Complete Solutions to Exercises 3.2

1. Checking **all** 10 axioms gives

   1.The vector addition $\mathbf{O}+\mathbf{O}=\mathbf{O}$ is also in the vector space S.

   2.Commutative law: $\mathbf{O}+\mathbf{O}=\mathbf{O}+\mathbf{O}$.

   3.Associative law: $(\mathbf{O}+\mathbf{O})+\mathbf{O}=\mathbf{O}+(\mathbf{O}+\mathbf{O})$.

   4.Neutral Element. Clearly $\mathbf{O}$ is in S.

   5.Additive Inverse. For every vector $\mathbf{O}$ there is a vector $\mathbf{O}$ which satisfies the following:

$$\mathbf{O}+(\mathbf{O})=\mathbf{O}$$

   6.Let k be a scalar then $k\mathbf{O}$ is also in S.

   7.Associative Law for scalar multiplication. Let k and c be real numbers then

$$k(c\mathbf{O})=(kc)\mathbf{O}$$

   8.Distributive Law for vectors. Let k be a real number then

$$k(\mathbf{O}+\mathbf{O})=k\mathbf{O}+k\mathbf{O}$$

   9.Distributive Law for scalars. Let k and c be real numbers then

$$(k+c)\mathbf{O}=k\mathbf{O}+c\mathbf{O}$$

   10. Identity Element. For every vector in S we have

$$1(\mathbf{O})=\mathbf{O}$$

Since ALL 10 axioms are satisfied therefore we have S is a vector space.

Assume in the remaining questions, the set S is non-empty. Ideally you should check for this.

2. We need to check conditions (a) and (b) of Proposition (3-4). These are

(a) If $\mathbf{u}$ and $\mathbf{v}$ are vectors in the set S then the vector addition $\mathbf{u}+\mathbf{v}$ is also in S.

(b) If $\mathbf{u}$ is a vector in S and k is any scalar then $k\mathbf{u}$ is also in S.

Let $\mathbf{u}=\begin{pmatrix}a\\0\end{pmatrix}$ and $\mathbf{v}=\begin{pmatrix}b\\0\end{pmatrix}$ be vectors in S.

$$\mathbf{u}+\mathbf{v}=\begin{pmatrix}a\\0\end{pmatrix}+\begin{pmatrix}b\\0\end{pmatrix}=\begin{pmatrix}a+b\\0\end{pmatrix}$$

Hence $\mathbf{u}+\mathbf{v}$ is in S.

Let k be a scalar and $k\mathbf{u}=k\begin{pmatrix}a\\0\end{pmatrix}=\begin{pmatrix}ka\\0\end{pmatrix}$. This is also in S.

Since conditions (a) and (b) of Proposition (3-4) is satisfied therefore we conclude S is a subspace of $\mathbb{R}^2$.

3. Again we need to check conditions (a) and (b) of Proposition (3-4). These are

(a) If $\mathbf{u}$ and $\mathbf{v}$ are vectors in the set S then the vector addition $\mathbf{u}+\mathbf{v}$ is also in S.

(b) If $\mathbf{u}$ is a vector in S and k is any scalar then $k\mathbf{u}$ is also in S.

Let $\mathbf{u}=\begin{pmatrix}a\\a\end{pmatrix}$ and $\mathbf{v}=\begin{pmatrix}b\\b\end{pmatrix}$ be vectors in S.

$$\mathbf{u}+\mathbf{v}=\begin{pmatrix}a\\a\end{pmatrix}+\begin{pmatrix}b\\b\end{pmatrix}=\begin{pmatrix}a+b\\a+b\end{pmatrix}$$

Hence $\mathbf{u}+\mathbf{v}$ is in S.

Let k be a scalar and $k\mathbf{u} = k\begin{pmatrix} a \\ a \end{pmatrix} = \begin{pmatrix} ka \\ ka \end{pmatrix}$. This is also in S.

Since conditions (a) and (b) of Proposition (3-4) is satisfied therefore we conclude S is a subspace of $\mathbb{R}^2$.

4. Again we need to check conditions (a) and (b) of Proposition (3-4). These are

(a) If $\mathbf{u}$ and $\mathbf{v}$ are vectors in the set S then the vector addition $\mathbf{u}+\mathbf{v}$ is also in S.

