Complete Solutions to Exercises 2.1

1. (a) We have

$$\mathbf{u}\cdot\mathbf{v}=\begin{pmatrix}-1\\3\end{pmatrix}\cdot\begin{pmatrix}2\\1\end{pmatrix}=(-1\times 2)+(3\times 1)=-2+3=1$$

(b) Since $\mathbf{u}\cdot\mathbf{v}=\mathbf{v}\cdot\mathbf{u}$ therefore by part (a) we have $\mathbf{v}\cdot\mathbf{u}=1$.
(c) The inner product of the vector $\mathbf{u}$ and $\mathbf{u}$ is given by

$$\mathbf{u}\cdot\mathbf{u}=\begin{pmatrix}-1\\3\end{pmatrix}\cdot\begin{pmatrix}-1\\3\end{pmatrix}=(-1)^2+3^2=10$$

(d) Similarly we have

$$\mathbf{v}\cdot\mathbf{v}=\begin{pmatrix}2\\1\end{pmatrix}\cdot\begin{pmatrix}2\\1\end{pmatrix}=2^2+1^2=5$$

(e) *What is* $\|\mathbf{u}\|^2$ *equal to?*

We know $\|\mathbf{u}\|=\sqrt{\mathbf{u}\cdot\mathbf{u}}$ therefore $\|\mathbf{u}\|^2=\mathbf{u}\cdot\mathbf{u}$ and by part (c) we have

$$\|\mathbf{u}\|^2=10$$

(f) Similarly we have $\|\mathbf{v}\|^2=5$.

(g) *What is* $\|\mathbf{u}\|$ *equal to?*

The norm (or length) $\|\mathbf{u}\|$ of the vector $\mathbf{u}$ is the square root of $\|\mathbf{u}\|^2=\mathbf{u}\cdot\mathbf{u}$:

$$\|\mathbf{u}\|=\sqrt{\mathbf{u}\cdot\mathbf{u}}=\sqrt{10}=3.16\ \text{(2dp)}$$

(h) Similarly to (g) we have

$$\|\mathbf{v}\|=\sqrt{\mathbf{v}\cdot\mathbf{v}}=\sqrt{5}=2.24\quad\text{(2dp)}$$

(i) To find $\|\mathbf{u}+\mathbf{v}\|^2$ we first add the vectors $\mathbf{u}$ and $\mathbf{v}$ and then determine the inner product

$$\mathbf{u}+\mathbf{v}=\begin{pmatrix}-1\\3\end{pmatrix}+\begin{pmatrix}2\\1\end{pmatrix}=\begin{pmatrix}-1+2\\3+1\end{pmatrix}=\begin{pmatrix}1\\4\end{pmatrix}$$

We have

$$\|\mathbf{u}+\mathbf{v}\|^2=\begin{pmatrix}1\\4\end{pmatrix}\cdot\begin{pmatrix}1\\4\end{pmatrix}=1^2+4^2=17$$

(j) Remember $d(\mathbf{u},\ \mathbf{v})$ is the distance between the vectors $\mathbf{u}$ and $\mathbf{v}$:

$$d(\mathbf{u},\ \mathbf{v})=\|\mathbf{u}-\mathbf{v}\|=\left\|\begin{pmatrix}-1\\3\end{pmatrix}-\begin{pmatrix}2\\1\end{pmatrix}\right\|$$

$$=\left\|\begin{pmatrix}-3\\2\end{pmatrix}\right\|=\sqrt{(-3)^2+2^2}=3.61\ \text{(2dp)}$$

2. (a) We have

$$\mathbf{u}\cdot\mathbf{v}=\begin{pmatrix}2\\3\\-1\end{pmatrix}\cdot\begin{pmatrix}5\\1\\-2\end{pmatrix}=(2\times 5)+(3\times 1)+((-1)\times(-2))=15$$

(b) Since $\mathbf{u}\cdot\mathbf{v}=\mathbf{v}\cdot\mathbf{u}$ therefore by part (a) we have $\mathbf{v}\cdot\mathbf{u}=15$.
(c) The inner product of the vector $\mathbf{u}$ and $\mathbf{u}$ is given by

$$\mathbf{u}\cdot\mathbf{u}=\begin{pmatrix}2\\3\\-1\end{pmatrix}\cdot\begin{pmatrix}2\\3\\-1\end{pmatrix}=2^2+3^2+(-1)^2=14$$

(d) Similarly we have

$$\mathbf{v}\cdot\mathbf{v}=\begin{pmatrix}5\\1\\-2\end{pmatrix}\cdot\begin{pmatrix}5\\1\\-2\end{pmatrix}=5^2+1^2+(-2)^2=30$$

(e) *What is* $\|\mathbf{u}\|^2$ *equal to?*

We know $\|\mathbf{u}\|=\sqrt{\mathbf{u}\cdot\mathbf{u}}$ therefore $\|\mathbf{u}\|^2=\mathbf{u}\cdot\mathbf{u}$ and by part (c) we have

$$\|\mathbf{u}\|^2=14$$

(f) Similarly by part (d) we have $\|\mathbf{v}\|^2=30$.

