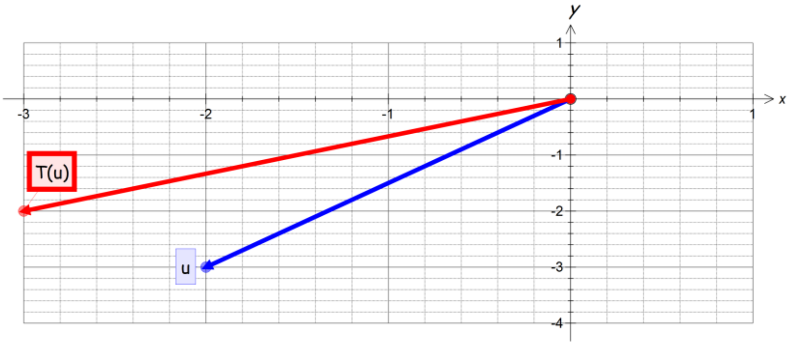
Plotting $\mathbf{u}$ and $T(\mathbf{u})$ on $\mathbb{R}^2$ gives:



(d) This time we apply the transformation to $\mathbf{u}=\begin{pmatrix}-2\\-3\end{pmatrix}$.

$$T\left(\begin{bmatrix}-2\\-3\end{bmatrix}\right)=\begin{pmatrix}-3\\-2\end{pmatrix}$$

The vectors are plotted as:



2. We are given the transformation $T:\mathbb{R}^3\to\mathbb{R}^2$ by $T\left(\begin{bmatrix}x\\y\\z\end{bmatrix}\right)=\begin{pmatrix}xy\\xz\end{pmatrix}$. T transforms a vector in 3d onto a plane.

(a) We transform the vector $\mathbf{u}=\begin{pmatrix}1\\3\\5\end{pmatrix}$ by substituting $x=1,\ y=3$ and $z=5$ into the above:

$$T\left(\begin{bmatrix}1\\3\\5\end{bmatrix}\right)=\begin{pmatrix}1\times3\\1\times5\end{pmatrix}=\begin{pmatrix}3\\5\end{pmatrix}$$

(b) Similarly we are given the vector $\mathbf{u}=\begin{pmatrix}-1\\-4\\4\end{pmatrix}$ and we substitute $x=-1,\ y=-4$ and $z=4$ into the above:

$$T\left(\begin{bmatrix}-1\\-4\\4\end{bmatrix}\right)=\begin{pmatrix}-1\times(-4)\\-1\times4\end{pmatrix}=\begin{pmatrix}4\\-4\end{pmatrix}$$

(c) For the vector $\mathbf{u}=\begin{pmatrix}\sqrt{2}\\\sqrt{8}\\\sqrt{18}\end{pmatrix}$ we have

$$T\left(\begin{bmatrix}\sqrt{2}\\\sqrt{8}\\\sqrt{18}\end{bmatrix}\right)=\begin{pmatrix}\sqrt{2}\sqrt{8}\\\sqrt{2}\sqrt{18}\end{pmatrix}$$

$$=\begin{pmatrix}\sqrt{16}\\\sqrt{36}\end{pmatrix}=\begin{pmatrix}4\\6\end{pmatrix}$$

(d) Similarly for the given vector $\mathbf{u}=\begin{pmatrix}-2\\-3\\-4\end{pmatrix}$ we have

$$T\left(\begin{bmatrix}-2\\-3\\-4\end{bmatrix}\right)=\begin{pmatrix}-2\times(-3)\\-2\times(-4)\end{pmatrix}=\begin{pmatrix}6\\8\end{pmatrix}$$

3. We are given the transformation $T\left(\begin{bmatrix}x\\y\\z\end{bmatrix}\right)=\begin{pmatrix}x+y\\y+z\\z+x\end{pmatrix}$. In each of the following cases we substitute the relevant *x*, *y* and *z* values:

(a) For $\mathbf{u}=\begin{pmatrix}2\\4\\7\end{pmatrix}$ we have $T\left(\begin{bmatrix}2\\4\\7\end{bmatrix}\right)=\begin{pmatrix}2+4\\4+7\\7+2\end{pmatrix}=\begin{pmatrix}6\\11\\9\end{pmatrix}$