(b) If $\mathbf{u}$ is a vector in S and k is any scalar then $k\mathbf{u}$ is also in S.

Let $\mathbf{u} = \begin{pmatrix} 0 \\ 0 \\ c \end{pmatrix}$ and $\mathbf{v} = \begin{pmatrix} 0 \\ 0 \\ d \end{pmatrix}$ be vectors in S.

$$\mathbf{u}+\mathbf{v} = \begin{pmatrix} 0 \\ 0 \\ c \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ d \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ c+d \end{pmatrix}$$

Hence $\mathbf{u}+\mathbf{v}$ is in S.

Let k be a scalar and $k\mathbf{u} = k\begin{pmatrix} 0 \\ 0 \\ c \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ kc \end{pmatrix}$. This is also in S.

Since conditions (a) and (b) of Proposition (3-4) is satisfied therefore we conclude S is a subspace of $\mathbb{R}^3$.

5. Let $\mathbf{u} = \begin{pmatrix} 1 \\ b \\ c \\ d \end{pmatrix}$ and $\mathbf{v} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$ then

$$\mathbf{u}+\mathbf{v} = \begin{pmatrix} 1 \\ b \\ c \\ d \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ b \\ c \\ d \end{pmatrix}$$

The first entry in the vector is 2 therefore $\mathbf{u}+\mathbf{v}$ is **not** in S, Hence S is **not** a subspace of $\mathbb{R}^4$.

6. We can use:

Proposition (3-5). A non-empty subset S is a subspace of a vector space V if and only if If $\mathbf{u}$ and $\mathbf{v}$ are vectors in S then any linear combination $k_1\mathbf{u}+k_2\mathbf{v}$ is also in S.

Clearly S is non-empty. Let $\mathbf{u} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$ with $a+b+c=0$ and $\mathbf{v} = \begin{pmatrix} d \\ e \\ f \end{pmatrix}$ with $d+e+f=0$.

Consider

$$k_1\mathbf{u}+k_2\mathbf{v}=k_1\begin{pmatrix}a\\b\\c\end{pmatrix}+k_2\begin{pmatrix}d\\e\\f\end{pmatrix}=\begin{pmatrix}k_1a+k_2d\\k_1b+k_2e\\k_1c+k_2f\end{pmatrix}$$

We have to check that the result is in S which means adding all the entries gives zero:

$$\begin{aligned}k_1a+k_2d+k_1b+k_2e+k_1c+k_2f&=k_1(a+b+c)+k_2(d+e+f)\\&=k_1(0)+k_2(0)=0\end{aligned}$$

Hence $k_1\mathbf{u}+k_2\mathbf{v}$ is also in S. By Proposition (3-5) we conclude that S is a subspace of the vector space V.

7. Let $\mathbf{u}=\begin{pmatrix}1\\1\end{pmatrix}$ and $k=\dfrac{1}{2}$ and we carry out scalar multiplication $k\mathbf{u}$ we have

$$k\mathbf{u}=\frac{1}{2}\begin{pmatrix}1\\1\end{pmatrix}=\begin{pmatrix}1/2\\1/2\end{pmatrix}$$

Since the entries ½ are **not integers** therefore $k\mathbf{u}$ is **not** a member of the set S. Hence S is **not** a subspace of the vector space V.

8. We are give $S=\left\{\begin{pmatrix}a\\b\\c\end{pmatrix}\middle|\ a,\ b\ \text{and c are rational numbers}\right\}$. Let $\mathbf{u}=\begin{pmatrix}1\\2\\3\end{pmatrix}$ and $k=$ ®

then

$$k\mathbf{u}=\text{®}\begin{pmatrix}1\\2\\3\end{pmatrix}=\begin{pmatrix}\text{®}\\2\text{®}\\3\text{®}\end{pmatrix}$$

Since ®, 2® and 3® are **not** rational numbers so $k\mathbf{u}$ is **not** in the set S. Therefore S is **not** a subspace of $\mathbb{R}^3$.

