Complete Solutions to Exercises 5.4

1. To show that a given linear transformation is one-to-one we use either of following two propositions:

(5-7) T is one-to-one $\Leftrightarrow$ $\mathbf{u} \neq \mathbf{v}$ implies $T(\mathbf{u}) \neq T(\mathbf{v})$

(5.2) T is one-to-one $\Leftrightarrow$ $T(\mathbf{u}) = T(\mathbf{v})$ implies $\mathbf{u} = \mathbf{v}$

(You could also show $\ker(T) = \{\mathbf{O}\}$ in each case but **not** in this question).

(a) Let **u** and **v** be distinct vectors in $\mathbb{R}^2$, that is $\mathbf{u} \neq \mathbf{v}$. We have

$$T(\mathbf{u}) = \mathbf{Iu} = \mathbf{u} \quad \text{and} \quad T(\mathbf{v}) = \mathbf{Iv} = \mathbf{v}$$

Since $\mathbf{u} \neq \mathbf{v}$ therefore $T(\mathbf{u}) \neq T(\mathbf{v})$ because we have $T(\mathbf{u}) = \mathbf{u}$ and $T(\mathbf{v}) = \mathbf{v}$. Since we have $\mathbf{u} \neq \mathbf{v}$ implies $T(\mathbf{u}) \neq T(\mathbf{v})$ therefore by Proposition (5-7) we conclude that T is one-to-one.

(b) We use (5.2) in this case. Let $\mathbf{u} = \begin{pmatrix} a \\ b \end{pmatrix}$ and $\mathbf{v} = \begin{pmatrix} c \\ d \end{pmatrix}$ be members of $\mathbb{R}^2$.

Applying the given transformation we have

$$T(\mathbf{u}) = T\left(\begin{pmatrix} a \\ b \end{pmatrix}\right) = \begin{pmatrix} b \\ a \end{pmatrix} \text{ and } T(\mathbf{v}) = T\left(\begin{pmatrix} c \\ d \end{pmatrix}\right) = \begin{pmatrix} d \\ c \end{pmatrix}$$

If $T(\mathbf{u}) = T(\mathbf{v})$ then $\begin{pmatrix} b \\ a \end{pmatrix} = \begin{pmatrix} d \\ c \end{pmatrix}$ which gives $b = d$ and $a = c$. Thus

$$\mathbf{u} = \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} c \\ d \end{pmatrix} = \mathbf{v}$$

We have $T(\mathbf{u}) = T(\mathbf{v})$ implies $\mathbf{u} = \mathbf{v}$ therefore by (5.2) we conclude that T is one-to-one.

(c) Let $\mathbf{u} = \begin{pmatrix} a \\ b \end{pmatrix}$ and . Applying the given transformation to vectors **u** and **v** we have

$$T(\mathbf{u}) = T\left(\begin{pmatrix} a \\ b \end{pmatrix}\right) = \begin{pmatrix} a+b \\ a-b \end{pmatrix} \text{ and } T(\mathbf{v}) = T\left(\begin{pmatrix} c \\ d \end{pmatrix}\right) = \begin{pmatrix} c+d \\ c-d \end{pmatrix}$$

If $T(\mathbf{u}) = T(\mathbf{v})$ then we have $\begin{pmatrix} a+b \\ a-b \end{pmatrix} = \begin{pmatrix} c+d \\ c-d \end{pmatrix}$ which gives

$$a + b = c + d$$
$$a - b = c - d$$

Adding these simultaneous equations together gives $2a = 2c \quad \Rightarrow \quad a = c$. Substituting $a = c$ into the first equation yields $c + b = c + d \quad \Rightarrow \quad b = d$. Thus the solutions of the above simultaneous equations is $a = c$ and $b = d$ which gives

$$\mathbf{u} = \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} c \\ d \end{pmatrix} = \mathbf{v}$$

We have $T(\mathbf{u}) = T(\mathbf{v})$ implies $\mathbf{u} = \mathbf{v}$. By (5.2) we conclude that the given linear transformation T is one-to-one.

2. This time we use the following Proposition

(5-7) T is one-to-one $\Leftrightarrow$ $\ker(T)=\{\mathbf{O}\}$

We need to show that kernel of T consists **only** of the zero matrix.