(b) For $\mathbf{u}=\begin{pmatrix}-3\\8\\-6\end{pmatrix}$ we substitute $x=-3,\ y=8$ and $z=-6$ into $T\left(\begin{bmatrix}x\\y\\z\end{bmatrix}\right)=\begin{pmatrix}x+y\\y+z\\z+x\end{pmatrix}$:

$$T\left(\begin{bmatrix}-3\\8\\-6\end{bmatrix}\right)=\begin{pmatrix}-3+8\\8+(-6)\\-6+(-3)\end{pmatrix}=\begin{pmatrix}5\\2\\-9\end{pmatrix}$$

(c) For $\mathbf{u}=\begin{pmatrix}\pi\\2\pi\\5\pi\end{pmatrix}$ we substitute $x=\pi,\ y=2\pi$ and $z=5\pi$ into $T\left(\begin{bmatrix}x\\y\\z\end{bmatrix}\right)=\begin{pmatrix}x+y\\y+z\\z+x\end{pmatrix}$:

$$T\left(\begin{bmatrix}\pi\\2\pi\\5\pi\end{bmatrix}\right)=\begin{pmatrix}\pi+2\pi\\2\pi+5\pi\\5\pi+\pi\end{pmatrix}=\begin{pmatrix}3\pi\\7\pi\\6\pi\end{pmatrix}$$

(d) Similarly for $\mathbf{u}=\begin{pmatrix}1/2\\2/3\\3/4\end{pmatrix}$ we have

$$T\left(\begin{bmatrix}1/2\\2/3\\3/4\end{bmatrix}\right)=\begin{pmatrix}\frac{1}{2}+\frac{2}{3}\\[4pt]\frac{2}{3}+\frac{3}{4}\\[4pt]\frac{3}{4}+\frac{1}{2}\end{pmatrix}=\begin{pmatrix}7/6\\17/12\\5/4\end{pmatrix}$$

4. $T(\mathbf{u})=\pm\sqrt{\mathbf{u}}$ is not a transformation because *T* takes the vector **u** to 2 different destinations. Remember for *T* to be a transformation we must have a unique output.

5. We need to find if the following transformations are linear or not.

(a) (i) $T:\mathbb{R}^2 \to \mathbb{R}^2$ given by $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right)=\begin{pmatrix} y \\ x \end{pmatrix}$. Let $\mathbf{u}=\begin{pmatrix} x \\ y \end{pmatrix}$ and $\mathbf{v}=\begin{pmatrix} s \\ t \end{pmatrix}$ then we check both conditions of definition (5-2):

(a) $T(\mathbf{u}+\mathbf{v})=T(\mathbf{u})+T(\mathbf{v})$ - T preserves vector addition.
(b) $T(k\mathbf{u})=kT(\mathbf{u})$ - T preserves scalar multiplication.

<u>Condition (a)</u>:

$$T(\mathbf{u}+\mathbf{v})=T\begin{pmatrix} x+s \\ y+t \end{pmatrix}=\begin{pmatrix} y+t \\ x+s \end{pmatrix} \qquad \begin{bmatrix}\text{Applying the} \\ \text{given Transformation}\end{bmatrix}$$

Also

$$T(\mathbf{u})+T(\mathbf{v})=T\begin{pmatrix} x \\ y \end{pmatrix}+T\begin{pmatrix} s \\ t \end{pmatrix}$$

$$\underset{\substack{\text{Applying the} \\ \text{given Transformation}}}{=}\begin{pmatrix} y \\ x \end{pmatrix}+\begin{pmatrix} t \\ s \end{pmatrix}=\begin{pmatrix} y+t \\ x+s \end{pmatrix}$$

Thus we have $T(\mathbf{u}+\mathbf{v})=T(\mathbf{u})+T(\mathbf{v})$ which is condition (a).

<u>Condition (b)</u>: We need to check $T(k\mathbf{u})=kT(\mathbf{u})$

$$T(k\mathbf{u})=T\begin{pmatrix} kx \\ ky \end{pmatrix}$$

$$\underset{\substack{\text{Applying the} \\ \text{given Transformation}}}{=}\begin{pmatrix} ky \\ kx \end{pmatrix}=k\begin{pmatrix} y \\ x \end{pmatrix}=kT(\mathbf{u})$$

Thus we have $T(k\mathbf{u})=kT(\mathbf{u})$ which is condition (b).