(g) *What is* $\|\mathbf{u}\|$ *equal to?*

By part (e) we take the square root of $\|\mathbf{u}\|^2=14$:

$$\|\mathbf{u}\|=\sqrt{\|\mathbf{u}\|^2}=\sqrt{14}=3.74 \text{ (2dp)}$$

(h) Similarly to (g) we have

$$\|\mathbf{v}\|=\sqrt{\|\mathbf{v}\|^2}=\sqrt{30}=5.48 \text{ (2dp)}$$

(i) To find $\|\mathbf{u}+\mathbf{v}\|^2$ we first add the vectors **u** and **v** and then determine the inner (dot) product

$$\mathbf{u}+\mathbf{v}=\begin{pmatrix}2\\3\\-1\end{pmatrix}+\begin{pmatrix}5\\1\\-2\end{pmatrix}=\begin{pmatrix}2+5\\3+1\\-1-2\end{pmatrix}=\begin{pmatrix}7\\4\\-3\end{pmatrix}$$

We have

$$\|\mathbf{u}+\mathbf{v}\|^2=\begin{pmatrix}7\\4\\-3\end{pmatrix}\cdot\begin{pmatrix}7\\4\\-3\end{pmatrix}=7^2+4^2+(-3)^2=74$$

(j) $d(\mathbf{u},\ \mathbf{v})$ is the distance between the vectors **u** and **v**:

$$d(\mathbf{u},\ \mathbf{v})=\|\mathbf{u}-\mathbf{v}\|=\left\|\begin{pmatrix}2\\3\\-1\end{pmatrix}-\begin{pmatrix}5\\1\\-2\end{pmatrix}\right\|$$

$$=\left\|\begin{pmatrix}-3\\2\\1\end{pmatrix}\right\|=\sqrt{(-3)^2+2^2+1^2}=3.74 \text{ (2dp)}$$

3. (a) We have

$$\mathbf{u}\cdot\mathbf{v}=\begin{pmatrix}-1\\2\\5\\-3\end{pmatrix}\cdot\begin{pmatrix}2\\-3\\-1\\5\end{pmatrix}=(-1\times 2)+(2\times(-3))+(5\times(-1))+((-3)\times 5)=-28$$

(b) Since $\mathbf{u}\cdot\mathbf{v}=\mathbf{v}\cdot\mathbf{u}$ therefore by part (a) we have $\mathbf{v}\cdot\mathbf{u}=-28$.

(c) The dot (inner) product of the vector $\mathbf{u}$ and **u** is given by

$$\mathbf{u}\cdot\mathbf{u}=\begin{pmatrix}-1\\2\\5\\-3\end{pmatrix}\cdot\begin{pmatrix}-1\\2\\5\\-3\end{pmatrix}=(-1)^2+2^2+5^2+(-3)^2=39$$

(d) Similarly we have

$$\mathbf{v}\cdot\mathbf{v}=\begin{pmatrix}2\\-3\\-1\\5\end{pmatrix}\cdot\begin{pmatrix}2\\-3\\-1\\5\end{pmatrix}=2^2+(-3)^2+(-1)^2+5^2=39$$

(e) *What is* $\|\mathbf{u}\|^2$ *equal to?*

We know $\|\mathbf{u}\|=\sqrt{\mathbf{u}\cdot\mathbf{u}}$ therefore $\|\mathbf{u}\|^2=\mathbf{u}\cdot\mathbf{u}$ and by part (c) we have

$$\|\mathbf{u}\|^2=39$$

(f) Similarly by part (d) we have $\|\mathbf{v}\|^2=39$.

(g) *What is* $\|\mathbf{u}\|$ *equal to?*

The square root of $\|\mathbf{u}\|^2$:

$$\|\mathbf{u}\|=\sqrt{39}=6.25\ \text{(2dp)}$$

(h) Similarly to (g) we have

$$\|\mathbf{v}\|=\sqrt{39}=6.25\quad\text{(2dp)}$$

(i) To find $\|\mathbf{u}+\mathbf{v}\|^2$ we first add the vectors **u** and **v** and then determine the inner (dot) product

$$\mathbf{u}+\mathbf{v}=\begin{pmatrix}-1\\2\\5\\-3\end{pmatrix}+\begin{pmatrix}2\\-3\\-1\\5\end{pmatrix}=\begin{pmatrix}-1+2\\2-3\\5-1\\-3+5\end{pmatrix}=\begin{pmatrix}1\\-1\\4\\2\end{pmatrix}$$

We have

$$\|\mathbf{u}+\mathbf{v}\|^2=\begin{pmatrix}1\\-1\\4\\2\end{pmatrix}\cdot\begin{pmatrix}1\\-1\\4\\2\end{pmatrix}=1^2+(-1)^2+4^2+2^2=22$$

(j) $d(\mathbf{u},\ \mathbf{v})$ is the distance between the vectors **u** and **v**:

$$d(\mathbf{u},\quad \mathbf{v}) = \|\mathbf{u}-\mathbf{v}\| = \left\| \begin{pmatrix} -1 \\ 2 \\ 5 \\ -3 \end{pmatrix} - \begin{pmatrix} 2 \\ -3 \\ -1 \\ 5 \end{pmatrix} \right\|$$

$$= \left\| \begin{pmatrix} -3 \\ 5 \\ 6 \\ -8 \end{pmatrix} \right\| = \sqrt{(-3)^2 + 5^2 + 6^2 + (-8)^2} = 11.58 \text{ (2dp)}$$

4. We are given that $\mathbf{i} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$, $\mathbf{j} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ and $\mathbf{k} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

(a) Need to show that $\mathbf{i}\cdot\mathbf{i} = 1$:

$$\mathbf{i}\cdot\mathbf{i} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = 1^2 + 0^2 + 0^2 = 1$$