Let $\mathbf{A}=\begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{pmatrix}$ then

$$T(\mathbf{A})=\begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{pmatrix}^T$$

$$\underset{\text{Transposing}}{=}\begin{pmatrix} a_{11} & \cdots & a_{m1} \\ \vdots & \ddots & \vdots \\ a_{1n} & \cdots & a_{mn} \end{pmatrix}=\begin{pmatrix} 0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \end{pmatrix}=\mathbf{O}$$

The kernel of T is the n by m zero matrix $\mathbf{O}$ because **all** the entries in the matrix are zero. Thus $\ker(T)=\{\mathbf{O}\}$ therefore by (5-7) we conclude T is one-to-one.

3. (a) For one-to-one we can check the kernel of T. Let $\mathbf{u}=\begin{pmatrix} x \\ y \end{pmatrix}$ then applying the given transformation we have

$$T\left(\begin{pmatrix} x \\ y \end{pmatrix}\right)=\begin{pmatrix} x \\ x \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \text{which gives } x=0$$

This means that y can be any real number r. Thus $\ker(T)=\left\{\begin{pmatrix} 0 \\ r \end{pmatrix} \middle| \ r\in\mathbb{R}\right\}$ which means that $\ker(T)\neq\{\mathbf{O}\}$ which implies that T is **not** one-to-one.

Is T onto?

No because T is **not** one-to-one so by Proposition (5-12):

> Proposition (5-12). If $T: V\to W$ is a linear transformation and $\dim(V)=\dim(W)$ then T is *both* one-to-one and onto $\Leftrightarrow$ $\ker(T)=\{\mathbf{O}\}$.

We conclude that T is **not** onto.

(b) We need to test for **one-to-one** and **onto** for the given transformation $T\left(\begin{pmatrix} x \\ y \\ z \end{pmatrix}\right)=\begin{pmatrix} 0 \\ y \\ z \end{pmatrix}$

<u>One-to-one</u>:

What is the kernel of T?

It is the vectors in $\mathbb{R}^3$ which give the zero vector under the given transformation T. We have

$$T\left(\begin{pmatrix} x \\ y \\ z \end{pmatrix}\right)=\begin{pmatrix} 0 \\ y \\ z \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad \text{gives} \quad x=r,\ y=0 \ \text{ and } \ z=0$$

where r is any real number.

Hence $\ker(T) \neq \{\mathbf{O}\}$ therefore T is **not** one-to-one.
By Proposition (5-12) we can say that T is **not** onto.

(c) For $T: \mathbb{R}^3 \to \mathbb{R}^3$ given by $T\left(\begin{pmatrix} x \\ y \\ z \end{pmatrix}\right) = \begin{pmatrix} z \\ y \\ x \end{pmatrix}$ we find the kernel.

$$T\left(\begin{pmatrix} x \\ y \\ z \end{pmatrix}\right) = \begin{pmatrix} z \\ y \\ x \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \text{ gives } x = 0,\ y = 0 \text{ and } z = 0$$

Hence $\ker(T) = \{\mathbf{O}\}$ therefore T is one-to-one and by Proposition (5-12):

> Proposition (5-12). If $T: V \to W$ is a linear transformation and $\dim(V) = \dim(W)$ then T is *both* one-to-one and onto $\Leftrightarrow$ $\ker(T) = \{\mathbf{O}\}$.

We conclude that T is onto.

(d) The given transformation is $T\left(\begin{pmatrix} x \\ y \end{pmatrix}\right) = \begin{pmatrix} 2x + 3y \\ x + y \\ 0 \end{pmatrix}$. We need to find the kernel to see if T is one-to-one.

$$T\left(\begin{pmatrix} x \\ y \end{pmatrix}\right) = \begin{pmatrix} 2x + 3y \\ x + y \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \text{ gives } x = 0 \text{ and } y = 0$$

Thus $\ker(T) = \{\mathbf{O}\}$ therefore T is one-to-one. *Is T onto?*

We **cannot** use Proposition (5-12) because we need $\dim(V) = \dim(W)$ and in this case we have $\dim(\mathbb{R}^2) = 2$ and $\dim(\mathbb{R}^3) = 3$.

Let $\mathbf{w} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$ where $c \neq 0$ in $\mathbb{R}^3$. Then there is **no** vector $\mathbf{u} = \begin{pmatrix} x \\ y \end{pmatrix}$ in $\mathbb{R}^2$ such that $T(\mathbf{u}) = \mathbf{w}$ because

$$T\left(\begin{pmatrix} x \\ y \end{pmatrix}\right) = \begin{pmatrix} 2x + 3y \\ x + y \\ 0 \end{pmatrix} \neq \begin{pmatrix} a \\ b \\ c \end{pmatrix} \text{ where } c \neq 0$$

Thus the linear transformation T is **not** onto because T does not fill the whole of $\mathbb{R}^3$.
In conclusion T is one-to-one but **not** onto.