We conclude that the given transformation $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right)=\begin{pmatrix} y \\ x \end{pmatrix}$ is linear.

(ii) We are given the transformation $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right)=\begin{pmatrix} x^2 \\ y^2 \end{pmatrix}$. Let $\mathbf{u}=\begin{pmatrix} x \\ y \end{pmatrix}$ and $\mathbf{v}=\begin{pmatrix} s \\ t \end{pmatrix}$. Need to check both conditions of definition (5-2) which are:

(a) $T(\mathbf{u}+\mathbf{v})=T(\mathbf{u})+T(\mathbf{v})$
(b) $T(k\mathbf{u})=kT(\mathbf{u})$

<u>Condition (a)</u>:

$$T(\mathbf{u}+\mathbf{v})=T\begin{pmatrix} x+s \\ y+t \end{pmatrix}$$

$$\underset{\substack{\text{Applying the} \\ \text{given Transformation}}}{=}\begin{pmatrix} (x+s)^2 \\ (y+t)^2 \end{pmatrix}=\begin{pmatrix} x^2+2xs+s^2 \\ y^2+2yt+t^2 \end{pmatrix} \qquad [\text{Expanding Brackets}]$$

Also

$$T(\mathbf{u})+T(\mathbf{v})=T\begin{pmatrix}x\\y\end{pmatrix}+T\begin{pmatrix}s\\t\end{pmatrix}$$

$$\underset{\substack{\text{Applying the}\\\text{given Transformation}}}{=}\begin{pmatrix}x^2\\y^2\end{pmatrix}+\begin{pmatrix}s^2\\t^2\end{pmatrix}=\begin{pmatrix}x^2+s^2\\y^2+t^2\end{pmatrix}$$

The entries are **not** equal, that is

$$x^2+2xs+s^2\neq x^2+s^2 \quad\text{and}\quad y^2+2yt+t^2\neq y^2+t^2$$

therefore

$$T(\mathbf{u}+\mathbf{v})\neq T(\mathbf{u})+T(\mathbf{v}) \qquad [\text{Not Equal}]$$

This means condition (a) of definition (5-2) fails for the given transformation so we conclude that $T\left(\begin{bmatrix}x\\y\end{bmatrix}\right)=\begin{pmatrix}x^2\\y^2\end{pmatrix}$ is **not** a linear transformation.

We only need to check one of the conditions of (5-2) for failure because for the transformation to be linear it has to satisfy **both** conditions (a) and (b).

(b) We need to check whether the transformation $T\left(\begin{bmatrix}x\\y\\z\end{bmatrix}\right)=\begin{pmatrix}xy\\xz\end{pmatrix}$ is linear. *What does this mean?*

Need to see whether the given transformation satisfies:

(a) $T(\mathbf{u}+\mathbf{v})=T(\mathbf{u})+T(\mathbf{v})$

(b) $T(k\mathbf{u})=kT(\mathbf{u})$

Let $\mathbf{u}=\begin{pmatrix}x\\y\\z\end{pmatrix}$ and $\mathbf{v}=\begin{pmatrix}r\\s\\t\end{pmatrix}$. We have

$$T(\mathbf{u}+\mathbf{v})=T\left(\begin{bmatrix}x\\y\\z\end{bmatrix}+\begin{bmatrix}r\\s\\t\end{bmatrix}\right)$$

$$=T\begin{pmatrix}x+r\\y+s\\z+t\end{pmatrix}\underset{\substack{\text{Applying the}\\\text{given Transformation}}}{=}\begin{pmatrix}(x+r)(y+s)\\(x+r)(z+t)\end{pmatrix}=\begin{pmatrix}xy+xs+ry+rs\\xz+xt+rz+rt\end{pmatrix}$$