9. Very similar to Example 7.

10. We are given $S=\left\{\begin{pmatrix}a&b\\c&d\end{pmatrix}\middle|\ a,\ b,\ c,\ d\ \text{ are all integers}\right\}$. Consider the matrix **A**

which is $\mathbf{A}=\begin{pmatrix}1&2\\3&4\end{pmatrix}$ and scalar $k=$ ® then scalar multiplication

$$k\mathbf{A}=\text{®}\begin{pmatrix}1&2\\3&4\end{pmatrix}=\begin{pmatrix}\text{®}&2\text{®}\\3\text{®}&4\text{®}\end{pmatrix}$$

Clearly the entries ®, 2®, 3® and 4® are **not** integers therefore $k\mathbf{A}$ is **not** in the set S. Hence S is **not** a subspace of M_{22}.

11. We can use:

Proposition (3-5). A nonempty subset S is a subspace of a vector space V if and only if $\mathbf{u}$ and $\mathbf{v}$ are vectors in S then any linear combination $k\mathbf{u}+c\mathbf{v}$ is also in S.

Let **A** and **B** be symmetric matrices, that is $\mathbf{A}^T = \mathbf{A}$ and $\mathbf{B}^T = \mathbf{B}$.
Consider $k\mathbf{A} + c\mathbf{B}$:

$$\begin{aligned}(k\mathbf{A}+c\mathbf{B})^T &= (k\mathbf{A})^T + (c\mathbf{B})^T \\ &= k\mathbf{A}^T + c\mathbf{B}^T = k\mathbf{A} + c\mathbf{B}\end{aligned}$$

Since $(k\mathbf{A}+c\mathbf{B})^T = k\mathbf{A}+c\mathbf{B}$ so $k\mathbf{A}+c\mathbf{B}$ is symmetrical which means it is in the subset S. Hence by Proposition (3-5) we have S is a subspace of the vector space V.

12. We need to find scalars which satisfy

$$k_1\mathbf{v}_1 + k_2\mathbf{v}_2 + k_3\mathbf{v}_3 = \mathbf{x}$$

where $\mathbf{v}_1 = t^2 - 1,\ \mathbf{v}_2 = t+1$ and $\mathbf{v}_3 = 2t^2 + t - 1$. Also $\mathbf{x} = 7t^2 + 8t + 1$. Substituting these into the above gives

$$\begin{aligned}k_1\mathbf{v}_1 + k_2\mathbf{v}_2 + k_3\mathbf{v}_3 &= k_1\left(t^2-1\right) + k_2\left(t+1\right) + k_3\left(2t^2+t-1\right) \\ &= \left(k_1+2k_3\right)t^2 + \left(k_2+k_3\right)t + \left(-k_1+k_2-k_3\right) \\ &= 7t^2 + 8t + 1\end{aligned}$$

Equating coefficients gives

$$\begin{aligned}k_1 \qquad + 2k_3 &= 7 \\ k_2 + k_3 &= 8 \\ -k_1 + k_2 - k_3 &= 1\end{aligned}$$

Writing the augmented matrix:

$$\begin{array}{l} \mathrm{R}_1 \\ \mathrm{R}_2 \\ \mathrm{R}_3 \end{array} \left(\begin{array}{ccc|c} 1 & 0 & 2 & 7 \\ 0 & 1 & 1 & 8 \\ -1 & 1 & -1 & 1 \end{array}\right)$$

Carrying out the row operation

$$\begin{array}{r} \mathrm{R}_1 \\ \mathrm{R}_2 \\ \mathrm{R}_3^* = \mathrm{R}_3 + \mathrm{R}_1 \end{array} \left(\begin{array}{ccc|c} 1 & 0 & 2 & 7 \\ 0 & 1 & 1 & 8 \\ 0 & 1 & 1 & 8 \end{array}\right)$$

Subtracting the last two rows:

$$\begin{array}{r} \mathrm{R}_1 \\ \mathrm{R}_2 \\ \mathrm{R}_3^{**} = \mathrm{R}_3^* - \mathrm{R}_2 \end{array} \left(\begin{array}{ccc|c} 1 & 0 & 2 & 7 \\ 0 & 1 & 1 & 8 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

From the middle row we have $k_2 + k_3 = 8$. Let $k_3 = 1$ then $k_2 = 7$. Also substituting $k_3 = 1$ into the top row we have

$$k_1 + 2 = 7 \text{ gives } k_1 = 5$$

Since we have scalars $k_1 = 5$, $k_2 = 7$ and $k_3 = 1$ which satisfy

$$\begin{aligned}k_1\mathbf{v}_1 + k_2\mathbf{v}_2 + k_3\mathbf{v}_3 &= 5\left(t^2-1\right) + 7\left(t+1\right) + \left(2t^2+t-1\right) \\ &= 7t^2 + 8t + 1 = \mathbf{x}\end{aligned}$$

x is a linear combination of $\mathbf{v}_1$, $\mathbf{v}_2$ and $\mathbf{v}_3$.