4. We are given the transformation $T(\mathbf{u}) = \mathbf{A}\mathbf{u}$ where $\mathbf{A} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$.

How do we show this transformation is one-to-one?

We need to show that the kernel consists only of the zero vector. Let $\mathbf{u} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ then

$$T\left(\begin{pmatrix} x \\ y \\ z \end{pmatrix}\right) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \text{ gives } x = 0,\ y = 0 \text{ and } z = 0$$

Thus the kernel is the zero vector, that is $\ker(T) = \{\mathbf{O}\}$. Since $\dim(\mathbb{R}^3) = \dim(\mathbb{R}^3) = 3$ therefore we can use the following:

Proposition (5-12). If $T: V \to W$ is a linear transformation and $\dim(V) = \dim(W)$ then T is *both* one-to-one and onto $\Leftrightarrow$ $\ker(T) = \{\mathbf{O}\}$.

Hence T is both one-to-one and onto.

5. We need to show that $T: P_3 \to P_2$ given by $T(\mathbf{p}) = \mathbf{p}'$ is **not** one-to-one but is onto. We use Definition (5-7) which says that $T(\mathbf{p}) = T(\mathbf{q})$ implies that $\mathbf{p} = \mathbf{q}$ to check for one-to-one.

Let $\mathbf{p} = ax^3 + bx^2 + cx + d$ and $\mathbf{q} = ex^3 + fx^2 + gx + h$ then

$$\begin{aligned} T(\mathbf{p}) &= \left(ax^3 + bx^2 + cx + d\right)' \\ &= 3ax^2 + 2bx + c \end{aligned}$$

and

$$\begin{aligned} T(\mathbf{q}) &= \left(ex^3 + fx^2 + gx + h\right)' \\ &= 3ex^2 + 2fx + g \end{aligned}$$

Remember we need to check that $T(\mathbf{p}) = T(\mathbf{q})$ implies that $\mathbf{p} = \mathbf{q}$:

$$T(\mathbf{p}) = 3ax^2 + 2bx + c = T(\mathbf{q}) = 3ex^2 + 2fx + g$$

Equating coefficients gives $a = e,\ b = f$ and $c = g$. However we do **not** need d to equal h. For $d \neq h$ therefore $\mathbf{p} \neq \mathbf{q}$. We have the same destination $T(\mathbf{p}) = T(\mathbf{q})$ but different start vectors $\mathbf{p} \neq \mathbf{q}$ therefore T is **not** one-to-one.

To show that $T: P_3 \to P_2$ given by $T(\mathbf{p}) = \mathbf{p}'$ is onto we only need to prove that the range of T is the vector space of quadratic polynomials. From above we have

$$T(\mathbf{p}) = 3ax^2 + 2bx + c$$

which means that $T(\mathbf{p})$ is a quadratic polynomial. Since we can write an arbitrary quadratic polynomial such as $ax^2 + bx + c$ in P_2 as

$$ax^2 + bx + c = \left(\frac{ax^3}{3} + \frac{bx^2}{2} + cx\right)'$$

therefore the range of T is P_2 which means that T is onto.

6. We need to show that $T: P_n \to P_n$ given by $T(\mathbf{p}) = \mathbf{p}''$ is **not** one-to-one nor onto. *How do we show that given transformation T is **not** one-to-one?*

We can find the kernel of *T*. The kernel of *T* is given by a polynomial **p** such that

$$T(\mathbf{p}) = \mathbf{p}'' = \mathbf{O}$$

Which polynomial gives zero after differentiating it twice?
The linear polynomial $\mathbf{p} = ax + b$ because

$$\mathbf{p}'' = (ax+b)'' = a' = 0$$

Thus a non-zero polynomial $\mathbf{p} = ax + b$ give zero under the transformation *T*. Therefore $\ker(T) = \{ax + b \mid a \in \mathbb{R},\ b \in \mathbb{R}\} \neq \{\mathbf{O}\}$. Since $\ker(T) \neq \{\mathbf{O}\}$ so using:

> Proposition (5-12). If $T:\ V \to W$ is a linear transformation and $\dim(V) = \dim(W)$ then *T* is *both* one-to-one and onto $\Leftrightarrow$ $\ker(T) = \{\mathbf{O}\}$.