Need to check $T(\mathbf{u}+\mathbf{v})=T(\mathbf{u})+T(\mathbf{v})$:

$$T(\mathbf{u})+T(\mathbf{v})=T\left(\begin{bmatrix}x\\y\\z\end{bmatrix}\right)+T\left(\begin{bmatrix}r\\s\\t\end{bmatrix}\right)$$

$$\underset{\substack{\text{Applying the}\\\text{given Transformation}}}{=}\begin{pmatrix}xy\\xz\end{pmatrix}+\begin{pmatrix}rs\\rt\end{pmatrix}$$

$$=\begin{pmatrix}xy+rs\\xz+rt\end{pmatrix}\neq\begin{pmatrix}xy+xs+ry+rs\\xz+xt+rz+rt\end{pmatrix} \qquad [\text{Not Equal}]$$

Thus $T(\mathbf{u}+\mathbf{v}) \neq T(\mathbf{u})+T(\mathbf{v})$ [Not Equal]. Therefore the given transformation T is **not** linear.

(c) (i) Need to check that the transformation $T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right)=\begin{pmatrix} x+y \\ y+z \\ z+x \end{pmatrix}$ is linear or not. Again we check the two conditions of definition (5-2) which are:

(a) $T(\mathbf{u}+\mathbf{v})=T(\mathbf{u})+T(\mathbf{v})$
(b) $T(k\mathbf{u})=kT(\mathbf{u})$

Let $\mathbf{u}=\begin{pmatrix} x \\ y \\ z \end{pmatrix}$ and $\mathbf{v}=\begin{pmatrix} r \\ s \\ t \end{pmatrix}$, then

$$T(\mathbf{u}+\mathbf{v})=T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}+\begin{bmatrix} r \\ s \\ t \end{bmatrix}\right)$$

$$=T\left(\begin{bmatrix} x+r \\ y+s \\ z+t \end{bmatrix}\right)=\begin{pmatrix} (x+r)+(y+s) \\ (y+s)+(z+t) \\ (z+t)+(x+r) \end{pmatrix} \qquad \left[\begin{array}{l}\text{Applying the} \\ \text{given Transformation}\end{array}\right]$$

Also we have

$$T(\mathbf{u})+T(\mathbf{v})=T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right)+T\left(\begin{bmatrix} r \\ s \\ t \end{bmatrix}\right)$$

$$\underset{\substack{\text{Applying the} \\ \text{given Transformation}}}{=}\begin{pmatrix} x+y \\ y+z \\ z+x \end{pmatrix}+\begin{pmatrix} r+s \\ s+t \\ t+r \end{pmatrix}=\begin{pmatrix} x+y+r+s \\ y+z+s+t \\ z+x+t+r \end{pmatrix}=T(\mathbf{u}+\mathbf{v})$$

Thus condition (a) is satisfied. Need to check condition (b) which is $T(k\mathbf{u})=kT(\mathbf{u})$:

$$T(k\mathbf{u})=T\left(k\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right)=T\left(\begin{bmatrix} kx \\ ky \\ kz \end{bmatrix}\right)$$

$$=\begin{pmatrix} kx+ky \\ ky+kz \\ kz+kx \end{pmatrix} \qquad \left[\begin{array}{l}\text{Applying the} \\ \text{given transformation}\end{array}\right]$$

$$=\begin{pmatrix} k(x+y) \\ k(y+z) \\ k(z+x) \end{pmatrix}=k\begin{pmatrix} x+y \\ y+z \\ z+x \end{pmatrix}=kT(\mathbf{u})$$

Thus (b) is satisfied therefore we conclude that the given transformation is linear.

(ii) We need to check whether transformation $T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{pmatrix} \sqrt{x} \\ \sqrt{y} \\ \sqrt{z} \end{pmatrix}$ is linear or not.

Let $\mathbf{u} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ and $\mathbf{v} = \begin{pmatrix} r \\ s \\ t \end{pmatrix}$. Then

$$T(\mathbf{u}+\mathbf{v}) = T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix} + \begin{bmatrix} r \\ s \\ t \end{bmatrix}\right)$$

$$= T\left(\begin{bmatrix} x+r \\ y+s \\ z+t \end{bmatrix}\right) = \begin{pmatrix} \sqrt{x+r} \\ \sqrt{y+s} \\ \sqrt{z+t} \end{pmatrix} \quad \begin{bmatrix} \text{Applying the} \\ \text{given Transformation} \end{bmatrix}$$

