Complete Solutions to Miscellaneous Exercises 7

1. (a) *What are the eigenvalues of* $\mathbf{A}=\begin{pmatrix}1 & 1\\ 0 & 3\end{pmatrix}$ *equal to?*

Since we have a (upper) triangular matrix therefore the eigenvalues are the entries on the leading diagonal of **A**. We have $\lambda_1=1$ and $\lambda_2=3$.

(b) Let **u** and **v** be the corresponding eigenvectors of $\lambda_1=1$ and $\lambda_2=3$ respectively. We can find these by

$$(\mathbf{A}-\mathbf{I})\mathbf{u}=\begin{pmatrix}1-1 & 1\\ 0 & 3-1\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix}=\begin{pmatrix}0 & 1\\ 0 & 2\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix}=\begin{pmatrix}0\\ 0\end{pmatrix}$$

This gives $x=1$ and $y=0$. Thus $\mathbf{u}=\begin{pmatrix}1\\ 0\end{pmatrix}$ is the eigenvector for $\lambda_1=1$. Similarly for $\lambda_2=3$ we have

$$(\mathbf{A}-3\mathbf{I})\mathbf{v}=\begin{pmatrix}1-3 & 1\\ 0 & 3-3\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix}=\begin{pmatrix}-2 & 1\\ 0 & 0\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix}=\begin{pmatrix}0\\ 0\end{pmatrix}$$

This gives $x=1$ and $y=2$. Thus $\mathbf{v}=\begin{pmatrix}1\\ 2\end{pmatrix}$. *What is the matrix* $\boldsymbol{Q}$ *equal to?*

$\mathbf{Q}=(\mathbf{u}\quad \mathbf{v})=\begin{pmatrix}1 & 1\\ 0 & 2\end{pmatrix}$. *What is the diagonal matrix* $\boldsymbol{D}$ *equal to?*

D is the diagonal matrix with entries along the leading diagonal given by the eigenvalues. We have $\mathbf{D}=\begin{pmatrix}\lambda_1 & 0\\ 0 & \lambda_2\end{pmatrix}=\begin{pmatrix}1 & 0\\ 0 & 3\end{pmatrix}$.

(c) *How do we find* $\mathbf{A}^5$ *?*

By applying (7-14) $\mathbf{A}^m=\mathbf{Q}\mathbf{D}^m\mathbf{Q}^{-1}$. We need to find the inverse of $\mathbf{Q}=\begin{pmatrix}1 & 1\\ 0 & 2\end{pmatrix}$:

$$\mathbf{Q}^{-1}=\frac{1}{2}\begin{pmatrix}2 & -1\\ 0 & 1\end{pmatrix}$$

Substituting $\mathbf{Q}=\begin{pmatrix}1 & 1\\ 0 & 2\end{pmatrix}$, $\mathbf{D}=\begin{pmatrix}1 & 0\\ 0 & 3\end{pmatrix}$ and $\mathbf{Q}^{-1}=\frac{1}{2}\begin{pmatrix}2 & -1\\ 0 & 1\end{pmatrix}$ into $\mathbf{A}^m=\mathbf{Q}\mathbf{D}^m\mathbf{Q}^{-1}$ with $m=5$ gives

$$\begin{aligned}
\mathbf{A}^5&=\begin{pmatrix}1 & 1\\ 0 & 2\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & 3\end{pmatrix}^5\frac{1}{2}\begin{pmatrix}2 & -1\\ 0 & 1\end{pmatrix}\\
&=\frac{1}{2}\begin{pmatrix}1 & 1\\ 0 & 2\end{pmatrix}\begin{pmatrix}1^5 & 0\\ 0 & 3^5\end{pmatrix}\begin{pmatrix}2 & -1\\ 0 & 1\end{pmatrix}\\
&=\frac{1}{2}\begin{pmatrix}1 & 1\\ 0 & 2\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & 243\end{pmatrix}\begin{pmatrix}2 & -1\\ 0 & 1\end{pmatrix}\\
&=\frac{1}{2}\begin{pmatrix}1 & 243\\ 0 & 486\end{pmatrix}\begin{pmatrix}2 & -1\\ 0 & 1\end{pmatrix}\\
&=\frac{1}{2}\begin{pmatrix}2 & 242\\ 0 & 486\end{pmatrix}=\begin{pmatrix}1 & 121\\ 0 & 243\end{pmatrix}
\end{aligned}$$

Hence $\mathbf{A}^5 = \begin{pmatrix} 1 & 121 \\ 0 & 243 \end{pmatrix}$.

2. To find $\mathbf{A}^{1,000,\ 001}$ of $\mathbf{A} = \begin{pmatrix} 4 & 5 \\ -3 & -4 \end{pmatrix}$ we need to determine the characteristic equation $p_A(\lambda)$.

$$\begin{aligned} p_A(\lambda) = \det(\mathbf{A} - \lambda\mathbf{I}) = \det\begin{pmatrix} 4-\lambda & 5 \\ -3 & -4-\lambda \end{pmatrix} \\ &= (4-\lambda)(-4-\lambda)+15 \\ &= (\lambda-4)(\lambda+4)+15 \\ &= \lambda^2 - 1 = 0 \end{aligned}$$

By the Cayley Hamilton Theorem we have

$$\mathbf{A}^2 - \mathbf{I} = \mathbf{O} \quad \text{which gives } \mathbf{A}^2 = \mathbf{I}$$

Using the rules of indices on matrices we have

$$\mathbf{A}^{1,000,\ 001} = \left(\mathbf{A}^2\right)^{500,000}\mathbf{A} = \mathbf{IA} = \mathbf{A} = \begin{pmatrix} 4 & 5 \\ -3 & -4 \end{pmatrix}$$

3. We need to find the eigenvalues and eigenvectors of $\mathbf{A} = \begin{pmatrix} 0 & 2 & 2 \\ 2 & 0 & 2 \\ 2 & 2 & 0 \end{pmatrix}$. Let λ be the eigenvalues:

$$\begin{aligned} \det(\mathbf{A} - \lambda\mathbf{I}) &= \det\begin{pmatrix} -\lambda & 2 & 2 \\ 2 & -\lambda & 2 \\ 2 & 2 & -\lambda \end{pmatrix} \\ &= -\lambda\det\begin{pmatrix} -\lambda & 2 \\ 2 & -\lambda \end{pmatrix} - 2\det\begin{pmatrix} 2 & 2 \\ 2 & -\lambda \end{pmatrix} + 2\det\begin{pmatrix} 2 & -\lambda \\ 2 & 2 \end{pmatrix} \\ &= -\lambda\left(\lambda^2 - 4\right) - 2(-2\lambda - 4) + 2(4 + 2\lambda) \\ &= -\lambda(\lambda-2)(\lambda+2) + 4(\lambda+2) + 4(\lambda+2) \\ &= -\lambda(\lambda-2)(\lambda+2) + 8(\lambda+2) \\ &= -(\lambda+2)\left[\lambda(\lambda-2) - 8\right] \\ &= -(\lambda+2)\left[\lambda^2 - 2\lambda - 8\right] \\ &= -(\lambda+2)(\lambda+2)(\lambda-4) = 0 \end{aligned}$$

We have $\lambda_{1,\ 2} = -2$ and $\lambda_3 = 4$. Let $\mathbf{u}$ be the eigenvector belonging to $\lambda_{1,\ 2} = -2$:

$$(\mathbf{A} + 2\mathbf{I})\mathbf{u} = \begin{pmatrix} 2 & 2 & 2 \\ 2 & 2 & 2 \\ 2 & 2 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Putting the matrix into reduced row echelon form gives

$$\begin{array}{c} \begin{array}{ccc} x & y & z \end{array} \\ \left(\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \end{array}$$

Since we have 3 unknowns and only 1 non-zero equation therefore there are $3-1=2$ free variables. The expansion of the above gives $x+y+z=0$. Let $z=s$ and $y=t$ then $x=-t-s$. Thus our general eigenvector is given by

$$\mathbf{u}=\begin{pmatrix} x \\ y \\ z \end{pmatrix}=\begin{pmatrix} -s-t \\ t \\ s \end{pmatrix}=s\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}+t\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$$

A basis for the eigenspace E_{-2} is $\left\{\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}\right\}$. [In this case we do **not** need to choose our eigenvectors so that they are orthogonal.]

Let $\mathbf{v}$ be the eigenvector for $\lambda_3=4$:

$$(\mathbf{A}-4\mathbf{I})\mathbf{v}=\begin{pmatrix} -4 & 2 & 2 \\ 2 & -4 & 2 \\ 2 & 2 & -4 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \text{ gives } x=y=z=1$$

A basis for the eigenspace E_4 is $\left\{\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}\right\}$.

4. We have the following answers:

(a) Let $\mathbf{A}$ be an invertible $n\times n$ matrix with eigenvalue λ. Then $\underline{\lambda^{-1}}$ is an eigenvalue of $\mathbf{A}^{-1}$.

(b) An $n\times n$ matrix $\mathbf{A}$ is diagonalizable if and only if $\mathbf{A}$ has n <u>linearly</u> <u>independent</u> <u>eigenvectors</u> .

5. We are given the matrix $\mathbf{A}=\begin{pmatrix} 7 & 5 \\ 3 & -7 \end{pmatrix}$. The eigenvalues λ are given by

$$\begin{aligned} \det(\mathbf{A}-\lambda\mathbf{I}) &= \det\begin{pmatrix} 7-\lambda & 5 \\ 3 & -7-\lambda \end{pmatrix} \\ &= (\lambda-7)(\lambda+7)-15 \\ &= \lambda^2-49-15=\lambda^2-64=(\lambda-8)(\lambda+8)=0 \end{aligned}$$

Our two eigenvalues are $\lambda_1=8$ and $\lambda_2=-8$. Let $\mathbf{u}$ and $\mathbf{v}$ be the corresponding eigenvectors of $\lambda_1=8$ and $\lambda_2=-8$ respectively. We have

$$(\mathbf{A}-8\mathbf{I})\mathbf{u}=\begin{pmatrix} 7-8 & 5 \\ 3 & -7-8 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} -1 & 5 \\ 3 & -15 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \end{pmatrix} \text{ gives } x=5 \text{ and } y=1$$

Similarly we have

$$(\mathbf{A}+8\mathbf{I})\mathbf{v}=\begin{pmatrix}7+8 & 5\\ 3 & -7+8\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix}=\begin{pmatrix}0\\ 0\end{pmatrix}$$

$$\begin{pmatrix}15 & 5\\ 3 & 1\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix}=\begin{pmatrix}0\\ 0\end{pmatrix} \text{ gives } x=1 \text{ and } y=-3$$

Thus $\lambda_1=8,\ \mathbf{u}=\begin{pmatrix}5\\ 1\end{pmatrix}$ and $\lambda_2=-8,\ \mathbf{v}=\begin{pmatrix}1\\ -3\end{pmatrix}$.