(b) Similarly we need to prove $\mathbf{j}\cdot\mathbf{j} = 1$:

$$\mathbf{j}\cdot\mathbf{j} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = 0^2 + 1^2 + 0^2 = 1$$

(c) Similarly we have

$$\mathbf{k}\cdot\mathbf{k} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = 0^2 + 0^2 + 1^2 = 1$$

(d) This time we need to show $\mathbf{i}\cdot\mathbf{j} = 0$:

$$\mathbf{i}\cdot\mathbf{j} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = (1\times 0) + (0\times 1) + (0\times 0) = 0$$

(e) Similarly we prove that $\mathbf{i}\cdot\mathbf{k} = 0$:

$$\mathbf{i}\cdot\mathbf{k} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = (1\times 0) + (0\times 0) + (0\times 1) = 0$$

(f) Also we can prove that $\mathbf{j}\cdot\mathbf{k} = 0$:

$$\mathbf{j}\cdot\mathbf{k} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = (0\times 0) + (1\times 0) + (0\times 1) = 0$$

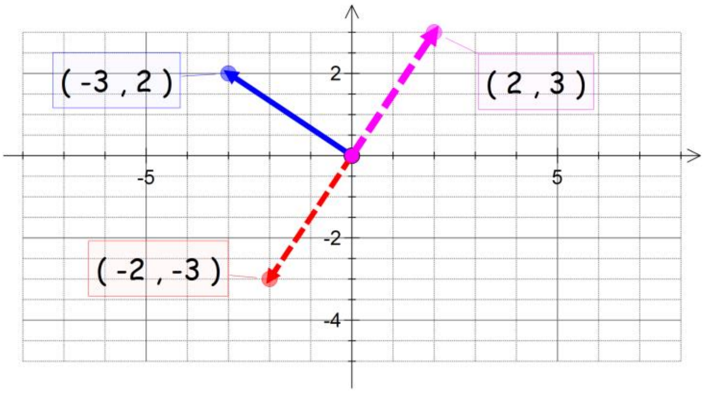
To show the given result we have

$$\mathbf{u} = a\mathbf{i} + b\mathbf{j} + c\mathbf{k} = a\begin{pmatrix}1\\0\\0\end{pmatrix} + b\begin{pmatrix}0\\1\\0\end{pmatrix} + c\begin{pmatrix}0\\0\\1\end{pmatrix} = \begin{pmatrix}a\\b\\c\end{pmatrix}$$

Similarly we have $\mathbf{v} = d\mathbf{i} + e\mathbf{j} + f\mathbf{k} = \begin{pmatrix}d\\e\\f\end{pmatrix}$. The dot product of these is

$$\mathbf{u}\cdot\mathbf{v} = \begin{pmatrix}a\\b\\c\end{pmatrix}\cdot\begin{pmatrix}d\\e\\f\end{pmatrix} = ad + be + cf$$

5. Plotting the given vectors we have



We can show that the vectors are orthogonal by evaluating their inner (dot) product and if this is zero then the vectors are orthogonal.

$$\mathbf{u}\cdot\mathbf{v} = \begin{pmatrix}-3\\2\end{pmatrix}\cdot\begin{pmatrix}-2\\-3\end{pmatrix} = \left(-3\times(-2)\right) + \left(2\times(-3)\right) = 6 - 6 = 0$$

Hence the vectors **u** and **v** are orthogonal.
By the above figure we see that every vector on the dashed line is perpendicular to the vector $\mathbf{u} = \begin{pmatrix}-3\\2\end{pmatrix}$ so we have an infinite number of solutions to the given equation $-3x + 2y = 0$:

$$x = -2,\ y = -3 \text{ or } x = 2,\ y = 3 \text{ or } x = 0,\ y = 0 \ \ldots$$

The general solution is $x = 2s,\ y = 3s$ where s is any real number.

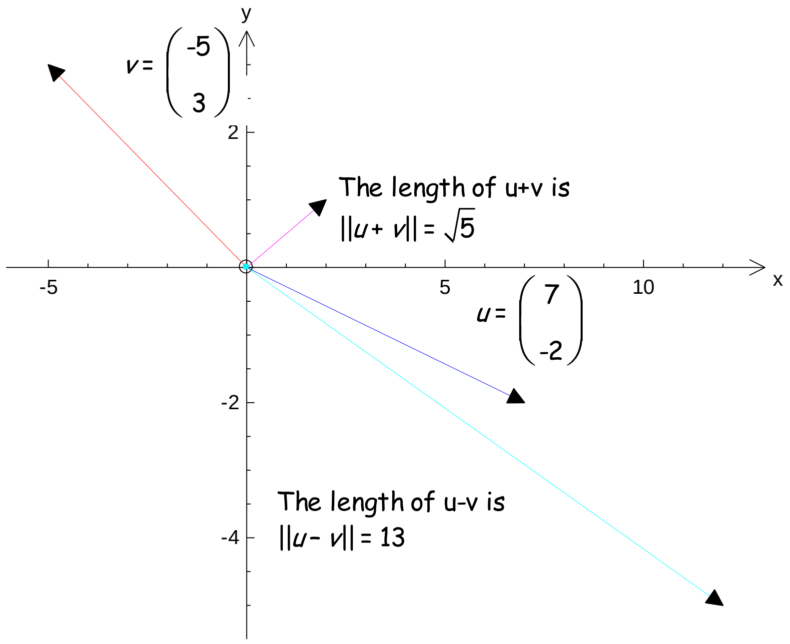
6. (a) We first evaluate $\|\mathbf{u}+\mathbf{v}\|$:

$$\|\mathbf{u}+\mathbf{v}\| = \left\|\begin{pmatrix}7\\-2\end{pmatrix} + \begin{pmatrix}-5\\3\end{pmatrix}\right\| = \left\|\begin{pmatrix}2\\1\end{pmatrix}\right\| = \sqrt{2^2 + 1^2} = \sqrt{5}$$