We also have to find $T(\mathbf{u}) + T(\mathbf{v})$:

$$T(\mathbf{u}) + T(\mathbf{v}) = T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) + T\left(\begin{bmatrix} r \\ s \\ t \end{bmatrix}\right)$$

$$= \begin{pmatrix} \sqrt{x} \\ \sqrt{y} \\ \sqrt{z} \end{pmatrix} + \begin{pmatrix} \sqrt{r} \\ \sqrt{s} \\ \sqrt{t} \end{pmatrix} \quad \begin{bmatrix} \text{Applying the} \\ \text{given Transformation} \end{bmatrix}$$

$$= \begin{pmatrix} \sqrt{x}+\sqrt{r} \\ \sqrt{y}+\sqrt{s} \\ \sqrt{z}+\sqrt{t} \end{pmatrix} \neq \begin{pmatrix} \sqrt{x+r} \\ \sqrt{y+s} \\ \sqrt{z+t} \end{pmatrix} \quad [\text{Not Equal}]$$

Since $T(\mathbf{u}+\mathbf{v}) \neq T(\mathbf{u}) + T(\mathbf{v})$ [Not Equal] therefore T is **not** linear.

6. (a) We need to check the transformation $T\left(c_2x^2 + c_1x + c_0\right) = c_0x^2 + c_1x + c_2$ is linear or not. Let $\mathbf{u} = a_2x^2 + a_1x + a_0$ and $\mathbf{v} = b_2x^2 + b_1x + b_0$ then

$$T(\mathbf{u}+\mathbf{v}) = T\left(\underbrace{a_2x^2 + a_1x + a_0}_{=\mathbf{u}} + \underbrace{b_2x^2 + b_1x + b_0}_{=\mathbf{v}}\right)$$

$$= T\left((a_2+b_2)x^2 + (a_1+b_1)x + (a_0+b_0)\right) \quad \begin{bmatrix} \text{Collecting} \\ \text{Like terms} \end{bmatrix}$$

$$= (a_0+b_0)x^2 + (a_1+b_1)x + (a_2+b_2) \quad \begin{bmatrix} \text{Applying the} \\ \text{given Transformation} \end{bmatrix}$$

What do we do next?

Check whether the above is equal to $T(\mathbf{u}) + T(\mathbf{v})$:

$$T(\mathbf{u})+T(\mathbf{v})=T\left(a_2x^2+a_1x+a_0\right)+T\left(b_2x^2+b_1x+b_0\right)$$

$$=a_0x^2+a_1x+a_2+b_0x^2+b_1x+b_2 \qquad \begin{bmatrix}\text{Applying the}\\ \text{given Transformation}\end{bmatrix}$$

$$=\left(a_0+b_0\right)x^2+\left(a_1+b_1\right)x+\left(a_2+b_2\right) \qquad [\text{Collecting Like terms}]$$

$$=T(\mathbf{u}+\mathbf{v}) \qquad [\text{From above}]$$

Condition (a) is satisfied. *What else do we need to check?*

Check condition (b) which is $T(k\mathbf{u})=kT(\mathbf{u})$:

$$T(k\mathbf{u})=T\left(k\left(a_2x^2+a_1x+a_0\right)\right)$$

$$=T\left(ka_2x^2+ka_1x+ka_0\right)$$

$$=ka_0x^2+ka_1x+ka_2 \qquad \begin{bmatrix}\text{Applying the}\\ \text{given Transformation}\end{bmatrix}$$

$$=k\left(a_0x^2+a_1x+a_2\right)=k\,T(\mathbf{u})$$

Hence condition (b) is satisfied which means that the given transformation is linear.

(b) *Is the transformation* $T\left(c_2x^2+c_1x+c_0\right)=c_2^2x^2+c_1^2x+c_0^2$ *linear?*

We need to check both conditions of definition (5-2) which says:

(a) $T(\mathbf{u}+\mathbf{v})=T(\mathbf{u})+T(\mathbf{v})$

(b) $T(k\mathbf{u})=kT(\mathbf{u})$

We check condition (b) first.