(a) The matrix $\mathbf{S}=\begin{pmatrix}\mathbf{u} & \mathbf{v}\end{pmatrix}=\begin{pmatrix}5 & 1\\ 1 & -3\end{pmatrix}$ and the diagonal matrix is $\Lambda=\begin{pmatrix}8 & 0\\ 0 & -8\end{pmatrix}$.

(b) Let **D** be the diagonal matrix for **B**. We know from part (a) that

$$\mathbf{B}^3=\mathbf{A}=\mathbf{S}\Lambda\mathbf{S}^{-1} \text{ where } \Lambda=\begin{pmatrix}8 & 0\\ 0 & -8\end{pmatrix}$$

*What is **B** equal to?*

$$\mathbf{B}=\mathbf{S}\Lambda^{1/3}\mathbf{S}^{-1} \quad (\dagger)$$

$\Lambda^{1/3}=\begin{pmatrix}8 & 0\\ 0 & -8\end{pmatrix}^{1/3}=\begin{pmatrix}2 & 0\\ 0 & -2\end{pmatrix}$. *What else do we need to find?*

$\mathbf{S}^{-1}$. Thus $\mathbf{S}^{-1}=\begin{pmatrix}5 & 1\\ 1 & -3\end{pmatrix}^{-1}=-\dfrac{1}{16}\begin{pmatrix}-3 & -1\\ -1 & 5\end{pmatrix}=\dfrac{1}{16}\begin{pmatrix}3 & 1\\ 1 & -5\end{pmatrix}$.

Substituting $\mathbf{S}=\begin{pmatrix}5 & 1\\ 1 & -3\end{pmatrix}$, $\Lambda^{1/3}=\begin{pmatrix}2 & 0\\ 0 & -2\end{pmatrix}$ and $\mathbf{S}^{-1}=\dfrac{1}{16}\begin{pmatrix}3 & 1\\ 1 & -5\end{pmatrix}$ into (†) gives

$$\begin{aligned}\mathbf{B}=\mathbf{S}\Lambda^{1/3}\mathbf{S}^{-1}&=\begin{pmatrix}5 & 1\\ 1 & -3\end{pmatrix}\begin{pmatrix}2 & 0\\ 0 & -2\end{pmatrix}\frac{1}{16}\begin{pmatrix}3 & 1\\ 1 & -5\end{pmatrix}\\ &=\frac{1}{16}\begin{pmatrix}10 & -2\\ 2 & 6\end{pmatrix}\begin{pmatrix}3 & 1\\ 1 & -5\end{pmatrix}\\ &=\frac{1}{16}\begin{pmatrix}28 & 20\\ 12 & -28\end{pmatrix}=\frac{1}{4}\begin{pmatrix}7 & 5\\ 3 & -7\end{pmatrix}\end{aligned}$$

Note that the given matrix $\mathbf{A}=\begin{pmatrix}7 & 5\\ 3 & -7\end{pmatrix}$ so $\mathbf{B}=\dfrac{1}{4}\begin{pmatrix}7 & 5\\ 3 & -7\end{pmatrix}=\dfrac{1}{4}\mathbf{A}$.

6. (a) (i) The eigenvalues and the corresponding eigenvectors of the given matrix $\mathbf{A}=\begin{pmatrix}4 & -3\\ 1 & 0\end{pmatrix}$ are

$$\lambda_1=3,\ \mathbf{u}=\begin{pmatrix}3\\ 1\end{pmatrix} \text{ and } \lambda_2=1,\ \mathbf{v}=\begin{pmatrix}1\\ 1\end{pmatrix}$$

(ii) The matrix **P** is given by $\mathbf{P}=\begin{pmatrix}\mathbf{u} & \mathbf{v}\end{pmatrix}=\begin{pmatrix}3 & 1\\ 1 & 1\end{pmatrix}$.

(iii) The eigenvalues of $\mathbf{A}^{2008}$ are 3^{2008} and $1^{2008}=1$. The determinant of $\mathbf{A}^{2008}$ is $3^{2008}\times 1=3^{2008}$.

(b) We need to determine whether $\mathbf{B}=\begin{pmatrix}1 & 1 & 0\\ 0 & 1 & 0\\ 0 & 0 & 0\end{pmatrix}$ is diagonalizable. Since we have an upper triangular matrix therefore the eigenvalues are the entries along the leading diagonal, that is $\lambda_{1,\ 2}=1$ and $\lambda_3=0$. By theorem (7-11) we have **B** is diagonalizable if and only if it has 3 linearly independent eigenvectors.
Let **u** be the eigenvector belonging to $\lambda_{1,\ 2}=1$:

$$(\mathbf{B}-\mathbf{I})\mathbf{u}=\begin{pmatrix}1-1 & 1 & 0\\ 0 & 1-1 & 0\\ 0 & 0 & 0\end{pmatrix}\begin{pmatrix}x\\ y\\ z\end{pmatrix}=\begin{pmatrix}0\\ 0\\ 0\end{pmatrix}$$

$$\begin{pmatrix}0 & 1 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{pmatrix}\begin{pmatrix}x\\ y\\ z\end{pmatrix}=\begin{pmatrix}0\\ 0\\ 0\end{pmatrix}$$

Since there is only 1 non-zero equation and 3 unknowns therefore there are $3-1=2$ free variables. Hence we have 2 linearly independent eigenvectors corresponding to $\lambda_{1,\ 2}=1$. Since $\lambda_3=0$ is a distinct eigenvalue from $\lambda_{1,\ 2}=1$ therefore it has a linearly independent eigenvector. We have 3 linearly independent eigenvectors for a 3 by 3 matrix so the matrix is diagonalizable. (Alternatively you can find the 3 linearly independent eigenvectors.)

7. (a) Since $\mathbf{A}=\begin{bmatrix}1 & 0\\ 10 & 2\end{bmatrix}$ is a triangular matrix therefore it's eigenvalues are given by the entries on the leading diagonal, that is $\lambda_1=1$ and $\lambda_2=2$. We have two distinct eigenvalues for a 2 by 2 matrix so the matrix **A** is diagonalizable.
Let **u** be the eigenvector belonging to $\lambda_1=1$:

$$(\mathbf{A}-\mathbf{I})\mathbf{u}=\begin{bmatrix}0 & 0\\ 10 & 1\end{bmatrix}\begin{bmatrix}x\\ y\end{bmatrix}=\begin{bmatrix}0\\ 0\end{bmatrix} \quad \text{gives} \quad \mathbf{u}=\begin{bmatrix}x\\ y\end{bmatrix}=\begin{bmatrix}-1\\ 10\end{bmatrix}$$

Let **v** be the eigenvector belonging to $\lambda_2=2$:

$$(\mathbf{A}-2\mathbf{I})\mathbf{v}=\begin{bmatrix}-1 & 0\\ 10 & 0\end{bmatrix}\begin{bmatrix}x\\ y\end{bmatrix}=\begin{bmatrix}0\\ 0\end{bmatrix} \quad \text{gives} \quad \mathbf{v}=\begin{bmatrix}x\\ y\end{bmatrix}=\begin{bmatrix}0\\ 1\end{bmatrix}$$

The diagonalizing matrix **S** is given by $\mathbf{S}=(\mathbf{u}\quad \mathbf{v})=\begin{pmatrix}-1 & 0\\ 10 & 1\end{pmatrix}$.

(b) The eigenvalues of the given matrix $\mathbf{A}=\begin{bmatrix}2 & 0\\ 0 & 3\end{bmatrix}$ are $\lambda_1=2$ and $\lambda_2=3$.

Let **u** be the eigenvector belonging to $\lambda_1=2$:

$$(\mathbf{A}-2\mathbf{I})\mathbf{u}=\begin{bmatrix}0 & 0\\ 0 & 1\end{bmatrix}\begin{bmatrix}x\\ y\end{bmatrix}=\begin{bmatrix}0\\ 0\end{bmatrix} \quad \text{gives} \quad \mathbf{u}=\begin{bmatrix}x\\ y\end{bmatrix}=\begin{bmatrix}1\\ 0\end{bmatrix}$$

Let **v** be the eigenvector belonging to $\lambda_2=3$:

$$(\mathbf{A}-3\mathbf{I})\mathbf{v}=\begin{bmatrix}-1 & 0\\ 0 & 0\end{bmatrix}\begin{bmatrix}x\\ y\end{bmatrix}=\begin{bmatrix}0\\ 0\end{bmatrix} \quad \text{gives} \quad \mathbf{v}=\begin{bmatrix}x\\ y\end{bmatrix}=\begin{bmatrix}0\\ 1\end{bmatrix}$$

The diagonalizing matrix **S** is given by $\mathbf{S}=\begin{pmatrix}\mathbf{v} & \mathbf{u}\end{pmatrix}=\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}$.

Check that this **S** is indeed the diagonalizing matrix:

$$\mathbf{S}^{-1}\mathbf{A}\mathbf{S}=\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix}2 & 0\\ 0 & 3\end{pmatrix}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix} \qquad \left[\mathbf{S}^{-1}=\mathbf{S}\right]$$

$$=\begin{pmatrix}0 & 3\\ 2 & 0\end{pmatrix}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}=\begin{pmatrix}3 & 0\\ 0 & 2\end{pmatrix}=\mathbf{B}$$

8. Note that the given matrix **B** is a symmetric matrix. The eigenvalues are given by

$$\det(\mathbf{B}-\lambda\mathbf{I})=\det\begin{pmatrix}1-\lambda & 1\\ 1 & -1-\lambda\end{pmatrix}$$

$$=(1-\lambda)(-1-\lambda)-1$$

$$=(\lambda-1)(\lambda+1)-1=\lambda^2-2=0$$

The eigenvalues are $\lambda_1=\sqrt{2}$ and $\lambda_2=-\sqrt{2}$. Since the eigenvalues are distinct therefore the matrix **B** is diagonalizable and the matrix **P** is given by $\mathbf{P}=\begin{pmatrix}\mathbf{u} & \mathbf{v}\end{pmatrix}$ where **u** and **v** are the eigenvectors belonging to $\lambda_1=\sqrt{2}$ and $\lambda_2=-\sqrt{2}$.