We have $T:\ P_n \to P_n$ which means we have $\dim(P_n) = \dim(P_n)$ therefore we can apply the above Proposition (5-12) and conclude that the given transformation *T* is **neither** onto nor one-to-one.

7. We are given the linear transformation $T:\ \mathbb{R}^3 \to \mathbb{R}^2$ defined by $T\left(\begin{pmatrix} x \\ y \\ z \end{pmatrix}\right) = \begin{pmatrix} y \\ z \end{pmatrix}$. *How do we show this is **not** one-to-one?*

Consider the 2 vectors $\mathbf{u} = \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}$ and $\mathbf{v} = \begin{pmatrix} 5 \\ 1 \\ 2 \end{pmatrix}$ then

$$T(\mathbf{u}) = T\left(\begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}\right) = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \quad \text{and} \quad T(\mathbf{v}) = T\left(\begin{pmatrix} 5 \\ 1 \\ 2 \end{pmatrix}\right) = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

We have $T(\mathbf{u}) = T(\mathbf{v}) = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ but $\mathbf{u} = \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} \neq \mathbf{v} = \begin{pmatrix} 5 \\ 1 \\ 2 \end{pmatrix}$. We arrive at the same destination $T(\mathbf{u}) = T(\mathbf{v})$ but with different starting vectors $\mathbf{u} \neq \mathbf{v}$. Hence *T* is not one-to-one.

How do we show T is onto?

Let $\mathbf{w} = \begin{pmatrix} a \\ b \end{pmatrix}$ be an arbitrary vector in $\mathbb{R}^2$ then we need to find a vector in $\mathbb{R}^3$ such that

$$T\left(\begin{pmatrix} x \\ y \\ z \end{pmatrix}\right) = \begin{pmatrix} a \\ b \end{pmatrix}$$

Let $\mathbf{u} = \begin{pmatrix} x \\ a \\ b \end{pmatrix}$ then $T(\mathbf{u}) = T\left(\begin{pmatrix} x \\ a \\ b \end{pmatrix}\right) = \begin{pmatrix} a \\ b \end{pmatrix} = \mathbf{w}$. Since we have found a vector **u** in the domain such that $T(\mathbf{u}) = \mathbf{w}$ therefore the range of *T* is $\mathbb{R}^2$ which means that *T* is onto.

8. We need to show that $T:\ P_2 \to P_3$ given by

$$T\left(ax^2+bx+c\right)=ax+\left(b+c\right)$$

is neither one-to-one nor onto.

<u>Not one-to-one</u>:

Let $\mathbf{u}=ax^2+bx+c$ and $\mathbf{v}=dx^2+ex+f$. If $T\left(\mathbf{u}\right)=T\left(\mathbf{v}\right)$ then we have

$$\left.\begin{aligned}T\left(\mathbf{u}\right)=T\left(ax^2+bx+c\right)=ax+\left(b+c\right)\\ T\left(\mathbf{v}\right)=T\left(dx^2+ex+f\right)=dx+\left(e+f\right)\end{aligned}\right\}\text{ gives } ax+\left(b+c\right)=dx+\left(e+f\right)$$

By equating the coefficients of $ax+\left(b+c\right)=dx+\left(e+f\right)$ we have

$$a=d \quad\text{and}\quad b+c=e+f$$

From $b+c=e+f \quad \not\Rightarrow \quad b=e$ and $c=f$ because we could have

$$b=e+f \quad\text{and}\quad c=0$$

This means that the vectors **u** and **v** are **not** identical, thus $\mathbf{u}\neq\mathbf{v}$. We have

$$T\left(\mathbf{u}\right)=T\left(\mathbf{v}\right) \text{ implies } \mathbf{u}\neq\mathbf{v}$$

By (5.2) we conclude that *T* is **not** one-to-one.

<u>Not Onto</u>:

Let $\mathbf{w}=dx^3$ be a vector in P_3. Then there is **no** vector **u** in P_2 such that

$$T\left(\mathbf{u}\right)=dx^3=\mathbf{w}$$

By (5-9) we conclude that the given transformation *T* is **not** onto.

9. We need to show $T:\ P_n \to P_n$ given by

$$T\left(\mathbf{p}\right)=\mathbf{p}'$$

is **neither** one-to-one nor onto. *How?*

Show that the kernel of *T* is **not** equal to the zero vector, $\{\mathbf{O}\}$. Let $\mathbf{p}=C$ where *C* is a non-zero constant then

$$T\left(\mathbf{p}\right)=C'=\mathbf{O}$$

The kernel of *T* is $\{C \mid C\in\mathbb{R}\}$ which means that $\ker\left(T\right)\neq\{\mathbf{O}\}$.