13. We need to find each **x** as a linear combination of the given vectors

$$\mathbf{p}_1 = t^2 + 2t - 1,\ \mathbf{p}_2 = 2t + 1 \text{ and } \mathbf{p}_3 = 5t^2 + 2t - 3$$

(a) We have to find the scalars which satisfy

$$\begin{aligned} k_1\mathbf{p}_1 + k_2\mathbf{p}_2 + k_3\mathbf{p}_3 &= k_1\left(t^2+2t-1\right)+k_2\left(2t+1\right)+k_3\left(5t^2+2t-3\right) \\ &= \left(k_1+5k_3\right)t^2+\left(2k_1+2k_2+2k_3\right)t+\left(-k_1+k_2-3k_3\right) \\ &= 4t^2-2t-3 \end{aligned}$$

Equating coefficients gives

$$\begin{aligned} k_1 \qquad\quad +5k_3 &= 4 \\ 2k_1+2k_2+2k_3 &= -2 \\ -k_1+k_2-3k_3 &= -3 \end{aligned}$$

Writing the augmented matrix gives

$$\begin{array}{c} \mathrm{R}_1 \\ \mathrm{R}_2 \\ \mathrm{R}_3 \end{array} \left(\begin{array}{ccc|c} 1 & 0 & 5 & 4 \\ 2 & 2 & 2 & -2 \\ -1 & 1 & -3 & -3 \end{array}\right)$$

Executing row operations:

$$\begin{array}{r} \mathrm{R}_1 \\ \mathrm{R}_2^* = \mathrm{R}_2 - 2\mathrm{R}_1 \\ \mathrm{R}_3^* = \mathrm{R}_3 + \mathrm{R}_1 \end{array} \left(\begin{array}{ccc|c} 1 & 0 & 5 & 4 \\ 0 & 2 & -8 & -10 \\ 0 & 1 & 2 & 1 \end{array}\right)$$

Carrying out row operations on the last two rows:

$$\begin{array}{r} \mathrm{R}_1 \\ \mathrm{R}_2^* \\ \mathrm{R}_3^{**} = 2\mathrm{R}_3^* - \mathrm{R}_2^* \end{array} \left(\begin{array}{ccc|c} 1 & 0 & 5 & 4 \\ 0 & 2 & -8 & -10 \\ 0 & 0 & 12 & 12 \end{array}\right)$$

From the last row we have $k_3 = 1$. Substituting this into the middle row gives

$$\begin{aligned} 2k_2 - 8k_3 &= -10 \\ 2k_2 - 8 = -10 \quad &\text{gives} \quad k_2 = -1 \end{aligned}$$

Substituting $k_3 = 1$ into the first row gives

$$\begin{aligned} k_1 + 5k_3 &= 4 \\ k_1 + 5(1) = 4 \quad &\text{gives} \quad k_1 = -1 \end{aligned}$$

Since we have found scalars, $k_1 = -1$, $k_2 = -1$ and $k_3 = 1$, therefore $\mathbf{x} = 4t^2 - 2t - 3$ is a linear combination of $\mathbf{p}_1$, $\mathbf{p}_2$ and $\mathbf{p}_3$. We can check this by:

$$(-1)\left(t^2+2t-1\right)+(-1)(2t+1)+5t^2+2t-3 = 4t^2-2t-3 = \mathbf{x}$$

(b) Similarly to (a) for $\mathbf{x} = -2t^2 - 2$ we have

$$\begin{aligned} k_1\mathbf{p}_1 + k_2\mathbf{p}_2 + k_3\mathbf{p}_3 &= k_1\left(t^2+2t-1\right)+k_2\left(2t+1\right)+k_3\left(5t^2+2t-3\right) \\ &= \left(k_1+5k_3\right)t^2+\left(2k_1+2k_2+2k_3\right)t+\left(-k_1+k_2-3k_3\right) \\ &= -2t^2-2 \end{aligned}$$