We have

$$\left(\mathbf{B}-\sqrt{2}\mathbf{I}\right)\mathbf{u}=\begin{pmatrix}1-\sqrt{2} & 1\\ 1 & -1-\sqrt{2}\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix}=\begin{pmatrix}0\\ 0\end{pmatrix} \quad \text{gives} \quad x=1+\sqrt{2},\ y=1$$

$$\left(\mathbf{B}+\sqrt{2}\mathbf{I}\right)\mathbf{v}=\begin{pmatrix}1+\sqrt{2} & 1\\ 1 & -1+\sqrt{2}\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix}=\begin{pmatrix}0\\ 0\end{pmatrix} \quad \text{gives} \quad x=1-\sqrt{2},\ y=1$$

Hence the eigenvalues and eigenvectors are given by

$$\lambda_1=\sqrt{2},\ \mathbf{u}=\begin{pmatrix}1+\sqrt{2}\\ 1\end{pmatrix} \text{ and } \lambda_2=-\sqrt{2},\ \mathbf{v}=\begin{pmatrix}1-\sqrt{2}\\ 1\end{pmatrix}$$

The matrix $\mathbf{P}=\begin{pmatrix}\mathbf{u} & \mathbf{v}\end{pmatrix}=\begin{pmatrix}1+\sqrt{2} & 1-\sqrt{2}\\ 1 & 1\end{pmatrix}$ and $\mathbf{D}=\begin{pmatrix}\sqrt{2} & 0\\ 0 & -\sqrt{2}\end{pmatrix}$.

9. (a) For the given matrix $\mathbf{A}=\begin{pmatrix}4 & 2 & -2\\ 0 & 3 & 1\\ 0 & 1 & 3\end{pmatrix}$ we are told that $\lambda_1=2$ and $\lambda_2=4$ are two eigenvalues of the matrix **A**. *How do we find the third eigenvalue?*
By Proposition (7-6) (b) we know that the trace (addition of all the entries on the leading diagonal) of a matrix $tr(\mathbf{A})=\lambda_1+\lambda_2+\lambda_3$. *What is the trace of* ***A****?*

$$tr(\mathbf{A})=4+3+3=10$$

Substituting $\lambda_1=2$, $\lambda_2=4$ and $tr(\mathbf{A})=10$ into $tr(\mathbf{A})=\lambda_1+\lambda_2+\lambda_3$ gives

$$2+4+\lambda_3=10 \quad \Rightarrow \quad \lambda_3=4$$

Let **w** be the eigenvector belonging to $\lambda_3 = 4$:

$$\left(\mathbf{A}-4\mathbf{I}\right)\mathbf{w} = \begin{pmatrix} 0 & 2 & -2 \\ 0 & -1 & 1 \\ 0 & 1 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \text{ gives } \mathbf{w} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$$

Note that **w** is linearly independent of the given eigenvector for $\lambda_2 = 4$. Hence the last pair is $\left(4, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}\right)$.

(b) The matrix **P** has entries along it's columns given by the eigenvectors of part (a).

$$\mathbf{P} = \begin{pmatrix} 2 & 1 & 0 \\ -1 & 0 & 1 \\ 1 & 0 & 1 \end{pmatrix} \text{ and } \mathbf{D} = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{pmatrix}$$

10. We need to prove that if λ_1 and λ_2 are *distinct* eigenvalues of a symmetric matrix **A**, then the corresponding eigenspaces are orthogonal.

Proof.

Let λ_1 and λ_2 be *distinct* eigenvalues with the corresponding eigenvectors **u** and **v** of the symmetric matrix **A**. By definition of eigenvalues and eigenvectors we have

$$\mathbf{Au} = \lambda_1\mathbf{u} \text{ and } \mathbf{Av} = \lambda_2\mathbf{v} \qquad (*)$$

Taking the transpose of $\mathbf{Au} = \lambda_1\mathbf{u}$ gives

$$\left(\mathbf{Au}\right)^T = \left(\lambda_1\mathbf{u}\right)^T$$

$$\mathbf{u}^T\mathbf{A}^T = \lambda_1\mathbf{u}^T \qquad \left[\text{Because } \left(\mathbf{XY}\right)^T = \mathbf{Y}^T\mathbf{X}^T \text{ and } \left(k\,\mathbf{X}\right)^T = k\mathbf{X}^T\right]$$

Multiple both sides of the last line by the eigenvector **v**:

$$\mathbf{u}^T\mathbf{A}^T\mathbf{v} = \lambda_1\mathbf{u}^T\mathbf{v} \qquad (\dagger)$$

Since the given matrix **A** is symmetric therefore $\mathbf{A}^T = \mathbf{A}$. Substituting this $\mathbf{A}^T = \mathbf{A}$ into (†) gives

$$\mathbf{u}^T\mathbf{Av} = \lambda_1\mathbf{u}^T\mathbf{v}$$

$$\mathbf{u}^T\lambda_2\mathbf{v} = \lambda_1\mathbf{u}^T\mathbf{v} \qquad \left[\text{By } (*)\right]$$

$$\lambda_2\mathbf{u}^T\mathbf{v} - \lambda_1\mathbf{u}^T\mathbf{v} = 0$$

$$\left(\lambda_2 - \lambda_1\right)\mathbf{u}^T\mathbf{v} = 0 \text{ gives } \mathbf{u}^T\mathbf{v} = 0$$

This last statement follows because we are given that λ_2 and λ_1 are **distinct**. Hence

$$\mathbf{u}^T\mathbf{v} = 0 \quad \Rightarrow \quad \mathbf{u}\cdot\mathbf{v} = 0$$

We conclude that **u** and **v** are orthogonal.

■

11. (a) The eigenvalues λ of $\mathbf{A} = \begin{bmatrix} 6 & 2 \\ 2 & 3 \end{bmatrix}$ are given by

$$\det(\mathbf{A}-\lambda\mathbf{I}) = \det\begin{bmatrix} 6-\lambda & 2 \\ 2 & 3-\lambda \end{bmatrix}$$
$$= (6-\lambda)(3-\lambda)-4$$
$$= (\lambda-6)(\lambda-3)-4$$
$$= \lambda^2-9\lambda+14 = (\lambda-2)(\lambda-7) = 0$$

The eigenvalues are $\lambda_1 = 2$ and $\lambda_2 = 7$. Let **u** be the eigenvector belonging to $\lambda_1 = 2$.

$$(\mathbf{A}-2\mathbf{I})\mathbf{u} = \begin{bmatrix} 6-2 & 2 \\ 2 & 3-2 \end{bmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{bmatrix} 4 & 2 \\ 2 & 1 \end{bmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \text{ gives } x = 1 \text{ and } y = -2$$

Let **v** be the eigenvector belonging to $\lambda_2 = 7$:

$$(\mathbf{A}-7\mathbf{I})\mathbf{v} = \begin{bmatrix} 6-7 & 2 \\ 2 & 3-7 \end{bmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{bmatrix} -1 & 2 \\ 2 & -4 \end{bmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \text{ gives } x = 2 \text{ and } y = 1$$

We have the eigenvalues and eigenvectors given by

$$\lambda_1 = 2,\ \mathbf{u} = \begin{pmatrix} 1 \\ -2 \end{pmatrix} \text{ and } \lambda_2 = 7,\ \mathbf{v} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

Note that since the given matrix **A** is symmetric therefore the eigenvectors **u** and **v** are orthogonal.

(b) Let $\mathbf{A} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix}$ then

$$\mathbf{A}\vec{v} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ 2 \\ -2 \end{bmatrix} = 2\begin{bmatrix} 1 \\ 1 \\ 1 \\ -1 \end{bmatrix} = 2\vec{v}$$

$$\mathbf{A}\vec{w} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ -1 \\ -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ -1 \\ 5 \end{bmatrix} \neq \lambda\begin{bmatrix} 1 \\ -1 \\ -1 \\ 2 \end{bmatrix}$$

Hence $\vec{v}$ is an eigenvector belonging to an eigenvalue of 2 but $\vec{w}$ is **not** an eigenvector of **A**.

12. (a) The eigenvalues and eigenvectors for the matrix $\mathbf{A} = \begin{bmatrix} 1 & 1 & -1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$ can be evaluated by the usual procedure outlined above. We obtain the following:

$$\lambda_1 = 1,\ \mathbf{u} = \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix},\ \ \lambda_2 = 2,\ \mathbf{v} = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \text{ and } \lambda_3 = 0,\ \mathbf{w} = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$$

(b) The given matrix **A** is diagonalizable because the eigenvectors **u**, **v** and **w** are linearly independent. It is easier to spot that we have distinct eigenvalues $\lambda_1 = 1,\ \lambda_2 = 2$ and $\lambda_3 = 0$ therefore the corresponding eigenvectors are linearly independent.

13. (a) (i) We have

$$\mathbf{Au} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} = 0\begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix}$$

$$\mathbf{Av} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} = 0\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$$

$$\mathbf{Aw} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ 3 \\ 3 \end{pmatrix} = 3\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

Hence the following are the eigenvalues and corresponding eigenvectors of **A**:

$$\lambda_1 = 0,\ \mathbf{u} = \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix},\ \lambda_2 = 0,\ \mathbf{v} = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} \text{ and } \lambda_3 = 3,\ \mathbf{w} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

(ii) Since **u**, **v** and **w** are orthogonal therefore the orthogonal matrix **Q** has its column entries given by the normalizing these eigenvectors. Normalizing a non-zero vector **x** is equal to $\dfrac{\mathbf{x}}{\|\mathbf{x}\|}$:

$$\mathbf{u} = \frac{1}{\sqrt{6}}\begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix},\ \mathbf{v} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} \text{ and } \mathbf{w} = \frac{1}{\sqrt{3}}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

Thus $\mathbf{Q} = \begin{pmatrix} \mathbf{u} & \mathbf{v} & \mathbf{w} \end{pmatrix} = \begin{pmatrix} 1/\sqrt{6} & 1/\sqrt{2} & 1/\sqrt{3} \\ 1/\sqrt{6} & -1/\sqrt{2} & 1/\sqrt{3} \\ -2/\sqrt{6} & 0 & 1/\sqrt{3} \end{pmatrix}$.

(iii) Since the eigenvalues are $\lambda_1 = 0,\ \lambda_2 = 0$ and $\lambda_3 = 3$ therefore the characteristic equation is given by

$$\begin{aligned} p_A(\lambda) &= (\lambda - 0)(\lambda - 0)(\lambda - 3) \\ &= \lambda^2(\lambda - 3) = \lambda^3 - 3\lambda^2 = 0 \end{aligned}$$

Cayley Hamilton Theorem states that the matrix **A** satisfies this characteristic equation. We have

$$p_A(\mathbf{A}) = \mathbf{A}^3 - 3\mathbf{A}^2 = \mathbf{O} \quad \text{which gives } \mathbf{A}^3 = 3\mathbf{A}^2$$

Substituting the entries for **A** into this $\mathbf{A}^3 = 3\mathbf{A}^2$ yields

$$\mathbf{A}^3 = 3\mathbf{A}^2 = 3\begin{pmatrix}1&1&1\\1&1&1\\1&1&1\end{pmatrix}\begin{pmatrix}1&1&1\\1&1&1\\1&1&1\end{pmatrix}$$
$$= 3\begin{pmatrix}3&3&3\\3&3&3\\3&3&3\end{pmatrix} = \begin{pmatrix}9&9&9\\9&9&9\\9&9&9\end{pmatrix}$$

(b) Need to prove that **B** and $\mathbf{B}^T$ have the same eigenvalues.
Proof.
The characteristic equation $p_{B^T}(\lambda)$ of the matrix $\mathbf{B}^T$ is given by

$$\begin{aligned} p_{B^T}(\lambda) &= \det\left(\mathbf{B}^T - \lambda\mathbf{I}\right) \\ &= \det\left(\mathbf{B}^T - (\lambda\mathbf{I})^T\right) && \left[\text{Because } (\lambda\mathbf{I})^T = \lambda\mathbf{I}\right] \\ &= \det\left(\mathbf{B} - (\lambda\mathbf{I})\right)^T && \left[\text{Because } \mathbf{X}^T \pm \mathbf{Y}^T = (\mathbf{X} \pm \mathbf{Y})^T\right] \\ &= \det\left(\mathbf{B} - \lambda\mathbf{I}\right) && \left[\text{Because } \det(\mathbf{X})^T = \det(\mathbf{X})\right] \\ &= p_B(\lambda) \end{aligned}$$

Hence $\mathbf{B}^T$ and **B** is given by the same characteristic equation $p_{B^T}(\lambda) = p_B(\lambda)$ therefore they have the same eigenvalues.