Since $\dim\left(P_n\right)=\dim\left(P_n\right)$ so by:

> Proposition (5-12). If $T:\ V\to W$ is a linear transformation and $\dim\left(V\right)=\dim\left(W\right)$ then *T* is *both* one-to-one and onto $\Leftrightarrow$ $\ker\left(T\right)=\{\mathbf{O}\}$.

We conclude that *T* is **neither** one-to-one nor onto.

10. Proposition (5-8) says that the linear transformation $T:\ V\to W$ is one-to-one $\Leftrightarrow$ $nullity\left(T\right)=0$.

Proof.

$\left(\Rightarrow\right)$. Assume *T* is one-to-one. Let $\mathbf{v}$ be in $\ker\left(T\right)$ then by the definition of the kernel we have $T\left(\mathbf{v}\right)=\mathbf{O}$. Since *T* is linear so by theorem (5-1) (part (a)) $T\left(\mathbf{O}\right)=\mathbf{O}$. Thus

$$T\left(\mathbf{v}\right)=T\left(\mathbf{O}\right)=\mathbf{O}$$

We are assuming that *T* is one-to-one therefore by (5-7)

$$T(\mathbf{v}) = T(\mathbf{O}) \text{ implies } \mathbf{v} = \mathbf{O}$$

Thus $\ker(T) = \{\mathbf{O}\}$ so we have $nullity(T) = 0$.

$(\Leftarrow)$. Assume $nullity(T) = 0$ then $\ker(T) = \{\mathbf{O}\}$ therefore by Proposition (5-7):

(5-7). Let $T: V \to W$ then T is one-to-one $\Leftrightarrow$ $\ker(T) = \{\mathbf{O}\}$

The given linear transformation is one-to-one.

■

11. We have to prove:

If $T: V \to W$ is a linear one-to-one transformation then for every vector **w** in $range(T)$ there exists a **unique** vector **v** in V such that $T(\mathbf{v}) = \mathbf{w}$.

Proof.

Let **w** be in $range(T)$ then by the definition of range (5.1):

(5.1)	$\text{range}(T) = \{T(\mathbf{v}) \mid \mathbf{v} \text{ in } V\}$

There exists a vector **v** in V such that $T(\mathbf{v}) = \mathbf{w}$. Suppose there is also a vector **u** in V such that

$$T(\mathbf{u}) = \mathbf{w}$$

Since T is one-to-one so by (5.2):

(5.2)	T is one-to-one $\Leftrightarrow$ $T(\mathbf{u}) = T(\mathbf{v})$ implies $\mathbf{u} = \mathbf{v}$

We have

$$T(\mathbf{v}) = T(\mathbf{u}) = \mathbf{w} \text{ implies } \mathbf{v} = \mathbf{u}$$

Thus the vector **v** is unique which completes our proof.

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12. Proposition (5-9) says a linear transformation $T: V \to W$ is an **onto** transformation $\Leftrightarrow$ $range(T) = W$.

Proof.

$(\Leftarrow)$. Assume $range(T) = W$ then by the definition of range (5.1):

(5.1)	$\text{range}(T) = \{T(\mathbf{v}) \mid \mathbf{v} \text{ in } V\}$

We have for every **w** in $range(T) = W$ there is a **v** in V such that $T(\mathbf{v}) = \mathbf{w}$. Thus T is onto.

$(\Rightarrow)$. We prove W is a subspace of $range(T)$ and $range(T)$ is subspace of W then $range(T) = W$.

Assume $T: V \to W$ is an **onto** linear transformation. Let **w** be an arbitrary vector in W. By Definition (5-8) :

Let $T: V \to W$ be a linear transform. The transform T is **onto** $\Leftrightarrow$ for every **w** in the arrival vector space W there exists at least one **v** in the start vector space V such that $\mathbf{w} = T(\mathbf{v})$.

There exists a vector **v** in V such that $T(\mathbf{v})=\mathbf{w}$. This means that **w** is in $range(T)$. Hence W is a subspace of $range(T)$.
Since $range(T)$ is a subspace of W so $range(T)=W$.