Condition (b):

Let $\mathbf{u}=a_2x^2+a_1x+a_0$ and k be any scalar, then

$$T(k\mathbf{u})=T\left(k\left(a_2x^2+a_1x+a_0\right)\right)$$

$$=T\left(ka_2x^2+ka_1x+ka_0\right) \qquad [\text{Expanding Brackets}]$$

$$=\left(ka_2\right)^2x^2+\left(ka_1\right)^2x+\left(ka_0\right)^2 \qquad \begin{bmatrix}\text{Applying the}\\ \text{given Transformation}\end{bmatrix}$$

$$=k^2a_2^2x^2+k^2a_1^2x+k^2a_0^2 \qquad [\text{Expanding Brackets}]$$

We need to see whether this is equal to $kT(\mathbf{u})$:

$$kT(\mathbf{u})=kT\left(a_2x^2+a_1x+a_0\right)$$

$$=k\left[a_2^2x^2+a_1^2x+a_0^2\right]=ka_2^2x^2+ka_1^2x+ka_0^2$$

Clearly this is **not equal** to $T(k\mathbf{u})$ above because the k's are **not squared**, therefore we conclude that the given transformation is **not** linear.

Remember to show that a given transformation is **not** linear we only need to show that one of the conditions of definition (5-2) fails.

7. (a) In this case we need to decide whether the following transformation $T(\mathbf{A})=\mathbf{A}^T$ is linear or not. Note that the given transformation is taking the transpose of a matrix.

How do we check this transformation is linear or not?

Check that

(a) $T(\mathbf{A}+\mathbf{B})=T(\mathbf{A})+T(\mathbf{B})$
(b) $T(k\,\mathbf{A})=k\,T(\mathbf{A})$

where $\mathbf{A}=\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$ and $\mathbf{B}=\begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}$.

<u>Condition (a)</u>:

$$\begin{aligned} T(\mathbf{A}+\mathbf{B}) &= T\left(\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}+\begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}\right) \\ &= T\left(\begin{pmatrix} a_{11}+b_{11} & a_{12}+b_{12} \\ a_{21}+b_{21} & a_{22}+b_{22} \end{pmatrix}\right) \\ &= \begin{pmatrix} a_{11}+b_{11} & a_{21}+b_{21} \\ a_{12}+b_{12} & a_{22}+b_{22} \end{pmatrix} \qquad [\text{Taking Transpose}] \end{aligned}$$

Need to see if this is equal to $T(\mathbf{A})+T(\mathbf{B})$:

$$\begin{aligned} T(\mathbf{A})+T(\mathbf{B}) &= T\left(\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}\right)+T\left(\begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}\right) \\ &= \begin{pmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \end{pmatrix}+\begin{pmatrix} b_{11} & b_{21} \\ b_{12} & b_{22} \end{pmatrix} \qquad [\text{Taking Transpose}] \\ &= \begin{pmatrix} a_{11}+b_{11} & a_{21}+b_{21} \\ a_{12}+b_{12} & a_{22}+b_{22} \end{pmatrix} = T(\mathbf{A}+\mathbf{B}) \end{aligned}$$

We have $T(\mathbf{A}+\mathbf{B})=T(\mathbf{A})+T(\mathbf{B})$. We also need to check $T(k\,\mathbf{A})=k\,T(\mathbf{A})$:

$$\begin{aligned} T(k\,\mathbf{A}) &= T\left(k\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}\right) \\ &= T\left(\begin{pmatrix} ka_{11} & ka_{12} \\ ka_{21} & ka_{22} \end{pmatrix}\right) \\ &= \begin{pmatrix} ka_{11} & ka_{21} \\ ka_{12} & ka_{22} \end{pmatrix} \qquad [\text{Taking Transpose}] \\ &= k\begin{pmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \end{pmatrix} = k\mathbf{A}^T = k\,T(\mathbf{A}) \end{aligned}$$

Thus condition (b) is satisfied therefore we conclude that the given transformation is linear. Hence the transpose of a 2 by 2 matrix is a linear transformation.