(b) Similarly we have

$$\|\mathbf{u}-\mathbf{v}\| = \left\|\begin{pmatrix}7\\-2\end{pmatrix} - \begin{pmatrix}-5\\3\end{pmatrix}\right\| = \left\|\begin{pmatrix}12\\-5\end{pmatrix}\right\| = \sqrt{12^2 + (-5)^2} = \sqrt{169} = 13$$

Plotting these in $\mathbb{R}^2$ gives



7. We need to prove the following:
Let **u**, **v** and **w** be vectors in $\mathbb{R}^n$ and k, c be real numbers (or real scalars). We have the following results:
(b) $(\mathbf{u}+\mathbf{v})+\mathbf{w}=\mathbf{u}+(\mathbf{v}+\mathbf{w})$ (Associative Law)
(e) $k(\mathbf{u}+\mathbf{v})=k\mathbf{u}+k\mathbf{v}$ (Distributive Law)
(g) $(kc)\mathbf{u}=k(c\mathbf{u})$
All these results are valid because a vector is a *n* by 1 matrix and we have proven these results for matrices in chapter 1. However we can show these for vectors as below.

Let $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ be vectors in $\mathbb{R}^n$ and $\mathbf{u}=\begin{pmatrix}u_1\\u_2\\\vdots\\u_n\end{pmatrix}$, $\mathbf{v}=\begin{pmatrix}v_1\\v_2\\\vdots\\v_n\end{pmatrix}$ and $\mathbf{w}=\begin{pmatrix}w_1\\w_2\\\vdots\\w_n\end{pmatrix}$.

Proof of (b). By examining the Left Hand Side we have

$$(\mathbf{u}+\mathbf{v})+\mathbf{w}=\left[\begin{pmatrix}u_1\\u_2\\\vdots\\u_n\end{pmatrix}+\begin{pmatrix}v_1\\v_2\\\vdots\\v_n\end{pmatrix}\right]+\begin{pmatrix}w_1\\w_2\\\vdots\\w_n\end{pmatrix}=\begin{pmatrix}u_1+v_1\\u_2+v_2\\\vdots+\vdots\\u_n+v_n\end{pmatrix}+\begin{pmatrix}w_1\\w_2\\\vdots\\w_n\end{pmatrix}$$

$$=\begin{pmatrix}u_1+v_1+w_1\\u_2+v_2+w_2\\\vdots+\vdots+\vdots\\u_n+v_n+w_n\end{pmatrix}=\begin{pmatrix}u_1\\u_2\\\vdots\\u_n\end{pmatrix}+\begin{pmatrix}v_1+w_1\\v_2+w_2\\\vdots+\vdots\\v_n+w_n\end{pmatrix}=\begin{pmatrix}u_1\\u_2\\\vdots\\u_n\end{pmatrix}+\left[\begin{pmatrix}v_1\\v_2\\\vdots\\v_n\end{pmatrix}+\begin{pmatrix}w_1\\w_2\\\vdots\\w_n\end{pmatrix}\right]$$

$$=\mathbf{u}+(\mathbf{v}+\mathbf{w})$$

Proof of (e). We need to show $k(\mathbf{u}+\mathbf{v})=k\mathbf{u}+k\mathbf{v}$:

$$k(\mathbf{u}+\mathbf{v}) = k\left(\begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{pmatrix} + \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix}\right) = k\begin{pmatrix} u_1+v_1 \\ u_2+v_2 \\ \vdots + \vdots \\ u_n+v_n \end{pmatrix}$$

$$\underset{\text{Scalar Multiplication}}{=} \begin{pmatrix} k(u_1+v_1) \\ k(u_2+v_2) \\ \vdots + \vdots \\ k(u_n+v_n) \end{pmatrix}$$

$$= \begin{pmatrix} ku_1+kv_1 \\ ku_2+kv_2 \\ \vdots + \vdots \\ ku_n+kv_n \end{pmatrix} = \begin{pmatrix} ku_1 \\ ku_2 \\ \vdots \\ ku_n \end{pmatrix} + \begin{pmatrix} kv_1 \\ kv_2 \\ \vdots \\ kv_n \end{pmatrix} = k\begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{pmatrix} + k\begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix} = k\mathbf{u}+k\mathbf{v}$$

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Proof of (g). Need to prove $(kc)\mathbf{u} = k(c\mathbf{u})$:

$$(kc)\mathbf{u} = kc\begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{pmatrix} = \begin{pmatrix} kcu_1 \\ kcu_2 \\ \vdots \\ kcu_n \end{pmatrix} = k\begin{pmatrix} cu_1 \\ cu_2 \\ \vdots \\ cu_n \end{pmatrix} = k\left(c\begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{pmatrix}\right) = k(c\mathbf{u})$$