Equating coefficients gives

$$\begin{aligned} k_1 \qquad\quad +5k_3 &= -2 \\ 2k_1+2k_2+2k_3 &= 0 \\ -k_1+k_2-3k_3 &= -2 \end{aligned}$$

Writing the augmented matrix gives

$$\begin{array}{c} \mathrm{R}_1 \\ \mathrm{R}_2 \\ \mathrm{R}_3 \end{array} \left(\begin{array}{ccc|c} 1 & 0 & 5 & -2 \\ 2 & 2 & 2 & 0 \\ -1 & 1 & -3 & -2 \end{array}\right)$$

Executing row operations:

$$\begin{array}{r} \mathrm{R}_1 \\ \mathrm{R}_2^* = \mathrm{R}_2 - 2\mathrm{R}_1 \\ \mathrm{R}_3^* = \mathrm{R}_3 + \mathrm{R}_1 \end{array} \left(\begin{array}{ccc|c} 1 & 0 & 5 & -2 \\ 0 & 2 & -8 & 4 \\ 0 & 1 & 2 & -4 \end{array}\right)$$

Carrying out row operations on the last two rows:

$$\begin{array}{r} \mathrm{R}_1 \\ \mathrm{R}_2^* \\ \mathrm{R}_3^{**} = 2\mathrm{R}_3^* - \mathrm{R}_2^* \end{array} \left(\begin{array}{ccc|c} 1 & 0 & 5 & -2 \\ 0 & 2 & -8 & 4 \\ 0 & 0 & 12 & -12 \end{array}\right)$$

From the last row we have $k_3 = -1$. Substituting $k_3 = -1$ into the top row:

$$k_1 \qquad + 5k_3 = -2 \text{ gives } k_1 = -2 + 5 = 3$$

Substituting $k_1 = 3$ and $k_3 = -1$ into the middle row:

$$2k_1 + 2k_2 + 2k_3 = 0 \;\Rightarrow\; 2(3) + 2k_2 + 2(-1) = 0 \;\Rightarrow\; k_2 = -2$$

We have scalars $k_1 = 3,\ k_2 = -2$ and $k_3 = -1$ which satisfies

$$\begin{aligned} k_1\mathbf{p}_1 + k_2\mathbf{p}_2 + k_3\mathbf{p}_3 &= 3\left(t^2 + 2t - 1\right) + (-2)(2t + 1) + (-1)\left(5t^2 + 2t - 3\right) \\ &= -2t^2 - 2 = \mathbf{x} \end{aligned}$$

Therefore $\mathbf{x}$ a linear combination of $\mathbf{p}_1$, $\mathbf{p}_2$ and $\mathbf{p}_3$.

(c) Similarly to (a) for $\mathbf{x} = 6$ we have

$$\begin{aligned} k_1\mathbf{p}_1 + k_2\mathbf{p}_2 + k_3\mathbf{p}_3 &= k_1\left(t^2 + 2t - 1\right) + k_2(2t + 1) + k_3\left(5t^2 + 2t - 3\right) \\ &= (k_1 + 5k_3)t^2 + (2k_1 + 2k_2 + 2k_3)t + (-k_1 + k_2 - 3k_3) \\ &= 6 \end{aligned}$$

Equating coefficients gives

$$\begin{aligned} k_1 \qquad\quad + 5k_3 &= 0 \\ 2k_1 + 2k_2 + 2k_3 &= 0 \\ -k_1 + k_2 - 3k_3 &= 6 \end{aligned}$$

Writing the augmented matrix gives

$$\begin{array}{c} \mathrm{R}_1 \\ \mathrm{R}_2 \\ \mathrm{R}_3 \end{array} \left(\begin{array}{ccc|c} 1 & 0 & 5 & 0 \\ 2 & 2 & 2 & 0 \\ -1 & 1 & -3 & 6 \end{array}\right)$$