(b) We need to check $T(\mathbf{A})=tr(\mathbf{A})$ is linear or not. In this case the transformation is given by finding the trace of the matrix which is adding the entries along the leading diagonal of the matrix. Let $\mathbf{A}=\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$ and $\mathbf{B}=\begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}$ then

$$\begin{aligned}T(\mathbf{A}+\mathbf{B}) &= T\left(\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22}\end{pmatrix} + \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22}\end{pmatrix}\right)\\ &= T\left(\begin{pmatrix} a_{11}+b_{11} & a_{12}+b_{12} \\ a_{21}+b_{21} & a_{22}+b_{22}\end{pmatrix}\right)\\ &= a_{11}+b_{11}+a_{22}+b_{22} \qquad \left[\text{Adding the diagonal entries}\right]\end{aligned}$$

What is $T(\mathbf{A})+T(\mathbf{B})$ equal to?

$$\begin{aligned}T(\mathbf{A})+T(\mathbf{B}) &= T\left(\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22}\end{pmatrix}\right) + T\left(\begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22}\end{pmatrix}\right)\\ &= a_{11}+a_{22}+b_{11}+b_{22} \qquad \left[\text{Adding the diagonal entries}\right]\end{aligned}$$

Thus $T(\mathbf{A}+\mathbf{B}) = T(\mathbf{A})+T(\mathbf{B})$. We also need to check condition (b) which is $T(k\,\mathbf{A}) = k\,T(\mathbf{A})$:

$$\begin{aligned}T(k\,\mathbf{A}) &= T\left(k\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22}\end{pmatrix}\right)\\ &= T\left(\begin{pmatrix} ka_{11} & ka_{12} \\ ka_{21} & ka_{22}\end{pmatrix}\right)\\ &= ka_{11}+ka_{22} \qquad \left[\text{Adding the diagonal entries}\right]\\ &= k\underbrace{\left(a_{11}+a_{22}\right)}_{=tr(\mathbf{A})=T(\mathbf{A})} = kT(\mathbf{A})\end{aligned}$$

Hence condition (b) is satisfied therefore we conclude that the given transformation is linear. Thus the trace of a 2 by 2 matrix is a linear transformation.

(c) *How do we check that $T(\mathbf{A}) = a_{11}a_{22}a_{33}\cdots a_{nn}$ is linear?*

Check the 2 conditions:

(a) $T(\mathbf{A}+\mathbf{B}) = T(\mathbf{A})+T(\mathbf{B})$
(b) $T(k\,\mathbf{A}) = k\,T(\mathbf{A})$

Let us check condition (b) first. Let $\mathbf{A} = \begin{pmatrix} a_{11} & \dots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn}\end{pmatrix}$.

<u>Condition (b)</u>:

$$\begin{aligned}T(k\,\mathbf{A}) &= T\left(k\begin{pmatrix} a_{11} & \dots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn}\end{pmatrix}\right)\\ &= T\begin{pmatrix} ka_{11} & \dots & ka_{1n} \\ \vdots & \ddots & \vdots \\ ka_{n1} & \cdots & ka_{nn}\end{pmatrix}\\ &= ka_{11}ka_{22}\cdots ka_{nn} \qquad \left[\text{Applying Transformation}\right]\\ &= k^n\left(a_{11}a_{22}\cdots a_{nn}\right) = k^n\,T(\mathbf{A}) \neq k\,T(\mathbf{A}) \quad \left[\text{Not Equal}\right]\end{aligned}$$

We have $T(k\,\mathbf{A}) \neq k\,T(\mathbf{A})$ $\quad$ [Not Equal] therefore T is **not** a linear transformation.
In general if our intuition says that a given transformation is **not** linear then it is easier to test condition (b) rather than condition (a) because we only have to concern ourselves with one argument, **A**.

8. We need to prove the transpose of any square matrix is a linear transformation.
Proof.
Let **A** and **B** be n by n matrices. We define the transformation $T:\ M_{nn} \to M_{nn}$ by $T(\mathbf{A}) = \mathbf{A}^T$. We check both conditions of definition (5-2):

$$\begin{aligned} T(\mathbf{A}+\mathbf{B}) &= (\mathbf{A}+\mathbf{B})^T \\ &= \mathbf{A}^T + \mathbf{B}^T && \text{[By Proposition (1-4) part (c)]} \\ &= T(\mathbf{A}) + T(\mathbf{B}) \end{aligned}$$

Condition (a) of (5-2) is satisfied. We need to check condition (b) of (5-2):

$$\begin{aligned} T(k\ \mathbf{A}) &= (k\ \mathbf{A})^T \\ &= k\mathbf{A}^T && \text{[By Proposition (1.19) part (b)]} \\ &= kT(\mathbf{A}) \end{aligned}$$

Hence the transpose of a square matrix is a linear transformation.