■

13. Proposition (5-10) says that a linear transformation $T: V \to W$ is onto $\Leftrightarrow$ $rank(T)=\dim(W)$.
Proof.
This follows from Proposition (5-9) because we have $T: V \to W$ is an **onto** transformation $\Leftrightarrow$ $range(T)=W$. Since $rank(T)$ is the dimension of the range of T so $rank(T)=\dim(W)$.

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14. Need to prove:

Proposition (5-12). If $T: V \to W$ is a linear transformation and $\dim(V)=\dim(W)$ then T is *both* one-to-one and onto $\Leftrightarrow$ $\ker(T)=\{\mathbf{O}\}$.

How do we prove this result?
By Proposition (5-7):

Proposition (5-7). Let $T: V \to W$ be a linear transformation between the vector spaces V and W. Then T is one-to-one $\Leftrightarrow$ $\ker(T)=\{\mathbf{O}\}$.

and Proposition (5-11):

Proposition (5-11). If $T: V \to W$ is a linear transformation and $\dim(V)=\dim(W)$ then T is a one-to-one transformation $\Leftrightarrow$ T is onto.

Proof.
Let $\dim(V)=\dim(W)$.
$(\Rightarrow)$. Let $T: V \to W$ be a linear transformation and T be *both* one-to-one and onto. Since T is one-to-one so by (5-7) we have $\ker(T)=\{\mathbf{O}\}$.
$(\Leftarrow)$. Assume $\ker(T)=\{\mathbf{O}\}$ so by (5-7) we have T is one-to-one and by (5-11) T is onto because $\dim(V)=\dim(W)$.

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15. We need to prove:

Proposition (5-13). Let $T: V \to W$ be a linear transform which is *both* one-to-one and onto. Then the inverse transform $T^{-1}: W \to V$ is also linear.

How do we prove this result?
By using the definition of the inverse transform:

Definition (5-9). Let $T: V \to W$ be a *bijective* linear transform. The inverse transformation $T^{-1}: W \to V$ is defined as:

$$\mathbf{v}=T^{-1}(\mathbf{w}) \quad \Leftrightarrow \quad T(\mathbf{v})=\mathbf{w}$$

Proof.

Since T is both *onto* and *one-to-one* (bijective), the inverse transform $T^{-1}: W \to V$ must exist. *What do we need to prove?*
We are required to prove that $T^{-1}: W \to V$ is linear. *How?*
By showing $T^{-1}(\mathbf{w}_1 + \mathbf{w}_2) = T^{-1}(\mathbf{w}_1) + T^{-1}(\mathbf{w}_2)$ and $T^{-1}(k\mathbf{w}) = kT^{-1}(\mathbf{w})$ where $\mathbf{w}_1$, $\mathbf{w}_2$ and $\mathbf{w}$ are arbitrary vectors of the arrival vector space W and k is a scalar.
Let $\mathbf{w}_1$, $\mathbf{w}_2$ and $\mathbf{w}$ be arbitrary vectors of the arrival vector space W.
Since T is onto there exists vectors $\mathbf{u}$ and $\mathbf{v}$ in the start vector space V such that

$$T(\mathbf{u}) = \mathbf{w}_1 \quad \text{and} \quad T(\mathbf{v}) = \mathbf{w}_2$$

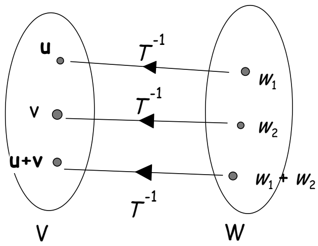
Applying the inverse transform definition (5-9):

(5-9) $$T(\mathbf{u}) = \mathbf{w} \quad \Leftrightarrow \quad \mathbf{u} = T^{-1}(\mathbf{w})$$

to these, $T(\mathbf{u}) = \mathbf{w}_1$ and $T(\mathbf{v}) = \mathbf{w}_2$, gives

$$\mathbf{u} = T^{-1}(\mathbf{w}_1) \quad \text{and} \quad \mathbf{v} = T^{-1}(\mathbf{w}_2)$$

This can be illustrated as:



Since T is linear we have

$$T(\mathbf{u} + \mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$$
$$= \mathbf{w}_1 + \mathbf{w}_2$$

Applying the inverse transform (5-9) to this $T(\mathbf{u} + \mathbf{v}) = \mathbf{w}_1 + \mathbf{w}_2$ gives

$$\mathbf{u} + \mathbf{v} = T^{-1}(\mathbf{w}_1 + \mathbf{w}_2) \qquad (\dagger)$$