Executing row operations:

$$\begin{array}{r} \mathrm{R}_1 \\ \mathrm{R}_2^* = \mathrm{R}_2 - 2\mathrm{R}_1 \\ \mathrm{R}_3^* = \mathrm{R}_3 + \mathrm{R}_1 \end{array} \left(\begin{array}{ccc|c} 1 & 0 & 5 & 0 \\ 0 & 2 & -8 & 0 \\ 0 & 1 & 2 & 6 \end{array}\right)$$

Carrying out row operations on the last two rows:

$$\begin{array}{r} \mathrm{R}_1 \\ \mathrm{R}_2^* \\ \mathrm{R}_3^{**} = 2\mathrm{R}_3^* - \mathrm{R}_2^* \end{array} \left(\begin{array}{ccc|c} 1 & 0 & 5 & 0 \\ 0 & 2 & -8 & 0 \\ 0 & 0 & 12 & 12 \end{array}\right)$$

From the last row we have $k_3 = 1$. Substituting $k_3 = 1$ into the middle row:

$$2k_2 - 8 = 0 \text{ gives } k_2 = 4$$

Substituting $k_3 = 1$ into the top row:

$$k_1 + 5 = 0 \text{ gives } k_1 = -5$$

Since we have scalars $k_1 = -5,\ k_2 = 2$ and $k_3 = 1$ which satisfies

$$\begin{aligned} k_1\mathbf{p}_1 + k_2\mathbf{p}_2 + k_3\mathbf{p}_3 &= k_1\left(t^2 + 2t - 1\right) + k_2\left(2t+1\right) + k_3\left(5t^2 + 2t - 3\right) \\ &= -5\left(t^2 + 2t - 1\right) + 4\left(2t+1\right) + \left(5t^2 + 2t - 3\right) \\ &= 6 = \mathbf{x} \end{aligned}$$

Therefore $\mathbf{x}$ a linear combination of $\mathbf{p}_1$, $\mathbf{p}_2$ and $\mathbf{p}_3$.

14. (a) Since from trigonometry we have $\cos^2(t) + \sin^2(t) = 1$ therefore 1 is a linear combination of $\mathbf{v}_1 = \sin^2(t)$ and $\mathbf{v}_2 = \cos^2(t)$ with both scalars equal to 1.

(b) Similarly we have

$$k_1\cos^2(t) + k_2\sin^2(t) = ®$$

$$® \cos^2(t) + ® \sin^2(t) = ®$$

Linear combination of $\mathbf{v}_1 = \sin^2(t)$ and $\mathbf{v}_2 = \cos^2(t)$ gives $\mathbf{x} = ®$.

(c) From our knowledge of trigonometry we have the identity

$$\cos^2(t) - \sin^2(t) = \cos(2t)$$

Therefore with scalars $k_1 = 1$ and $k_2 = -1$ we have

$$k_1\cos^2(t) + k_2\sin^2(t) = \cos^2(t) - \sin^2(t) = \cos(2t)$$

Hence the vector $\mathbf{x} = \cos(2t)$ is a linear combination of $\mathbf{v}_1 = \sin^2(t)$ and $\mathbf{v}_2 = \cos^2(t)$.

15. We need to show that $\mathbf{u} = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}$ and $\mathbf{v} = \begin{pmatrix} -1 \\ 5 \\ 0 \end{pmatrix}$ span the subspace S which is set of vectors $\begin{pmatrix} a \\ b \\ 0 \end{pmatrix}$. Let $\begin{pmatrix} a \\ b \\ 0 \end{pmatrix}$ be an arbitrary vector and k and c be scalars.

$$\begin{aligned} k\mathbf{u} + c\mathbf{v} &= k\begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} + c\begin{pmatrix} -1 \\ 5 \\ 0 \end{pmatrix} \\ &= \begin{pmatrix} k \\ 2k \\ 0 \end{pmatrix} + \begin{pmatrix} -c \\ 5c \\ 0 \end{pmatrix} = \begin{pmatrix} k - c \\ 2k + 5c \\ 0 \end{pmatrix} = \begin{pmatrix} a \\ b \\ 0 \end{pmatrix} \end{aligned}$$