■

14. (a) An eigenvalue λ of a square matrix **A** is a scalar such that

$$\mathbf{Au} = \lambda\mathbf{u}$$

where $\mathbf{u} \neq \mathbf{O}$ is a column vector called the eigenvector belonging to eigenvalue λ.

(b) Need to show that $\mathbf{u} = \begin{pmatrix}1\\1\\1\end{pmatrix}$ is an eigenvector of $\mathbf{A} = \begin{pmatrix}0&2&1\\1&1&1\\1&2&0\end{pmatrix}$:

$$\mathbf{Au} = \begin{pmatrix}0&2&1\\1&1&1\\1&2&0\end{pmatrix}\begin{pmatrix}1\\1\\1\end{pmatrix} = \begin{pmatrix}3\\3\\3\end{pmatrix} = 3\begin{pmatrix}1\\1\\1\end{pmatrix}$$

Hence the corresponding eigenvalue is $\lambda_1 = 3$.

(c) To show that **A** is diagonalizable we need to find the other eigenvalues and eigenvectors. We can use the trace and determinant of **A** to find these other two eigenvalues λ_2 and λ_3:

$$3 + \lambda_2 + \lambda_3 = 1 \quad \text{[Because trace of } \mathbf{A} \text{ is 1]}$$
$$3\lambda_2\lambda_3 = 3 \quad \text{[Because determinant of } \mathbf{A} \text{ is 3]}$$

Solving these equations gives $\lambda_2 = \lambda_3 = -1$. Hence we have $\lambda_1 = 3$ and $\lambda_{2,\,3} = -1$. We are given the eigenvector belonging to $\lambda_1 = 3$ in part (b). Let **v** be the eigenvector belonging to $\lambda_{2,\,3} = -1$:

$$\left(\mathbf{A}+\mathbf{I}\right)\mathbf{v}=\begin{pmatrix}1&2&1\\1&2&1\\1&2&1\end{pmatrix}\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix}$$

Placing this into row echelon form gives

$$\left(\begin{array}{ccc|c}1&2&1&0\\0&0&0&0\\0&0&0&0\end{array}\right)$$

We have 1 non-zero equation and 3 unknowns therefore there are $3-1=2$ free variables. Let $z=s$ and $y=t$ then $x=-2t-s$. Thus the general eigenvector **v** belonging to $\lambda_{2,\,3}=-1$ is given by

$$\mathbf{v}=\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}-2t-s\\t\\s\end{pmatrix}=s\begin{pmatrix}-1\\0\\1\end{pmatrix}+t\begin{pmatrix}-2\\1\\0\end{pmatrix}$$

A basis for the eigenspace E_{-1} is $\left\{\begin{pmatrix}-1\\0\\1\end{pmatrix},\begin{pmatrix}-2\\1\\0\end{pmatrix}\right\}$. Since we have 3 linearly independent vectors $\left\{\begin{pmatrix}1\\1\\1\end{pmatrix},\begin{pmatrix}-1\\0\\1\end{pmatrix},\begin{pmatrix}-2\\1\\0\end{pmatrix}\right\}$ for a 3 by 3 matrix therefore the matrix **A** is diagonalizable.

The non-singular matrix $\mathbf{P}=\begin{pmatrix}1&-1&-2\\1&0&1\\1&1&0\end{pmatrix}$. To find the inverse of this matrix we apply row operations but have a look at early chapters to find the inverse. It is given by

$$\mathbf{P}^{-1}=\frac{1}{4}\begin{pmatrix}1&2&1\\-1&-2&3\\-1&2&-1\end{pmatrix}$$

Checking $\mathbf{P}^{-1}\mathbf{AP}$ is a diagonal. (You are **not** asked to check but if time allows carry out the following check.)

$$\mathbf{P}^{-1}\mathbf{AP}=\frac{1}{4}\begin{pmatrix}1&2&1\\-1&-2&3\\-1&2&-1\end{pmatrix}\begin{pmatrix}0&2&1\\1&1&1\\1&2&0\end{pmatrix}\begin{pmatrix}1&-1&-2\\1&0&1\\1&1&0\end{pmatrix}$$

$$=\frac{1}{4}\begin{pmatrix}3&6&3\\1&2&-3\\1&-2&1\end{pmatrix}\begin{pmatrix}1&-1&-2\\1&0&1\\1&1&0\end{pmatrix}$$

$$=\frac{1}{4}\begin{pmatrix}12&0&0\\0&-4&0\\0&0&-4\end{pmatrix}=\begin{pmatrix}3&0&0\\0&-1&0\\0&0&-1\end{pmatrix}$$

15. (a) For the given matrix $\mathbf{A}=\begin{pmatrix}1&-2&2\\8&11&-8\\4&4&-1\end{pmatrix}$ the characteristic equation $p(\lambda)$ is

$$p(\lambda)=(\lambda-5)(\lambda-3)^2=0$$

The eigenvalues and eigenvectors are given by

$\lambda_1=5$ has eigenvector $\mathbf{u}=\begin{pmatrix}-1\\4\\2\end{pmatrix}$ and $\lambda_{2,\,3}=3$ has eigenvectors $\mathbf{v}_1=\begin{pmatrix}1\\0\\1\end{pmatrix}$, $\mathbf{v}_2=\begin{pmatrix}-1\\1\\0\end{pmatrix}$

The eigenvector **u** is a basis for the eigenspace E_5 and $\{\mathbf{v}_1,\ \mathbf{v}_2\}$ is a basis for E_3.

The matrix $\mathbf{P}=\begin{pmatrix}\mathbf{u} & \mathbf{v}_1 & \mathbf{v}_2\end{pmatrix}=\begin{pmatrix}-1&1&-1\\4&0&1\\2&1&0\end{pmatrix}$ and from the theory of early chapters

$\mathbf{P}^{-1}=\begin{pmatrix}1&1&-1\\-2&-2&3\\-4&-3&4\end{pmatrix}$.

The diagonal matrix $\mathbf{P}^{-1}\mathbf{AP}=\mathbf{D}$ has the entries on the leading diagonal of the eigenvalues:

$$\mathbf{D}=\begin{pmatrix}5&0&0\\0&3&0\\0&0&3\end{pmatrix}$$

(b) Cayley Hamilton theorem states that the given matrix **A** satisfies its characteristic equation. This means that $p(\mathbf{A})=(\mathbf{A}-5\mathbf{I})(\mathbf{A}-3\mathbf{I})^2=\mathbf{O}$. Verifying this result gives:

$$(\mathbf{A}-5\mathbf{I})(\mathbf{A}-3\mathbf{I})^2=\begin{pmatrix}1-5&-2&2\\8&11-5&-8\\4&4&-1-5\end{pmatrix}\begin{pmatrix}1-3&-2&2\\8&11-3&-8\\4&4&-1-3\end{pmatrix}^2$$

$$=\begin{pmatrix}-4&-2&2\\8&6&-8\\4&4&-6\end{pmatrix}\begin{pmatrix}-2&-2&2\\8&8&-8\\4&4&-4\end{pmatrix}\begin{pmatrix}-2&-2&2\\8&8&-8\\4&4&-4\end{pmatrix}$$

$$=\begin{pmatrix}-4&-2&2\\8&6&-8\\4&4&-6\end{pmatrix}\begin{pmatrix}-4&-4&4\\16&16&-16\\8&8&-8\end{pmatrix}$$

$$=\begin{pmatrix}0&0&0\\0&0&0\\0&0&0\end{pmatrix}=\mathbf{O}$$

Hence Cayley Hamilton theorem is satisfied.

16. (a) We need to find the characteristic equation $P_A(t)=\det(\mathbf{A}-t\mathbf{I})$:

$$\det(\mathbf{A}-t\mathbf{I}) = \det\begin{pmatrix} -t & -3 & 0 & 0 \\ 1 & -1-t & 0 & 0 \\ 1 & -1 & -t & -9 \\ 1 & -1 & 1 & 10-t \end{pmatrix}$$

$$= -t\det\begin{pmatrix} -1-t & 0 & 0 \\ -1 & -t & -9 \\ -1 & 1 & 10-t \end{pmatrix} + 3\det\begin{pmatrix} 1 & 0 & 0 \\ 1 & -t & -9 \\ 1 & 1 & 10-t \end{pmatrix}$$

$$= -t(-1-t)\left[-t(10-t)+9\right] + 3\left[-t(10-t)+9\right]$$

$$= \left(t^2+t\right)\left[t^2-10t+9\right] + 3\left[t^2-10t+9\right]$$

$$= \left(t^2+t+3\right)\left[t^2-10t+9\right]$$

Hence we have our result $P_A(t) = \left(t^2+t+3\right)\left(t^2-10t+9\right)$.

(b) The eigenvalues are given by $P_A(t) = \left(t^2+t+3\right)\left(t^2-10t+9\right) = 0$. We need to solve this equation. *How?*

The first quadratic $t^2+t+3=0$ with $a=1,\ b=1$ and $c=3$ gives the roots

$$t = \frac{-b \pm \sqrt{b^2-4ac}}{2a} = \frac{-1\pm\sqrt{1^2-12}}{2} = \frac{-1\pm\sqrt{-11}}{2}$$

In this case t is **not** real because we have $\sqrt{-11}$. (You might have come across the square root of negative numbers in your other modules. A number of this type is called a complex number.)

Solving the other quadratic gives

$$t^2-10t+9 = (t-1)(t-9) = 0$$

Real eigenvalues of **A** are $t_1 = 1$ and $t_2 = 9$.