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8. *Proof.* We need to prove $(\mathbf{u}+\mathbf{v})\cdot\mathbf{w} = \mathbf{u}\cdot\mathbf{w}+\mathbf{v}\cdot\mathbf{w}$ where $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ are vectors in $\mathbb{R}^n$. We have

$$(\mathbf{u}+\mathbf{v})\cdot\mathbf{w} = \left[\begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{pmatrix} + \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix}\right]\cdot\begin{pmatrix} w_1 \\ w_2 \\ \vdots \\ w_n \end{pmatrix}$$

$$= \begin{pmatrix} u_1+v_1 \\ u_2+v_2 \\ \vdots + \vdots \\ u_n+v_n \end{pmatrix}\cdot\begin{pmatrix} w_1 \\ w_2 \\ \vdots \\ w_n \end{pmatrix}$$

$$= (u_1+v_1)w_1 + (u_2+v_2)w_2 + \cdots + (u_n+v_n)w_n$$

$$= u_1w_1+v_1w_1+u_2w_2+v_2w_2+\cdots+u_nw_n+v_nw_n \qquad [\text{Expanding}]$$

$$= \underbrace{u_1w_1+u_2w_2+\cdots+u_nw_n}_{=\mathbf{u}\cdot\mathbf{w}} + \underbrace{v_1w_1+v_2w_2+\cdots+v_nw_n}_{=\mathbf{v}\cdot\mathbf{w}} \qquad [\text{Rearranging}]$$

$$= \mathbf{u}\cdot\mathbf{w}+\mathbf{v}\cdot\mathbf{w}$$

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9. (a) We need to find $\|\mathbf{u}\|$:

$$\|\mathbf{u}\| = \sqrt{\begin{pmatrix} 2 \\ -7 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ -7 \end{pmatrix}} = \sqrt{2^2 + (-7)^2} = \sqrt{53}$$

For $\frac{1}{\|\mathbf{u}\|}\mathbf{u}$ we have $\frac{1}{\|\mathbf{u}\|}\mathbf{u} = \frac{1}{\sqrt{53}}\begin{pmatrix} 2 \\ -7 \end{pmatrix}$.

(b) In this case we have

$$\|\mathbf{u}\| = \sqrt{\begin{pmatrix} -9 \\ 3 \\ 7 \end{pmatrix} \cdot \begin{pmatrix} -9 \\ 3 \\ 7 \end{pmatrix}} = \sqrt{(-9)^2 + 3^2 + 7^2} = \sqrt{139}$$

Hence $\frac{1}{\|\mathbf{u}\|}\mathbf{u} = \frac{1}{\sqrt{139}}\begin{pmatrix} -9 \\ 3 \\ 7 \end{pmatrix}$

(c) Similarly we have

$$\|\mathbf{u}\| = \sqrt{\begin{pmatrix} -3 \\ 5 \\ 8 \\ 6 \end{pmatrix} \cdot \begin{pmatrix} -3 \\ 5 \\ 8 \\ 6 \end{pmatrix}} = \sqrt{(-3)^2 + 5^2 + 8^2 + 6^2} = \sqrt{134}$$

Hence $\frac{1}{\|\mathbf{u}\|}\mathbf{u} = \frac{1}{\sqrt{134}}\begin{pmatrix} -3 \\ 5 \\ 8 \\ 6 \end{pmatrix}$

(d) Lastly we have

$$\|\mathbf{u}\| = \sqrt{\begin{pmatrix} -6 \\ 2 \\ 8 \\ 3 \\ 5 \end{pmatrix} \cdot \begin{pmatrix} -6 \\ 2 \\ 8 \\ 3 \\ 5 \end{pmatrix}} = \sqrt{(-6)^2 + 2^2 + 8^2 + 3^2 + 5^2} = \sqrt{138}$$

Hence $\frac{1}{\|\mathbf{u}\|}\mathbf{u} = \frac{1}{\sqrt{138}}\begin{pmatrix} -6 \\ 2 \\ 8 \\ 3 \\ 5 \end{pmatrix}$.

We need to find $\left\|\frac{1}{\|\mathbf{u}\|}\mathbf{u}\right\|$ in each case. Generally it is easier to find $\left\|\frac{1}{\|\mathbf{u}\|}\mathbf{u}\right\|^2$ and then take the square root of your result to give $\left\|\frac{1}{\|\mathbf{u}\|}\mathbf{u}\right\|$.

(a) We have

$$\left\|\frac{1}{\|\mathbf{u}\|}\mathbf{u}\right\|^2 = \frac{1}{\sqrt{53}}\begin{pmatrix}2\\-7\end{pmatrix}\cdot\frac{1}{\sqrt{53}}\begin{pmatrix}2\\-7\end{pmatrix} \qquad \left[\text{Using } \|\mathbf{v}\|^2 = \mathbf{v}\cdot\mathbf{v} \text{ with } \mathbf{v} = \frac{1}{\sqrt{53}}\begin{pmatrix}2\\-7\end{pmatrix}\right]$$

$$= \frac{1}{\sqrt{53}}\frac{1}{\sqrt{53}}\begin{pmatrix}2\\-7\end{pmatrix}\cdot\begin{pmatrix}2\\-7\end{pmatrix}$$

$$= \frac{1}{53}\left(2^2+(-7)^2\right) = \frac{1}{53}(53) = 1$$

Taking the square root gives $\left\|\frac{1}{\|\mathbf{u}\|}\mathbf{u}\right\| = \sqrt{1} = 1$.