■

9. We need to show that $T(f) = \int_0^1 f(x)\,dx$ is a linear transformation. Let f and g be functions of $C[0,1]$.

<u>Condition (a)</u>:

$$\begin{aligned} T(f+g) &= \int_0^1 \left[f(x) + g(x)\right]dx \\ &= \int_0^1 f(x)\,dx + \int_0^1 g(x)\,dx = T(f) + T(g) \end{aligned}$$

Hence condition (a) is satisfied. Need to show condition (b).

<u>Condition (b)</u>:

$$\begin{aligned} T(k\ f) &= \int_0^1 \left[kf(x)\right]dx \\ &= k\int_0^1 f(x)\,dx = k\,T(f) \end{aligned}$$

Thus condition (b) is satisfied which means that integration is linear.

10. We need to prove that if $T:\ V \to W$ is a transform such that $T(\mathbf{O}) \neq \mathbf{O}$ then T is **not** a linear transform.
Proof.
Suppose T is a linear transform. Then we have

$$\begin{aligned}T(\mathbf{u}) &= T(\mathbf{u}+\mathbf{O}) \\ &= T(\mathbf{u}) + \underbrace{T(\mathbf{O})}_{\neq \mathbf{O}} \\ &\neq T(\mathbf{u})\end{aligned}$$

We have a contradiction because $T(\mathbf{u}) \neq T(\mathbf{u})$ therefore T is **not** a linear transformation.

■

11. We need to prove that $T(\mathbf{v}) = \mathbf{O}$ for all vectors $\mathbf{v} \in V$ is a linear transformation.
Need to check the 2 conditions of definition (5-2):

(a) $T(\mathbf{u}+\mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$
(b) $T(k\mathbf{u}) = kT(\mathbf{u})$

We have

$$T(\mathbf{u}+\mathbf{v}) = \mathbf{O} \text{ and } T(\mathbf{u}) + T(\mathbf{v}) = \mathbf{O} + \mathbf{O} = \mathbf{O}$$

Thus condition (a) is satisfied, that is $T(\mathbf{u}+\mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$. Also

$$T(k\mathbf{u}) = \mathbf{O} \text{ and } kT(\mathbf{u}) = k\,\mathbf{O} = \mathbf{O}$$

Hence condition (b) is satisfied therefore we conclude that zero transformation $T(\mathbf{v}) = \mathbf{O}$ is a linear transformation.

12. We need to prove that if **u** is any vector in V then we can write $T(\mathbf{u})$ as a linear combination of

$$\{T(\mathbf{v}_1),\ T(\mathbf{v}_2),\ T(\mathbf{v}_3),\ \cdots,\ T(\mathbf{v}_n)\}$$

where the vectors $\mathbf{v}_{\text{subscript}}$ form a basis for V.

Proof.

Since $\{\mathbf{v}_1,\ \mathbf{v}_2,\ \mathbf{v}_3,\ \cdots,\ \mathbf{v}_n\}$ is a basis for V and **u** is in V therefore we can write this as

$$\mathbf{u} = k_1\mathbf{v}_1 + k_2\mathbf{v}_2 + k_3\mathbf{v}_3 + \cdots + k_n\mathbf{v}_n \qquad \text{where the } k\text{'s are scalars}$$

Since T is a linear transformation therefore

$$\begin{aligned}T(\mathbf{u}) &= T(k_1\mathbf{v}_1 + k_2\mathbf{v}_2 + k_3\mathbf{v}_3 + \cdots + k_n\mathbf{v}_n) \\ &\underset{\text{By theorem (5-2)}}{=} k_1T(\mathbf{v}_1) + k_2T(\mathbf{v}_2) + k_3T(\mathbf{v}_3) + \cdots + k_nT(\mathbf{v}_n)\end{aligned}$$

Thus $T(\mathbf{u})$ is a linear combination of $\{T(\mathbf{v}_1),\ T(\mathbf{v}_2),\ T(\mathbf{v}_3),\ \cdots,\ T(\mathbf{v}_n)\}$.

■