Let $\mathbf{u}$ be the eigenvector belonging to $t_1 = 1$:

$$(\mathbf{A}-\mathbf{I})\mathbf{u} = \begin{pmatrix} -1 & -3 & 0 & 0 \\ 1 & -1-1 & 0 & 0 \\ 1 & -1 & -1 & -9 \\ 1 & -1 & 1 & 10-1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} -1 & -3 & 0 & 0 \\ 1 & -2 & 0 & 0 \\ 1 & -1 & -1 & -9 \\ 1 & -1 & 1 & 9 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \text{ gives } \mathbf{u} = \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 9 \\ -1 \end{pmatrix}$$

A basis for E_1 is $\begin{pmatrix} 0 \\ 0 \\ 9 \\ -1 \end{pmatrix}$.

17. You need to be very careful with this question. The eigenvalues λ of the given matrix are:

$$\det(\mathbf{A}-\lambda\mathbf{I})=\det\begin{pmatrix}0-\lambda & 1 & 2 & 0 & 0\\ -1 & -1-\lambda & 1 & 0 & 1\\ 0 & 0 & 1-\lambda & 0 & 0\\ 0 & 0 & 2 & -3-\lambda & 0\\ 1 & 2 & 3 & 4 & 0-\lambda\end{pmatrix}$$

$$=(1-\lambda)\det\begin{pmatrix}-\lambda & 1 & 0 & 0\\ -1 & -1-\lambda & 0 & 1\\ 0 & 0 & -3-\lambda & 0\\ 1 & 2 & 4 & -\lambda\end{pmatrix}\qquad\begin{bmatrix}\text{Expanding along}\\ \text{the middle row}\end{bmatrix}$$

$$=(1-\lambda)(-3-\lambda)\det\begin{pmatrix}-\lambda & 1 & 0\\ -1 & -1-\lambda & 1\\ 1 & 2 & -\lambda\end{pmatrix}\qquad\begin{bmatrix}\text{Expanding along}\\ \text{the third row}\end{bmatrix}$$

$$=(1-\lambda)(-3-\lambda)\Big[-\big[(-2\lambda)-1\big]-\lambda\big(\lambda[1+\lambda]+1\big)\Big]\qquad\begin{bmatrix}\text{Expanding along}\\ \text{the last column}\end{bmatrix}$$

$$=(1-\lambda)(-3-\lambda)\Big[2\lambda+1-\lambda\big(\lambda^2+\lambda+1\big)\Big]$$

Taking out the negative signs and simplifying we have

$$\begin{aligned}\det(\mathbf{A}-\lambda\mathbf{I})&=(1-\lambda)(-3-\lambda)\Big[2\lambda+1-\lambda\big(\lambda^2+\lambda+1\big)\Big]\\ &=(1-\lambda)(3+\lambda)\big[\lambda^3+\lambda^2-\lambda-1\big]\\ &=-(\lambda-1)(3+\lambda)\Big[(\lambda-1)(\lambda+1)^2\Big]\\ &=-(\lambda-1)^2(\lambda+1)^2(3+\lambda)=0\end{aligned}$$

The eigenvalues are $\lambda_{1,\ 2}=1$, $\lambda_{3,\ 4}=-1$ and $\lambda_5=-3$.

18. (a) The characteristic equation of the given matrix is:

$$\det(\mathbf{A}-\lambda\mathbf{I})=\det\begin{pmatrix}1-\lambda & 1 & 0\\ 1 & 2-\lambda & 1\\ 0 & 1 & 1-\lambda\end{pmatrix}$$
$$=\lambda(\lambda-1)(\lambda-3)=0$$

The eigenvalues are $\lambda_1=0$, $\lambda_2=1$ and $\lambda_3=3$.

(b) Let **u**, **v** and **w** be the eigenvectors of $\lambda_1=0$, $\lambda_2=1$ and $\lambda_3=3$ respectively. We have

$$\mathbf{Au}=\begin{pmatrix}1&1&0\\1&2&1\\0&1&1\end{pmatrix}\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix}\text{ gives }\mathbf{u}=\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}-1\\1\\-1\end{pmatrix}$$

$$(\mathbf{A}-\mathbf{I})\mathbf{v}=\begin{pmatrix}0&1&0\\1&1&1\\0&1&0\end{pmatrix}\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix}\text{ gives }\mathbf{v}=\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}1\\0\\-1\end{pmatrix}$$

$$(\mathbf{A}-3\mathbf{I})\mathbf{w}=\begin{pmatrix}-2 & 1 & 0\\ 1 & -1 & 1\\ 0 & 1 & -2\end{pmatrix}\begin{pmatrix}x\\ y\\ z\end{pmatrix}=\begin{pmatrix}0\\ 0\\ 0\end{pmatrix} \text{ gives } \mathbf{w}=\begin{pmatrix}x\\ y\\ z\end{pmatrix}=\begin{pmatrix}1\\ 2\\ 1\end{pmatrix}$$

(c) The basis $\beta=\left\{\begin{pmatrix}1\\ 2\\ 1\end{pmatrix},\begin{pmatrix}1\\ 0\\ -1\end{pmatrix},\begin{pmatrix}-1\\ 1\\ -1\end{pmatrix}\right\}$. The diagonal matrix $\mathbf{D}=\begin{pmatrix}3 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 0\end{pmatrix}$

(d) $\mathbf{S}=\begin{pmatrix}\mathbf{w} & \mathbf{v} & \mathbf{u}\end{pmatrix}=\begin{pmatrix}1 & 1 & -1\\ 2 & 0 & 1\\ 1 & -1 & -1\end{pmatrix}$.

(e) The inverse of the above matrix **S** can be found using row operations. We have

$$\mathbf{S}^{-1}=\frac{1}{6}\begin{pmatrix}1 & 2 & 1\\ 3 & 0 & -3\\ -2 & 2 & -2\end{pmatrix}$$

(f) The vectors in $\beta=\left\{\begin{pmatrix}1\\ 2\\ 1\end{pmatrix},\begin{pmatrix}1\\ 0\\ -1\end{pmatrix},\begin{pmatrix}-1\\ 1\\ -1\end{pmatrix}\right\}$ are orthogonal so we need to normalize them.

Normalizing a non-zero vector **v** is equal to $\dfrac{\mathbf{v}}{\|\mathbf{v}\|}$. Call this new basis β':

$$\beta'=\left\{\frac{1}{\sqrt{6}}\begin{pmatrix}1\\ 2\\ 1\end{pmatrix},\ \frac{1}{\sqrt{2}}\begin{pmatrix}1\\ 0\\ -1\end{pmatrix},\ \frac{1}{\sqrt{3}}\begin{pmatrix}-1\\ 1\\ -1\end{pmatrix}\right\}$$

(g) Let the new **S** be called **S**'. We have $\mathbf{S}'=\begin{pmatrix}1/\sqrt{6} & 1/\sqrt{2} & -1/\sqrt{3}\\ 2/\sqrt{6} & 0 & 1/\sqrt{3}\\ 1/\sqrt{6} & -1/\sqrt{2} & -1/\sqrt{3}\end{pmatrix}$.

(h) Since **S**' is an orthogonal matrix therefore

$$(\mathbf{S}')^{-1}=(\mathbf{S}')^{T}=\begin{pmatrix}1/\sqrt{6} & 1/\sqrt{2} & -1/\sqrt{3}\\ 2/\sqrt{6} & 0 & 1/\sqrt{3}\\ 1/\sqrt{6} & -1/\sqrt{2} & -1/\sqrt{3}\end{pmatrix}^{T}$$

$$=\begin{pmatrix}1/\sqrt{6} & 2/\sqrt{6} & 1/\sqrt{6}\\ 1/\sqrt{2} & 0 & -1/\sqrt{2}\\ -1/\sqrt{3} & 1/\sqrt{3} & -1/\sqrt{3}\end{pmatrix}$$

19. (a) *What are the eigenvalues λ of the given matrix **A** equal to?*

Since $\mathbf{A}=\begin{pmatrix}a & b\\ 0 & c\end{pmatrix}$ is a triangular matrix therefore the eigenvalues are the entries on the leading diagonal. We have $\lambda_1=a$ and $\lambda_2=c$.

Let **u** be the eigenvector belonging to $\lambda_1 = a$:

$$(\mathbf{A}-a\mathbf{I})\mathbf{u} = \begin{pmatrix} a-a & b \\ 0 & c-a \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$
$$= \begin{pmatrix} 0 & b \\ 0 & c-a \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \text{ gives } \mathbf{u} = \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

Let **v** be the eigenvector belonging to $\lambda_2 = c$:

$$(\mathbf{A}-c\mathbf{I})\mathbf{v} = \begin{pmatrix} a-c & b \\ 0 & c-c \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$
$$= \begin{pmatrix} a-c & b \\ 0 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \text{ gives } \mathbf{v} = \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} b \\ c-a \end{pmatrix}$$

The eigenvector matrix $\mathbf{S} = \begin{pmatrix} 1 & b \\ 0 & c-a \end{pmatrix}$ and the inverse of this matrix is

$$\mathbf{S}^{-1} = \frac{1}{c-a}\begin{pmatrix} c-a & -b \\ 0 & 1 \end{pmatrix}$$

The eigenvalue matrix $\Lambda = \begin{pmatrix} a & 0 \\ 0 & c \end{pmatrix}$.

(b) From part (a) we have $\mathbf{A} = \mathbf{S}\Lambda\mathbf{S}^{-1}$. By applying (7-14) $\mathbf{A}^m = \mathbf{S}\Lambda^m\mathbf{S}^{-1}$ with $m = 1000$ we have

$$\mathbf{A}^{1000} = \mathbf{S}\Lambda^{1000}\mathbf{S}^{-1}$$
$$= \begin{pmatrix} 1 & b \\ 0 & c-a \end{pmatrix}\begin{pmatrix} a & 0 \\ 0 & c \end{pmatrix}^{1000} \frac{1}{c-a}\begin{pmatrix} c-a & -b \\ 0 & 1 \end{pmatrix}$$
$$= \frac{1}{c-a}\begin{pmatrix} 1 & b \\ 0 & c-a \end{pmatrix}\begin{pmatrix} a^{1000} & 0 \\ 0 & c^{1000} \end{pmatrix}\begin{pmatrix} c-a & -b \\ 0 & 1 \end{pmatrix}$$
$$= \frac{1}{c-a}\begin{pmatrix} a^{1000} & bc^{1000} \\ 0 & (c-a)c^{1000} \end{pmatrix}\begin{pmatrix} c-a & -b \\ 0 & 1 \end{pmatrix}$$
$$= \frac{1}{c-a}\begin{pmatrix} (c-a)a^{1000} & -a^{1000}b + bc^{1000} \\ 0 & (c-a)c^{1000} \end{pmatrix}$$
$$= \begin{pmatrix} a^{1000} & b\left(c^{1000} - a^{1000}\right)/(c-a) \\ 0 & c^{1000} \end{pmatrix} \qquad \left[\text{Taking in } \frac{1}{c-a}\right]$$

By using the algebraic identity

$$x^n - y^n = (x-y)\left(x^{n-1} + x^{n-2}y + \cdots + xy^{n-2} + y^{n-1}\right)$$

we can write

$$c^{1000} - a^{1000} = (c-a)\left(c^{999} + c^{998}a + c^{997}a^2 + \cdots + ca^{998} + a^{999}\right)$$