(b) In this case we have

$$\left\|\frac{1}{\|\mathbf{u}\|}\mathbf{u}\right\|^2 = \frac{1}{\sqrt{139}}\begin{pmatrix}-9\\3\\7\end{pmatrix}\cdot\frac{1}{\sqrt{139}}\begin{pmatrix}-9\\3\\7\end{pmatrix} \qquad \left[\text{Using } \|\mathbf{v}\|^2 = \mathbf{v}\cdot\mathbf{v} \text{ with } \mathbf{v} = \frac{1}{\sqrt{139}}\begin{pmatrix}-9\\3\\7\end{pmatrix}\right]$$

$$= \frac{1}{\sqrt{139}}\frac{1}{\sqrt{139}}\begin{pmatrix}-9\\3\\7\end{pmatrix}\cdot\begin{pmatrix}-9\\3\\7\end{pmatrix}$$

$$= \frac{1}{139}\left((-9)^2+3^2+7^2\right) = \frac{1}{139}(139) = 1$$

Taking the square root gives $\left\|\frac{1}{\|\mathbf{u}\|}\mathbf{u}\right\| = \sqrt{1} = 1$.

(c) Similarly we have

$$\left\|\frac{1}{\|\mathbf{u}\|}\mathbf{u}\right\|^2 = \frac{1}{\sqrt{134}}\begin{pmatrix}-3\\5\\8\\6\end{pmatrix}\cdot\frac{1}{\sqrt{134}}\begin{pmatrix}-3\\5\\8\\6\end{pmatrix} = \frac{1}{\sqrt{134}}\frac{1}{\sqrt{134}}\begin{pmatrix}-3\\5\\8\\6\end{pmatrix}\cdot\begin{pmatrix}-3\\5\\8\\6\end{pmatrix}$$

$$= \frac{1}{134}\left((-3)^2+5^2+8^2+6^2\right) = \frac{1}{134}(134) = 1$$

Taking the square root gives $\left\|\frac{1}{\|\mathbf{u}\|}\mathbf{u}\right\| = \sqrt{1} = 1$.

(d) As above we have

$$\left\|\frac{1}{\|\mathbf{u}\|}\mathbf{u}\right\|^2 = \frac{1}{\sqrt{138}}\begin{pmatrix}-6\\2\\8\\3\\5\end{pmatrix}\cdot\frac{1}{\sqrt{138}}\begin{pmatrix}-6\\2\\8\\3\\5\end{pmatrix} = \frac{1}{\sqrt{138}}\frac{1}{\sqrt{138}}\begin{pmatrix}-6\\2\\8\\3\\5\end{pmatrix}\cdot\begin{pmatrix}-6\\2\\8\\3\\5\end{pmatrix}$$

$$= \frac{1}{138}\left((-6)^2+2^2+8^2+3^2+5^2\right) = \frac{1}{138}(138) = 1$$

Taking the square root gives $\left\|\frac{1}{\|\mathbf{u}\|}\mathbf{u}\right\| = \sqrt{1} = 1$.

In each case $\left\|\frac{1}{\|\mathbf{u}\|}\mathbf{u}\right\| = 1$.

10. *Proof*. We have $\|\mathbf{u}\|$ is a scalar therefore $\frac{1}{\|\mathbf{u}\|}$ is also a scalar and so by letting $k = \frac{1}{\|\mathbf{u}\|}$ and applying Proposition (2-7) property (b) which says $\|k\mathbf{u}\| = |k|\|\mathbf{u}\|$ where k is a scalar we have

$$\left\|\frac{1}{\|\mathbf{u}\|}\mathbf{u}\right\| = \left|\frac{1}{\|\mathbf{u}\|}\right|\|\mathbf{u}\|$$

$$= \frac{1}{\|\mathbf{u}\|}\|\mathbf{u}\| \underset{\text{Cancelling}}{=} 1 \qquad \left[\begin{array}{l}\text{Because } \|\mathbf{u}\| > 0 \text{ therefore } \frac{1}{\|\mathbf{u}\|} > 0 \\ \text{which means that } \left|\frac{1}{\|\mathbf{u}\|}\right| = \frac{1}{\|\mathbf{u}\|}\end{array}\right]$$

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11. (a) We need to produce one counter example and such an example is Example 6 in the main text. Let $\mathbf{u} = \begin{pmatrix} 2 \\ 5 \end{pmatrix} \neq \mathbf{O}$ and $\mathbf{v} = \begin{pmatrix} 5 \\ -2 \end{pmatrix} \neq \mathbf{O}$ then

$$\mathbf{u}\cdot\mathbf{v} = \begin{pmatrix} 2 \\ 5 \end{pmatrix}\cdot\begin{pmatrix} 5 \\ -2 \end{pmatrix} = (2\times 5) + (5\times(-2)) = 10 - 10 = 0$$

Hence we can have non-zero vectors whose inner (dot) product is 0.

(b) Again we only need a counter example. Let $\mathbf{u} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ and $\mathbf{v} = \begin{pmatrix} -2 \\ 5 \\ 6 \end{pmatrix}$. We have

$$\|\mathbf{u}\| = \sqrt{\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}\cdot\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}} = \sqrt{1^2 + 2^2 + 3^2} = \sqrt{14}$$

$$\|\mathbf{v}\| = \sqrt{\begin{pmatrix} -2 \\ 5 \\ 6 \end{pmatrix}\cdot\begin{pmatrix} -2 \\ 5 \\ 6 \end{pmatrix}} = \sqrt{(-2)^2 + 5^2 + 6^2} = \sqrt{65}$$

$$\|\mathbf{u}+\mathbf{v}\| = \left\|\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + \begin{pmatrix} -2 \\ 5 \\ 6 \end{pmatrix}\right\| = \left\|\begin{pmatrix} -1 \\ 7 \\ 9 \end{pmatrix}\right\| = \sqrt{(-1)^2 + 7^2 + 9^2} = \sqrt{131}$$

To show these are **not** equal:

$$\|\mathbf{u}+\mathbf{v}\| = \sqrt{131} = 11.45$$

$$\|\mathbf{u}\| + \|\mathbf{v}\| = \sqrt{14} + \sqrt{65} = 11.80$$

We conclude that $\|\mathbf{u}+\mathbf{v}\| \neq \|\mathbf{u}\| + \|\mathbf{v}\|$ [Not Equal].