Substituting the above result of $\mathbf{u} = T^{-1}(\mathbf{w}_1)$ and $\mathbf{v} = T^{-1}(\mathbf{w}_2)$ into the Left Hand Side of (†) yields

$$T^{-1}(\mathbf{w}_1) + T^{-1}(\mathbf{w}_2) = T^{-1}(\mathbf{w}_1 + \mathbf{w}_2)$$

which shows that T^{-1} preserves vector addition.
We also need to prove $T^{-1}(k\mathbf{w}) = kT^{-1}(\mathbf{w})$.
Let $T(\mathbf{u}) = \mathbf{w}$ then by the inverse transform definition (5-9) we have $T^{-1}(\mathbf{w}) = \mathbf{u}$.
T is linear therefore

$$T(k\mathbf{u}) = kT(\mathbf{u}) = k\mathbf{w} \qquad (*)$$

Applying the inverse transform to this (*) we have

$$k\mathbf{u} = T^{-1}(k\mathbf{w})$$

and since $T^{-1}(\mathbf{w}) = \mathbf{u}$ we have our result, that is $kT^{-1}(\mathbf{w}) = T^{-1}(k\mathbf{w})$. Hence T^{-1} preserves scalar multiplication.
Thus we have shown *both* conditions of linearity therefore the inverse transform is linear.

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16. We need to prove:

Lemma (5-14).
Let V and W be finite dimensional real vector spaces of equal dimension and $T: V \to W$ be an isomorphism.
If $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \cdots, \mathbf{v}_n\}$ is a basis (axes) for V then $\{T(\mathbf{v}_1), T(\mathbf{v}_2), T(\mathbf{v}_3), \cdots, T(\mathbf{v}_n)\}$ is a basis (axes) for W.

Proof.
Since $T: V \to W$ is an isomorphism so T is a bijection which means both vector spaces V and W are of the same dimension, say n. Also $\ker(T) = \{\mathbf{O}\}$ so only the zero vector is transformed to the zero vector.
We can write the zero vector in V as a linear combination of the basis vectors $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \cdots, \mathbf{v}_n\}$:

$$k_1\mathbf{v}_1 + k_2\mathbf{v}_2 + \cdots + k_n\mathbf{v}_n = \mathbf{O} \quad \text{implies} \quad k_1 = k_2 = k_3 = \cdots = k_n = 0$$

Because $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \cdots, \mathbf{v}_n\}$ is a basis for V.
Since T is linear we have $T(\mathbf{O}) = \mathbf{O}$ and

$$\begin{aligned} T(\mathbf{O}) &= T(k_1\mathbf{v}_1 + k_2\mathbf{v}_2 + \cdots + k_n\mathbf{v}_n) \\ &= k_1T(\mathbf{v}_1) + k_2T(\mathbf{v}_2) + \cdots + k_nT(\mathbf{v}_n) = \mathbf{O} \end{aligned}$$

All the scalars $k_1 = k_2 = k_3 = \cdots = k_n = 0$ so $\{T(\mathbf{v}_1), T(\mathbf{v}_2), T(\mathbf{v}_3), \cdots, T(\mathbf{v}_n)\}$ is linearly independent in W. The dimension of W is n so this set forms a basis for W.

■

17. We need to show that $T\left(\begin{pmatrix} x \\ y \end{pmatrix}\right) = \begin{pmatrix} 2x+y \\ x-y \end{pmatrix}$ is one-to-one and onto for T to have an inverse linear transformation.

How do we show that T is one-to-one and onto?

Since $\dim(\mathbb{R}^2) = \dim(\mathbb{R}^2)$ therefore we only need to show that kernel of T is the zero vector. We have

$$T\left(\begin{pmatrix} x \\ y \end{pmatrix}\right) = \begin{pmatrix} 2x+y \\ x-y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \text{ gives } x = y = 0$$

Thus $\ker(T) = \{\mathbf{O}\}$ therefore T is **both** one-to-one and onto.
To find T^{-1} let $a = 2x + y$ and $b = x - y$. We need to write x and y in terms of a and b:

$$\begin{aligned} & a = 2x + y \\ + \quad & b = x - y \\ \hline & a + b = 3x \quad \text{which gives} \quad x = \frac{a+b}{3} \end{aligned}$$

Substituting $x = \dfrac{a+b}{3}$ into $b = x - y$ yields

$$b = \underbrace{\frac{a+b}{3}}_{=x} - y \quad \text{which gives} \quad y = \frac{a+b}{3} - b = \frac{a+b}{3} - \frac{3b}{3} = \frac{a-2b}{3}$$