The entry

$$\frac{b\left(c^{1000} - a^{1000}\right)}{c-a} = \frac{b(c-a)\left(c^{999} + c^{998}a + c^{997}a^2 + \cdots + ca^{998} + a^{999}\right)}{c-a}$$
$$= b\left(c^{999} + c^{998}a + c^{997}a^2 + \cdots + ca^{998} + a^{999}\right) \qquad [\text{Cancelling } c-a]$$

Hence

$$\mathbf{A}^{1000}=\begin{pmatrix} a^{1000} & b\left(c^{999}+c^{998}a+c^{997}a^2+\cdots+ca^{998}+a^{999}\right) \\ 0 & c^{1000} \end{pmatrix}$$

20. (a) A square matrix **A** is diagonalizable if there exists an invertible (non-singular) matrix **P** such that $\mathbf{P}^{-1}\mathbf{AP}=\mathbf{D}$ where **D** is a diagonal matrix.
(b) An eigenvalue λ of a square matrix **A** is a scalar such that

$$\mathbf{Au}=\lambda\mathbf{u}$$

where $\mathbf{u}\neq\mathbf{O}$ is a column vector called the eigenvector belonging to eigenvalue λ.
Eigenvalues and eigenvectors come in pairs like chalk and cheese because one is vector and the other is a scalar.
(c) If an eigenvalue λ_0 occurs *m* times where $m>1$ then we say λ_0 is an eigenvalue with **multiplicity** of *m* or λ_0 has **multiplicity** *m*. This is called algebraic multiplicity.
We do **not cover** geometric multiplicity of an eigenvalue in this book.
However geometric multiplicity of an eigenvalue λ_0 of a square matrix **A** is the dimension of the subspace of eigenvector **u** belonging to λ_0.
(d) As noted in part (c) geometric multiplicity of an eigenvalue λ is **not** covered in the book.
We are given that a 3×3 matrix **A** has characteristic polynomial $(\lambda-1)(\lambda-2)^2$.
The algebraic multiplicity of $\lambda_1=1$ is one and $\lambda_{2,\,3}=2$ is two.
A 3×3 matrix **A** is diagonalizable if and only if it has 3 linearly independent eigenvectors. Matrix **A** is diagonalizable if and only if $\lambda_1=1$ has geometric multiplicity of 1 and $\lambda_{2,\,3}=2$ has geometric multiplicity of 2.

(e) (i) We are given the matrix $\mathbf{A}=\begin{bmatrix}1&-2&1\\1&1&2\\1&0&3\end{bmatrix}$ and it's characteristic polynomial $(\lambda-1)(\lambda-2)^2$. The eigenvalues are $\lambda_1=1$ and $\lambda_{2,\,3}=2$.
Let **u** be the eigenvector belonging to $\lambda_1=1$. We have

$$(\mathbf{A}-\mathbf{I})\mathbf{u}=\begin{bmatrix}1-1&-2&1\\1&1-1&2\\1&0&3-1\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$$

$$\begin{bmatrix}0&-2&1\\1&0&2\\1&0&2\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix} \text{ gives } x=-4,\; y=1 \text{ and } z=2$$

Let **v** be the eigenvector belonging to $\lambda_{2,\,3}=2$:

$$\left(\mathbf{A}-2\mathbf{I}\right)\mathbf{v}=\begin{bmatrix}1-2 & -2 & 1\\ 1 & 1-2 & 2\\ 1 & 0 & 3-2\end{bmatrix}\begin{bmatrix}x\\ y\\ z\end{bmatrix}=\begin{bmatrix}0\\ 0\\ 0\end{bmatrix}$$

$$\begin{bmatrix}-1 & -2 & 1\\ 1 & -1 & 2\\ 1 & 0 & 1\end{bmatrix}\begin{bmatrix}x\\ y\\ z\end{bmatrix}=\begin{bmatrix}0\\ 0\\ 0\end{bmatrix} \text{ gives } x=-1,\ y=1 \text{ and } z=1$$

We have $\lambda_1=1$, $\mathbf{u}=\begin{pmatrix}-4\\ 1\\ 2\end{pmatrix}$ and $\lambda_{2,\,3}=2$, $\mathbf{v}=\begin{pmatrix}-1\\ 1\\ 1\end{pmatrix}$. The eigenvectors of $\lambda_{2,\,3}=2$ are all multiples of **v** which means they are linearly dependent on **v** so we only have 2 linearly independent eigenvectors **u** and **v** of the given 3×3 matrix. Hence **A** is **not** diagonalizable.

(ii) We are given the matrix $\mathbf{B}=\begin{bmatrix}1 & -1 & 1\\ -1 & 1 & 1\\ -1 & -1 & 3\end{bmatrix}$ and it's characteristic polynomial

$(\lambda-1)(\lambda-2)^2$ which gives the eigenvalues $\lambda_1=1$ and $\lambda_{2,\,3}=2$. Let **u** be the eigenvector belonging to $\lambda_1=1$. We have

$$\left(\mathbf{B}-\mathbf{I}\right)\mathbf{u}=\begin{bmatrix}1-1 & -1 & 1\\ -1 & 1-1 & 1\\ -1 & -1 & 3-1\end{bmatrix}\begin{bmatrix}x\\ y\\ z\end{bmatrix}=\begin{bmatrix}0\\ 0\\ 0\end{bmatrix}$$

$$\begin{bmatrix}0 & -1 & 1\\ -1 & 0 & 1\\ -1 & -1 & 2\end{bmatrix}\begin{bmatrix}x\\ y\\ z\end{bmatrix}=\begin{bmatrix}0\\ 0\\ 0\end{bmatrix} \text{ gives } \mathbf{u}=\begin{bmatrix}x\\ y\\ z\end{bmatrix}=\begin{bmatrix}1\\ 1\\ 1\end{bmatrix}$$

Let **v** be the eigenvector belonging to $\lambda_{2,\,3}=2$:

$$\left(\mathbf{B}-2\mathbf{I}\right)\mathbf{v}=\begin{bmatrix}1-2 & -1 & 1\\ -1 & 1-2 & 1\\ -1 & -1 & 3-2\end{bmatrix}\begin{bmatrix}x\\ y\\ z\end{bmatrix}=\begin{bmatrix}0\\ 0\\ 0\end{bmatrix}$$

$$\begin{bmatrix}-1 & -1 & 1\\ -1 & -1 & 1\\ -1 & -1 & 1\end{bmatrix}\begin{bmatrix}x\\ y\\ z\end{bmatrix}=\begin{bmatrix}0\\ 0\\ 0\end{bmatrix}$$

Putting the last matrix in row echelon form gives

$$\begin{array}{c} \begin{array}{ccc} x & y & z \end{array} \\ \left[\begin{array}{ccc|c}-1 & -1 & 1 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\end{array}\right]\end{array}$$

Since we have 1 non-zero equation with 3 unknowns therefore there are $3-1=2$ free variables. Let $z=s$ and $y=t$ then from the first row we have

$$-x-y+z=0 \quad\Rightarrow\quad x=z-y$$

Substituting $y=t$ and $z=s$ we have $x=s-t$ and the general eigenvector is given by

$$\mathbf{v}=\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}s-t\\t\\s\end{pmatrix}=s\begin{pmatrix}1\\0\\1\end{pmatrix}+t\begin{pmatrix}-1\\1\\0\end{pmatrix}$$

Hence the matrix **B** is diagonalizable because it has 3 linearly independent eigenvectors given by $\left\{\begin{pmatrix}1\\1\\1\end{pmatrix},\ \begin{pmatrix}1\\0\\1\end{pmatrix},\ \begin{pmatrix}-1\\1\\0\end{pmatrix}\right\}$. The diagonalizing matrix **P** is given by $\mathbf{P}=\begin{pmatrix}1&1&-1\\1&0&1\\1&1&0\end{pmatrix}$.

21. (a) The vectors $\mathbf{w}_1,\ \mathbf{w}_2,\ \mathbf{w}_3$ in $\mathbb{R}^3$ are said to be orthonormal if and only if

(i) $\mathbf{w}_1,\ \mathbf{w}_2,\ \mathbf{w}_3$ are orthogonal, that is $\mathbf{w}_1\cdot\mathbf{w}_2=\mathbf{w}_2\cdot\mathbf{w}_3=\mathbf{w}_2\cdot\mathbf{w}_3=0$.

(ii) $\mathbf{w}_1,\ \mathbf{w}_2,\ \mathbf{w}_3$ are unit vectors, that is $\|\mathbf{w}_1\|=\|\mathbf{w}_2\|=\|\mathbf{w}_3\|=1$.

(b) By the Gram Schmidt process we have $\mathbf{w}_1=\mathbf{v}_1=\begin{bmatrix}0\\1\\1\end{bmatrix}$ and

$$\mathbf{w}_2=\mathbf{v}_2-\frac{\mathbf{v}_2\cdot\mathbf{w}_1}{\|\mathbf{w}_1\|^2}\mathbf{w}_1 \qquad (*)$$

We have $\mathbf{v}_2\cdot\mathbf{w}_1=\begin{bmatrix}1\\1\\0\end{bmatrix}\cdot\begin{bmatrix}0\\1\\1\end{bmatrix}=0+1+0=1$, $\|\mathbf{w}_1\|^2=\begin{bmatrix}0\\1\\1\end{bmatrix}\cdot\begin{bmatrix}0\\1\\1\end{bmatrix}=0^2+1^2+1^2=2$.