12. (a) *Proof. How do we prove* $\|\mathbf{u}_1+\mathbf{u}_2\|^2=\|\mathbf{u}_1\|^2+\|\mathbf{u}_2\|^2$ *for orthogonal vectors* $\mathbf{u}_1$ *and* $\mathbf{u}_2$*?*

$$\begin{aligned}\|\mathbf{u}_1+\mathbf{u}_2\|^2 &= (\mathbf{u}_1+\mathbf{u}_2)\cdot(\mathbf{u}_1+\mathbf{u}_2)\\ &= (\mathbf{u}_1+\mathbf{u}_2)\cdot\mathbf{u}_1+(\mathbf{u}_1+\mathbf{u}_2)\cdot\mathbf{u}_2\\ &= \mathbf{u}_1\cdot\mathbf{u}_1+\mathbf{u}_2\cdot\mathbf{u}_1+\mathbf{u}_1\cdot\mathbf{u}_2+\mathbf{u}_2\cdot\mathbf{u}_2\\ &= \|\mathbf{u}_1\|^2+\underbrace{\mathbf{u}_2\cdot\mathbf{u}_1}_{=0}+\underbrace{\mathbf{u}_1\cdot\mathbf{u}_2}_{=0}+\|\mathbf{u}_2\|^2 \qquad \begin{bmatrix}\text{Vectors } \mathbf{u}_1 \text{ and } \mathbf{u}_2 \text{ are orthogonal}\\ \text{therefore } \mathbf{u}_1\cdot\mathbf{u}_2=0 \text{ and } \mathbf{u}_2\cdot\mathbf{u}_1=0\end{bmatrix}\\ &= \|\mathbf{u}_1\|^2+\|\mathbf{u}_2\|^2\end{aligned}$$

■

(b) Need to prove $\|\mathbf{u}_1+\mathbf{u}_2+\cdots+\mathbf{u}_n\|^2=\|\mathbf{u}_1\|^2+\|\mathbf{u}_2\|^2+\cdots+\|\mathbf{u}_n\|^2$.

Proof. We use mathematical induction. The 3 steps are:

<u>Step 1</u>. Check the result for $n=2$ say. This follows from part (a).

<u>Step 2</u>. Assume the result is true for $n=k$, that is

$$\|\mathbf{u}_1+\mathbf{u}_2+\cdots+\mathbf{u}_k\|^2=\|\mathbf{u}_1\|^2+\|\mathbf{u}_2\|^2+\cdots+\|\mathbf{u}_k\|^2 \qquad \textbf{(*)}$$

<u>Step 3</u>. Required to prove this result for $n=k+1$. We need to prove that

$$\|\mathbf{u}_1+\mathbf{u}_2+\cdots+\mathbf{u}_k+\mathbf{u}_{k+1}\|^2=\|\mathbf{u}_1\|^2+\|\mathbf{u}_2\|^2+\cdots+\|\mathbf{u}_k\|^2+\|\mathbf{u}_{k+1}\|^2$$

Consider the Left Hand Side of the last line:

$$\begin{aligned}\|\mathbf{u}_1+\mathbf{u}_2+\cdots+\mathbf{u}_k+\mathbf{u}_{k+1}\|^2 &= \|(\mathbf{u}_1+\mathbf{u}_2+\cdots+\mathbf{u}_k)+\mathbf{u}_{k+1}\|^2\\ &\underset{\text{By Part (a)}}{=} \|(\mathbf{u}_1+\mathbf{u}_2+\cdots+\mathbf{u}_k)\|^2+\|\mathbf{u}_{k+1}\|^2\\ &\underset{\text{By (*)}}{=} \|\mathbf{u}_1\|^2+\|\mathbf{u}_2\|^2+\cdots+\|\mathbf{u}_k\|^2+\|\mathbf{u}_{k+1}\|^2\end{aligned}$$

Hence by mathematical induction we have our result.

■

13. (a) *Proof*. Need to prove $\|\mathbf{u}+\mathbf{v}\|^2+\|\mathbf{u}-\mathbf{v}\|^2=2\|\mathbf{u}\|^2+2\|\mathbf{v}\|^2$. Consider each part of the Left Hand Side:

$$\begin{aligned}\|\mathbf{u}+\mathbf{v}\|^2 &= \|\mathbf{u}\|^2+\underbrace{\mathbf{u}\cdot\mathbf{v}+\mathbf{u}\cdot\mathbf{v}}_{=2\mathbf{u}\cdot\mathbf{v}}+\|\mathbf{v}\|^2\\ &= \|\mathbf{u}\|^2+2(\mathbf{u}\cdot\mathbf{v})+\|\mathbf{v}\|^2\end{aligned}$$