We have $x=\dfrac{a+b}{3}$ and $y=\dfrac{a-2b}{3}$. Thus

$$T^{-1}\left(\begin{pmatrix} a \\ b \end{pmatrix}\right)=\begin{pmatrix} x \\ y \end{pmatrix}=\begin{pmatrix} \dfrac{a+b}{3} \\ \dfrac{a-2b}{3} \end{pmatrix}=\frac{1}{3}\begin{pmatrix} a+b \\ a-2b \end{pmatrix}$$

18. We need to show that $T\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right)=\begin{pmatrix} a & b & c & d \end{pmatrix}^T$ is an isomorphism. *How?*

By showing $\ker(T)=\{\mathbf{O}\}$. Remember the kernel is all the elements in M_{22} which are transformed to the zero vector $\begin{pmatrix} 0 & 0 & 0 & 0 \end{pmatrix}^T=\mathbf{O}$. Hence we have $a=b=c=d=0$ which means the kernel of T is the $\mathbf{O}_{22}$ matrix. Hence $\ker(T)=\{\mathbf{O}\}$ therefore T is an isomorphism.

19. (a) Required to prove the following:
A linear transform $T:V\to W$ is an isomorphism $\Leftrightarrow$ $\ker(T)=\{\mathbf{O}\}$.
Proof.
Since $T:V\to W$ is an isomorphism $\Leftrightarrow$ T is invertible $\Leftrightarrow$ T is a bijection and therefore one-to-one. By

Proposition (5-7). Let $T:\ V\to W$ be a linear transformation between the vector spaces V and W. Then T is one-to-one $\Leftrightarrow$ $\ker(T)=\{\mathbf{O}\}$.

We have our result that T is an isomorphism $\Leftrightarrow$ $\ker(T)=\{\mathbf{O}\}$.

■

(b) If $T:V\to W$ is an isomorphism then $T^{-1}:\ W\to V$ is also an isomorphism.
Proof.
We have $T:V\to W$ is an isomorphism which means that T is invertible. By

Proposition (5-13). Let $T:\ V\to W$ be a linear transform which is *both* one-to-one and onto. Then the inverse transform $T^{-1}:\ W\to V$ is also linear.

We can say that $T^{-1}:\ W\to V$ is linear. Since $\ker(T)=\{\mathbf{O}\}$ so $\ker\left(T^{-1}\right)=\{\mathbf{O}\}$. *Why?*

Suppose $\ker\left(T^{-1}\right)\neq\{\mathbf{O}\}$ and let $\mathbf{w}\neq\mathbf{O}$ be in $\ker\left(T^{-1}\right)$. Then

$$T^{-1}(\mathbf{w})=\mathbf{O}$$

Using Definition (5-9):

Let $T:\ V\to W$ be a *bijective* linear transform. The inverse transformation $T^{-1}:\ W\to V$ is defined as:

$$\mathbf{v}=T^{-1}(\mathbf{w}) \quad\Leftrightarrow\quad T(\mathbf{v})=\mathbf{w}$$

We have $T(\mathbf{O})=\mathbf{w}\neq\mathbf{O}$. This cannot be the case because $\ker(T)=\{\mathbf{O}\}$. Hence our supposition $\ker\left(T^{-1}\right)\neq\{\mathbf{O}\}$ must be false so $\ker\left(T^{-1}\right)=\{\mathbf{O}\}$ which means T^{-1} is an isomorphism by result (a).

■

(c) Required to prove that:
If vector spaces V and W are isomorphic then $\dim(V)=\dim(W)$.

Proof.
Let $\dim(V)=n$. We are given that V and W are isomorphic so there is an invertible transformation $T:V\to W$. This is a bijection which means it is one-to-one and onto.
Using:

Proposition (5-8). Let $T: V\to W$ be a linear transformation. T is one-to-one $\Leftrightarrow$ $nullity(T)=0$.

We have $nullity(T)=0$. By the Dimension Theorem:

$$rank(T)+nullity(T)=n \text{ (where } n \text{ is the dimension of } V)$$

Substituting $nullity(T)=0$ into this gives

$$rank(T)=n$$

By Proposition (5-10):

Let $T: V\to W$ be a linear transformation. Then T is onto $\Leftrightarrow$

$$rank(T)=\dim(W)$$

We have $\dim(W)=n=\dim(V)$. This completes our proof.

■