Substituting these $\mathbf{v}_2\cdot\mathbf{w}_1=1$, $\|\mathbf{w}_1\|^2=2$ into the above (*) gives

$$\mathbf{w}_2=\begin{bmatrix}1\\1\\0\end{bmatrix}-\frac{1}{2}\begin{bmatrix}0\\1\\1\end{bmatrix}=\begin{bmatrix}1\\1-1/2\\0-1/2\end{bmatrix}=\begin{bmatrix}1\\1/2\\-1/2\end{bmatrix}=\frac{1}{2}\begin{bmatrix}2\\1\\-1\end{bmatrix}$$

Remember to simplify our arithmetic we can ignore the fraction ½:

Let $\mathbf{w}_2'=\begin{bmatrix}2\\1\\-1\end{bmatrix}$. *How do we find* $\mathbf{w}_3$*?*

$$\mathbf{w}_3=\mathbf{v}_3-\frac{\mathbf{v}_3\cdot\mathbf{w}_1}{\|\mathbf{w}_1\|^2}\mathbf{w}_1-\frac{\mathbf{v}_3\cdot\mathbf{w}_2'}{\|\mathbf{w}_2'\|^2}\mathbf{w}_2'$$

We have $\mathbf{v}_3\cdot\mathbf{w}_1=\begin{bmatrix}5\\4\\6\end{bmatrix}\cdot\begin{bmatrix}0\\1\\1\end{bmatrix}=0+(4\times1)+(6\times1)=10$ and

$$\mathbf{v}_3\cdot\mathbf{w}_2'=\begin{bmatrix}5\\4\\6\end{bmatrix}\cdot\begin{bmatrix}2\\1\\-1\end{bmatrix}=(5\times2)+(4\times1)+\left[6\times(-1)\right]=8$$

$$\|\mathbf{w}_2'\|^2 = \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix} = 2^2 + 1^2 + (-1)^2 = 6$$

Substituting these $\mathbf{v}_3 = \begin{bmatrix} 5 \\ 4 \\ 6 \end{bmatrix}$, $\mathbf{v}_3 \cdot \mathbf{w}_1 = 10$, $\mathbf{v}_3 \cdot \mathbf{w}_2' = 8$, $\|\mathbf{w}_1\|^2 = 2$ and $\|\mathbf{w}_2'\|^2 = 6$ into the above $\mathbf{w}_3 = \mathbf{v}_3 - \dfrac{\mathbf{v}_3 \cdot \mathbf{w}_1}{\|\mathbf{w}_1\|^2}\mathbf{w}_1 - \dfrac{\mathbf{v}_3 \cdot \mathbf{w}_2'}{\|\mathbf{w}_2'\|^2}\mathbf{w}_2'$ gives

$$\mathbf{w}_3 = \begin{bmatrix} 5 \\ 4 \\ 6 \end{bmatrix} - \frac{10}{2}\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} - \frac{8}{6}\begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}$$

$$= \begin{bmatrix} 5 \\ 4 \\ 6 \end{bmatrix} - 5\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} - \frac{4}{3}\begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 5-0-8/3 \\ 4-5-4/3 \\ 6-5+4/3 \end{bmatrix} = \begin{bmatrix} 7/3 \\ -7/3 \\ 7/3 \end{bmatrix} = \frac{7}{3}\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

Again ignore the fraction, we have $\mathbf{w}_3' = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$. Our orthogonal vectors are

$$\mathbf{w}_1 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \ \mathbf{w}_2' = \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix} \text{ and } \mathbf{w}_3' = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

Since we want an orthonormal set so we need to normalise these orthogonal vectors:

$$\mathbf{w}_1 = \frac{1}{\sqrt{2}}\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \ \mathbf{w}_2' = \frac{1}{\sqrt{6}}\begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix} \text{ and } \mathbf{w}_3' = \frac{1}{\sqrt{3}}\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

(c) (i) The characteristic equation of the given symmetric matrix $\mathbf{A} = \begin{pmatrix} 3 & 2 & 4 \\ 2 & 0 & 2 \\ 4 & 2 & 3 \end{pmatrix}$ is

$$\lambda^3 - 6\lambda^2 - 15\lambda - 8 = 0$$

where λ is an eigenvalue. Factorising this gives

$$\lambda^3 - 6\lambda^2 - 15\lambda - 8 = (\lambda + 1)^2(\lambda - 8) = 0$$

The eigenvalues are $\lambda_{1,\,2} = -1$ and $\lambda_3 = 8$.

(ii) Let **u** be the eigenvector corresponding to $\lambda_{1,\,2} = -1$:

$$(\mathbf{A}+\mathbf{I})\mathbf{u}=\begin{pmatrix}3+1 & 2 & 4\\ 2 & 0+1 & 2\\ 4 & 2 & 3+1\end{pmatrix}\begin{pmatrix}x\\ y\\ z\end{pmatrix}=\begin{pmatrix}0\\ 0\\ 0\end{pmatrix}$$

$$\begin{pmatrix}4 & 2 & 4\\ 2 & 1 & 2\\ 4 & 2 & 4\end{pmatrix}\begin{pmatrix}x\\ y\\ z\end{pmatrix}=\begin{pmatrix}0\\ 0\\ 0\end{pmatrix}$$

Putting this into row echelon form gives

$$\begin{array}{c} \begin{array}{ccc} x & y & z \end{array} \\ \left(\begin{array}{ccc|c}2 & 1 & 2 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\end{array}\right) \end{array} \qquad (2x+y+2z=0)$$

We have 1 non-zero equation and 3 unknowns therefore there are $3-1=2$ free variables. This means that we have 2 linearly independent eigenvectors for $\lambda_{1,\,2}=-1$. We select our vectors $\mathbf{u}_1$ and $\mathbf{u}_2$ so that they are **orthogonal**:

$$\mathbf{u}_1=\begin{pmatrix}1\\ -4\\ 1\end{pmatrix} \text{ and } \mathbf{u}_2=\begin{pmatrix}1\\ 0\\ -1\end{pmatrix}$$

A basis for the eigenspace E_{-1} is $\left\{\begin{pmatrix}1\\ -4\\ 1\end{pmatrix}, \begin{pmatrix}1\\ 0\\ -1\end{pmatrix}\right\}$.

Let $\mathbf{v}$ be the eigenvector for $\lambda_3=8$. We have

$$(\mathbf{A}-8\mathbf{I})\mathbf{v}=\begin{pmatrix}3-8 & 2 & 4\\ 2 & 0-8 & 2\\ 4 & 2 & 3-8\end{pmatrix}\begin{pmatrix}x\\ y\\ z\end{pmatrix}=\begin{pmatrix}0\\ 0\\ 0\end{pmatrix}$$

$$\begin{pmatrix}-5 & 2 & 4\\ 2 & -8 & 2\\ 4 & 2 & -5\end{pmatrix}\begin{pmatrix}x\\ y\\ z\end{pmatrix}=\begin{pmatrix}0\\ 0\\ 0\end{pmatrix} \text{ gives } x=z=2 \text{ and } y=1$$

The eigenvector corresponding to $\lambda_3=8$ is $\mathbf{v}=\begin{pmatrix}2\\ 1\\ 2\end{pmatrix}$. The set of linearly independent vectors are $\left\{\begin{pmatrix}1\\ -4\\ 1\end{pmatrix}, \begin{pmatrix}1\\ 0\\ -1\end{pmatrix}, \begin{pmatrix}2\\ 1\\ 2\end{pmatrix}\right\}$. [We have actually found an orthogonal set of eigenvectors.]

(d) We need to prove that if $\mathbf{v}_1,\ \mathbf{v}_2,\ \mathbf{v}_3 \in \mathbb{R}^3$ is an orthogonal set then they are linearly independent.

We assume that none of the vectors $\mathbf{v}_1,\ \mathbf{v}_2,\ \mathbf{v}_3 \in \mathbb{R}^3$ are the zero because if any one of them is then the set is linearly dependent.

Let $k_1,\ k_2$ and k_3 be scalars such that

$$k_1\mathbf{v}_1 + k_2\mathbf{v}_2 + k_3\mathbf{v}_3 = \mathbf{O} \qquad (*)$$

How do we show linear independence?

Required to prove that $k_1 = k_2 = k_3 = 0$. Consider the inner product

$$\begin{aligned} 0 &= \langle k_1\mathbf{v}_1,\ \mathbf{O}\rangle \\ &= \langle k_1\mathbf{v}_1,\ k_1\mathbf{v}_1 + k_2\mathbf{v}_2 + k_3\mathbf{v}_3\rangle \qquad [\text{By } (*)] \\ &= \langle k_1\mathbf{v}_1,\ k_1\mathbf{v}_1\rangle + \langle k_1\mathbf{v}_1,\ k_2\mathbf{v}_2\rangle + \langle k_1\mathbf{v}_1,\ k_3\mathbf{v}_3\rangle \\ &= k_1^2\langle \mathbf{v}_1,\ \mathbf{v}_1\rangle + k_1k_2 \underbrace{\langle \mathbf{v}_1,\ \mathbf{v}_2\rangle}_{=0 \text{ because orthogonality}} + k_1k_3 \underbrace{\langle \mathbf{v}_1,\ \mathbf{v}_3\rangle}_{=0 \text{ because orthogonality}} \\ &= k_1^2\|\mathbf{v}_1\|^2 = 0 \end{aligned}$$

Since the vector $\mathbf{v}_1$ is in the set of orthonormal vectors therefore it is a unit vector so we have $\|\mathbf{v}_1\| = 1$. Hence from above $k_1^2\|\mathbf{v}_1\|^2 = 0$ we have $k_1 = 0$.

Similarly $k_2 = k_3 = 0$. Hence $\mathbf{v}_1,\ \mathbf{v}_2,\ \mathbf{v}_3 \in \mathbb{R}^3$ are linearly independent.

■

22. (a) A set of vectors $\{\mathbf{v}_1,\ \mathbf{v}_2,\ \cdots,\ \mathbf{v}_n\}$ are orthogonal if and only if

$$\langle \mathbf{v}_i,\ \mathbf{v}_j\rangle = 0 \text{ provided } i \neq j$$

This set $\{\mathbf{v}_1,\ \mathbf{v}_2,\ \cdots,\ \mathbf{v}_n\}$ is orthonormal if it is orthogonal and each of these is a unit vector, that is for each $i = 1,\ 2,\ 3,\ \cdots,\ n$ we have

$$\langle \mathbf{v}_i,\ \mathbf{v}_i\rangle = \|\mathbf{v}_i\|^2 = 1$$

(b) A square matrix $\mathbf{Q}$ is orthogonal if and only if $\mathbf{Q}^T = \mathbf{Q}^{-1}$.

To prove that the set of columns of an orthogonal matrix forms an orthonormal set we have to show that

$$\mathbf{Q}\mathbf{Q}^T = \mathbf{I} = [\mathbf{I}]_{ij} = \begin{cases} 0 & \text{if } i \neq j \\ 1 & \text{if } i = j \end{cases}$$

For the rest of the proof see Chapter 4??

(c) (i) Need to show that $\mathbf{u} = (1,\ 0,\ 1,\ 0)$ and $\mathbf{v} = (1,\ 0,\ -1,\ 0)$ are eigenvectors of the matrix $\mathbf{A}$:

$$\mathbf{Au} = \begin{pmatrix} 5 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \\ -1 & 0 & 5 & 0 \\ 0 & -1 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 4 \\ 0 \\ 4 \\ 0 \end{pmatrix} = 4\begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} = 4\mathbf{u}$$

and

$$\mathbf{Av} = \begin{pmatrix} 5 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \\ -1 & 0 & 5 & 0 \\ 0 & -1 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ -1 \\ 0 \end{pmatrix} = \begin{pmatrix} 6 \\ 0 \\ -6 \\ 0 \end{pmatrix} = 6\begin{pmatrix} 1 \\ 0 \\ -1 \\ 0 \end{pmatrix} = 6\mathbf{v}$$

Hence the eigenvalues are $\lambda_1 = 4$ and $\lambda_2 = 6$.