Similarly we have

$$\begin{aligned}\|\mathbf{u}-\mathbf{v}\|^2 &= (\mathbf{u}-\mathbf{v})\cdot(\mathbf{u}-\mathbf{v})\\ &= (\mathbf{u}-\mathbf{v})\cdot\mathbf{u}+(\mathbf{u}-\mathbf{v})\cdot(-\mathbf{v})\\ &= \mathbf{u}\cdot\mathbf{u}-\mathbf{v}\cdot\mathbf{u}+\mathbf{u}\cdot(-\mathbf{v})-\mathbf{v}\cdot(-\mathbf{v})\\ &= \|\mathbf{u}\|^2\underbrace{-\mathbf{u}\cdot\mathbf{v}-\mathbf{u}\cdot\mathbf{v}}_{-2\mathbf{u}\cdot\mathbf{v}}+\|\mathbf{v}\|^2\\ &= \|\mathbf{u}\|^2-2(\mathbf{u}\cdot\mathbf{v})+\|\mathbf{v}\|^2\end{aligned}$$

Adding both these parts together gives

$$\|\mathbf{u}+\mathbf{v}\|^2+\|\mathbf{u}-\mathbf{v}\|^2=\underbrace{\|\mathbf{u}\|^2+2(\mathbf{u}\cdot\mathbf{v})+\|\mathbf{v}\|^2}_{=\|\mathbf{u}+\mathbf{v}\|^2}+\underbrace{\|\mathbf{u}\|^2-2(\mathbf{u}\cdot\mathbf{v})+\|\mathbf{v}\|^2}_{=\|\mathbf{u}-\mathbf{v}\|^2}$$
$$=2\|\mathbf{u}\|^2+2\|\mathbf{v}\|^2$$

Hence we have our result.

■

(b) *Proof*. We need to prove $\|\mathbf{u}+\mathbf{v}\|^2-\|\mathbf{u}-\mathbf{v}\|^2=4\mathbf{u}\cdot\mathbf{v}$. We use the solution of part (a).

$$\|\mathbf{u}+\mathbf{v}\|^2-\|\mathbf{u}-\mathbf{v}\|^2=\|\mathbf{u}\|^2+2(\mathbf{u}\cdot\mathbf{v})+\|\mathbf{v}\|^2-\left[\|\mathbf{u}\|^2-2(\mathbf{u}\cdot\mathbf{v})+\|\mathbf{v}\|^2\right]$$
$$=\|\mathbf{u}\|^2+2(\mathbf{u}\cdot\mathbf{v})+\|\mathbf{v}\|^2-\|\mathbf{u}\|^2+2(\mathbf{u}\cdot\mathbf{v})-\|\mathbf{v}\|^2$$
$$=2(\mathbf{u}\cdot\mathbf{v})+2(\mathbf{u}\cdot\mathbf{v})=4(\mathbf{u}\cdot\mathbf{v})$$

■

14. (a) *Proof*. We need to show $d(\mathbf{u},\ \mathbf{v})=\sqrt{(u_1-v_1)^2+(u_2-v_2)^2+\cdots+(u_n-v_n)^2}$.
By using the definition $d(\mathbf{u},\ \mathbf{v})=\|\mathbf{u}-\mathbf{v}\|$ we have

$$d(\mathbf{u},\ \mathbf{v})=\|\mathbf{u}-\mathbf{v}\|$$
$$=\sqrt{(\mathbf{u}-\mathbf{v})\cdot(\mathbf{u}-\mathbf{v})} \qquad (\dagger)$$

What is $\mathbf{u}-\mathbf{v}$ *equal to?*

$$\mathbf{u}-\mathbf{v}=\begin{pmatrix}u_1\\u_2\\\vdots\\u_n\end{pmatrix}-\begin{pmatrix}v_1\\v_2\\\vdots\\v_n\end{pmatrix}=\begin{pmatrix}u_1-v_1\\u_2-v_2\\\vdots-\vdots\\u_n-v_n\end{pmatrix}$$

Substituting this into (†) gives

$$d(\mathbf{u},\ \mathbf{v})=\sqrt{\begin{pmatrix}u_1-v_1\\u_2-v_2\\\vdots-\vdots\\u_n-v_n\end{pmatrix}\cdot\begin{pmatrix}u_1-v_1\\u_2-v_2\\\vdots-\vdots\\u_n-v_n\end{pmatrix}}=\sqrt{(u_1-v_1)^2+(u_2-v_2)^2+\cdots+(u_n-v_n)^2}$$

(b) *Proof*. To prove $d(\mathbf{u},\ \mathbf{v})=d(\mathbf{v},\ \mathbf{u})$ we use the result of part (a) above:

$$d(\mathbf{u},\ \mathbf{v})=\sqrt{(u_1-v_1)^2+(u_2-v_2)^2+\cdots+(u_n-v_n)^2}$$
$$=\sqrt{(v_1-u_1)^2+(v_2-u_2)^2+\cdots+(v_n-u_n)^2}=d(\mathbf{v},\ \mathbf{u})$$

■

(c) *Proof*. We need to show $d(\mathbf{u},\ \mathbf{v})\ge 0$ and $d(\mathbf{u},\ \mathbf{v})=0 \Leftrightarrow \mathbf{u}=\mathbf{v}$. Since $d(\mathbf{u},\ \mathbf{v})=\|\mathbf{u}-\mathbf{v}\|$, the proof follows by Proposition (2-7) property (a).

$$d(\mathbf{u},\ \mathbf{v})=\|\mathbf{u}-\mathbf{v}\|\ge 0$$
$$d(\mathbf{u},\ \mathbf{v})=\|\mathbf{u}-\mathbf{v}\|=0 \Leftrightarrow \mathbf{u}-\mathbf{v}=\mathbf{O} \Leftrightarrow \mathbf{u}=\mathbf{v}$$

■