(ii) *How do we find the other two eigenvalues of* $\boldsymbol{A}$*?*

We can use the trace and determinant of the matrix **A** to find remaining eigenvalues λ_3 and λ_4. We have

$$tr(\mathbf{A}) = 5+1+5+1 = 12$$
$$\det(\mathbf{A}) = 5[5-1(0+5)]-1[-1(1-1)] = 0$$

We need to solve the equations

$$4+6+\lambda_3+\lambda_4 = 12 \qquad [\text{Because } \lambda_1 = 4 \text{ and } \lambda_2 = 6]$$
$$4\times 6\times \lambda_3 \times \lambda_4 = 24\lambda_3\lambda_4 = 0$$

From the bottom equation one of the eigenvalues is zero. Let $\lambda_3 = 0$. Substituting this $\lambda_3 = 0$ into the top equation gives $\lambda_4 = 2$. Hence the other two eigenvalues are $\lambda_3 = 0$ and $\lambda_4 = 2$. Let **w** be the eigenvector belonging to $\lambda_3 = 0$:

$$\mathbf{Aw} = \begin{pmatrix} 5 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \\ -1 & 0 & 5 & 0 \\ 0 & -1 & 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \\ r \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \text{ gives } \mathbf{w} = \begin{pmatrix} x \\ y \\ z \\ r \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix}$$

Let **x** be the eigenvector belonging to $\lambda_4 = 2$:

$$(\mathbf{A}-2\mathbf{I})\mathbf{x} = \begin{pmatrix} 5-2 & 0 & -1 & 0 \\ 0 & 1-2 & 0 & -1 \\ -1 & 0 & 5-2 & 0 \\ 0 & -1 & 0 & 1-2 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \\ r \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 3 & 0 & -1 & 0 \\ 0 & -1 & 0 & -1 \\ -1 & 0 & 3 & 0 \\ 0 & -1 & 0 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \\ r \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \text{ gives } \mathbf{x} = \begin{pmatrix} x \\ y \\ z \\ r \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \\ -1 \end{pmatrix}$$

(iii) Since we have distinct eigenvalues therefore the eigenvectors are linearly independent. Note that the given matrix **A** is symmetrical therefore our invertible (nonsingular) matrix **P** is an orthogonal matrix given by

$$\mathbf{P} = \begin{pmatrix} \mathbf{u} & \mathbf{v} & \mathbf{w} & \mathbf{x} \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{pmatrix}$$

where $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$, $\mathbf{x}$ are normalised eigenvectors. Hence the matrix **Q** is

$$\mathbf{Q} = \mathbf{P}^T = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 0 & 1 & 0 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & -1 \end{pmatrix} \qquad \begin{bmatrix} \text{Because } \mathbf{P} \text{ is an} \\ \text{orthogonal matrix} \\ \text{so } \mathbf{P}^{-1} = \mathbf{P}^T = \mathbf{Q} \end{bmatrix}$$

Our diagonal matrix Λ is given by

$$\Lambda = \begin{pmatrix} 4 & 0 & 0 & 0 \\ 0 & 6 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix}$$

23. We need to convert the given quadratic $3x^2 + 4xy + 6y^2$ into diagonal form. Let
$\mathbf{x} = \begin{pmatrix} x \\ y \end{pmatrix}$

$$3x^2 + 4xy + 6y^2 = \mathbf{x}^T\mathbf{A}\mathbf{x} \text{ where } \mathbf{A} = \begin{pmatrix} 3 & 2 \\ 2 & 6 \end{pmatrix}$$

The characteristic equation of **A** is given by $(\lambda - 2)(\lambda - 7) = 0$. The eigenvalues are

$$\lambda_1 = 2 \text{ and } \lambda_2 = 7$$

Let **u** be the eigenvector belonging to $\lambda_1 = 2$:

$$(\mathbf{A} - 2\mathbf{I})\mathbf{u} = \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \text{ gives } \mathbf{u} = \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$$

Let **v** be the eigenvector belonging to $\lambda_2 = 7$:

$$(\mathbf{A} - 7\mathbf{I})\mathbf{v} = \begin{pmatrix} -4 & 2 \\ 2 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \text{ gives } \mathbf{v} = \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

Normalizing the eigenvectors **u** and **v**:

$$\mathbf{u} = \frac{1}{\sqrt{5}}\begin{pmatrix} 2 \\ -1 \end{pmatrix} \text{ and } \mathbf{v} = \frac{1}{\sqrt{5}}\begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

The orthogonal matrix **Q** is given by

$$\mathbf{Q} = \begin{pmatrix} \mathbf{u} & \mathbf{v} \end{pmatrix} = \frac{1}{\sqrt{5}}\begin{pmatrix} 2 & 1 \\ -1 & 2 \end{pmatrix}$$

Let $\mathbf{y} = \begin{pmatrix} X \\ Y \end{pmatrix}$ be the new axes. By (7-22) we have $\mathbf{x} = \mathbf{Q}\mathbf{y}$ which gives $\mathbf{y} = \mathbf{Q}^{-1}\mathbf{x}$. Since **Q** is orthogonal therefore

$$\mathbf{Q}^{-1} = \mathbf{Q}^T = \frac{1}{\sqrt{5}}\begin{pmatrix} 2 & 1 \\ -1 & 2 \end{pmatrix}^T = \frac{1}{\sqrt{5}}\begin{pmatrix} 2 & -1 \\ 1 & 2 \end{pmatrix}$$

Thus using $\mathbf{y} = \mathbf{Q}^{-1}\mathbf{x}$ we have

$$\begin{pmatrix} X \\ Y \end{pmatrix} = \frac{1}{\sqrt{5}}\begin{pmatrix} 2 & -1 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{\sqrt{5}}\begin{pmatrix} 2x - y \\ x + 2y \end{pmatrix}$$

Hence we have $X = \frac{1}{\sqrt{5}}(2x - y)$ and $Y = \frac{1}{\sqrt{5}}(x + 2y)$. The diagonal form is

$$3x^2 + 4xy + 6y^2 = \lambda_1{}^2 X + \lambda_2{}^2 Y = 2X^2 + 7Y^2 \quad [\text{Because } \lambda_1 = 2 \text{ and } \lambda_2 = 7\,]$$

24. We are given the quadratic $2xy + 4xz + 4yz + 3z^2 = aX^2 + bY^2 + cZ^2$. We need to write this in matrix form:

$$2xy+4xz+4yz+3z^2=\mathbf{x}^T\mathbf{A}\mathbf{x} \quad \text{where } \mathbf{A}=\begin{pmatrix}0&1&2\\1&0&2\\2&2&3\end{pmatrix},\ \mathbf{x}=\begin{pmatrix}x\\y\\z\end{pmatrix}$$

The eigenvalues of the matrix **A** are $\lambda_{1,\,2}=-1$ and $\lambda_3=5$.

Let **u** be the eigenvector belonging to $\lambda_{1,\,2}=-1$:

$$(\mathbf{A}+\mathbf{I})\mathbf{u}=\begin{pmatrix}1&1&2\\1&1&2\\2&2&4\end{pmatrix}\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix}$$

Putting this into reduced row echelon form gives

$$\left(\begin{array}{ccc|c}1&1&2&0\\0&0&0&0\\0&0&0&0\end{array}\right)$$

We have two linearly independent eigenvectors:

$$\mathbf{u}=\begin{pmatrix}1\\1\\-1\end{pmatrix} \text{ and } \mathbf{v}=\begin{pmatrix}-1\\1\\0\end{pmatrix}$$

Choose your eigenvectors so that they are orthogonal, that is $\mathbf{u}\cdot\mathbf{v}=0$.
Let **w** be the other eigenvector belonging to $\lambda_3=5$:

$$(\mathbf{A}-5\mathbf{I})\mathbf{w}=\begin{pmatrix}-5&1&2\\1&-5&2\\2&2&-2\end{pmatrix}\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix} \text{ gives } \mathbf{w}=\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}1\\1\\2\end{pmatrix}$$

Normalizing the eigenvectors **u**, **v** and **w** gives

$$\mathbf{u}=\frac{1}{\sqrt{3}}\begin{pmatrix}1\\1\\-1\end{pmatrix},\quad \mathbf{v}=\frac{1}{\sqrt{2}}\begin{pmatrix}-1\\1\\0\end{pmatrix} \text{ and } \mathbf{w}=\frac{1}{\sqrt{6}}\begin{pmatrix}1\\1\\2\end{pmatrix}$$

Our orthogonal vector is $\mathbf{Q}=\begin{pmatrix}\mathbf{u}&\mathbf{v}&\mathbf{w}\end{pmatrix}=\begin{pmatrix}1/\sqrt{3}&-1/\sqrt{2}&1/\sqrt{6}\\1/\sqrt{3}&1/\sqrt{2}&1/\sqrt{6}\\-1/\sqrt{3}&0&2/\sqrt{6}\end{pmatrix}$.

Let $\mathbf{y}=\begin{pmatrix}X\\Y\\Z\end{pmatrix}$ be the new axes. By (7-22) we have $\mathbf{x}=\mathbf{Q}\mathbf{y}$ which gives $\mathbf{y}=\mathbf{Q}^{-1}\mathbf{x}$. Since **Q** is orthogonal therefore

$$\mathbf{Q}^{-1}=\mathbf{Q}^T=\begin{pmatrix}1/\sqrt{3}&-1/\sqrt{2}&1/\sqrt{6}\\1/\sqrt{3}&1/\sqrt{2}&1/\sqrt{6}\\-1/\sqrt{3}&0&2/\sqrt{6}\end{pmatrix}^T=\begin{pmatrix}1/\sqrt{3}&1/\sqrt{3}&-1/\sqrt{3}\\-1/\sqrt{2}&1/\sqrt{2}&0\\1/\sqrt{6}&1/\sqrt{6}&2/\sqrt{6}\end{pmatrix}$$

Thus using $\mathbf{y}=\mathbf{Q}^{-1}\mathbf{x}$ we have

$$\begin{pmatrix} X \\ Y \\ Z \end{pmatrix} = \begin{pmatrix} 1/\sqrt{3} & 1/\sqrt{3} & -1/\sqrt{3} \\ -1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 1/\sqrt{6} & 1/\sqrt{6} & 2/\sqrt{6} \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{3}}\left(x+y-z\right) \\ \frac{1}{\sqrt{2}}\left(-x+y\right) \\ \frac{1}{\sqrt{6}}\left(x+y+2z\right) \end{pmatrix}$$

Hence we have $X = \frac{1}{\sqrt{3}}\left(x+y-z\right)$, $Y = \frac{1}{\sqrt{2}}\left(y-x\right)$ and $Z = \frac{1}{\sqrt{6}}\left(x+y+2z\right)$.

Since the eigenvalues are $\lambda_{1,\ 2} = -1$ and $\lambda_3 = 5$ therefore the diagonal form is

$$2xy + 4xz + 4yz + 3z^2 = \lambda_1{}^2 X + \lambda_2{}^2 Y + \lambda_3 Z^2 = -X^2 - Y^2 + 5Z^2$$