We have the simultaneous equations

$$k - c = a$$
$$2k + 5c = b$$

From the first equation we have $k = c + a$. Substituting this into the second equation:

$$2(c + a) + 5c = b$$
$$2c + 2a + 5c = b$$
$$7c = b - 2a \text{ gives } c = \frac{b - 2a}{7}$$

Placing $c = \frac{b-2a}{7}$ into the first equation $k - c = a$ yields

$$k - \frac{b - 2a}{7} = a$$
$$k = a + \frac{b - 2a}{7} = \frac{7a}{7} + \frac{b - 2a}{7} = \frac{5a + b}{7}$$

Since we have $c = \frac{b-2a}{7}$ and $k = \frac{5a+b}{7}$ therefore we conclude that the given vectors span *S*.

16. You need to find scalars k_1, k_2 and k_3 which satisfy

$$k_1\mathbf{p}_1 + k_2\mathbf{p}_2 + k_3\ \mathbf{p}_3 = k_1\left(t^2 + 3\right) + k_2\left(2t^2 + 5t + 6\right) + k_3\left(5t\right)$$
$$= \left(k_1 + 2k_2\right)t^2 + \left(5k_2 + 5k_3\right)t + \left(3k_1 + 6k_2\right)$$
$$= t^2 + 3$$

Equating coefficients:

$$k_1 + 2k_2 = 1 \qquad (*)$$
$$5k_2 + 5k_3 = 0 \qquad (**)$$
$$3k_1 + 6k_2 = 3 \qquad (***)$$

From the first equation (*) we have $k_1 = 1 - 2k_2$. Let $k_2 = 1$ then $k_1 = -1$. From the middle equation (**) we have $k_2 = -k_3$ which means that $k_3 = -1$.

We have scalars, $k_1 = -1$, $k_2 = 1$ and $k_3 = -1$:

$$k_1\mathbf{p}_1 + k_2\mathbf{p}_2 + k_3\ \mathbf{p}_3 = (-1)\left(t^2 + 3\right) + 2t^2 + 5t + 6 + (-1)(5t)$$
$$= t^2 + 3$$

Therefore the vector **x** belongs to span $\{\mathbf{p}_1,\ \mathbf{p}_2,\ \mathbf{p}_3\}$.

17. We need to solve the linear combination

$$k_1\mathbf{A} + k_2\mathbf{B} + k_3\mathbf{C} = \mathbf{D}$$

for scalars k_1, k_2 and k_3:

$$k_1\mathbf{A}+k_2\mathbf{B}+k_3\mathbf{C}=k_1\begin{pmatrix}1&1\\1&1\end{pmatrix}+k_2\begin{pmatrix}-1&2\\5&7\end{pmatrix}+k_3\begin{pmatrix}2&6\\8&0\end{pmatrix}$$
$$=\begin{pmatrix}k_1&k_1\\k_1&k_1\end{pmatrix}+\begin{pmatrix}-k_2&2k_2\\5k_2&7k_2\end{pmatrix}+\begin{pmatrix}2k_3&6k_3\\8k_3&0\end{pmatrix}$$
$$=\begin{pmatrix}k_1-k_2+2k_3&k_1+2k_2+6k_3\\k_1+5k_2+8k_3&k_1+7k_2+0\end{pmatrix}=\begin{pmatrix}7&-3\\-14&-26\end{pmatrix}$$

Equating entries gives

$$k_1-k_2+2k_3=7$$
$$k_1+2k_2+6k_3=-3$$
$$k_1+5k_2+8k_3=-14$$
$$k_1+7k_2+0=-26$$

Solving these equations gives $k_1=2,\ k_2=-4$ and $k_3=\frac{1}{2}$.

Therefore **D** belongs to span $\{\mathbf{A},\ \mathbf{B},\ \mathbf{C}\}$.

18. (a) We need to test if 0 is in the span of $\{\mathbf{f},\ \mathbf{g}\}$ where $\mathbf{f}=\cos(2x)$, $\mathbf{g}=\sin(2x)$:

$$k\mathbf{f}+c\mathbf{g}=k\cos(2x)+c\sin(2x)=0 \quad\Rightarrow\quad k=c=0$$

Hence 0 is in the span of $\{\mathbf{f},\ \mathbf{g}\}$.

(b) Similarly we have

$$k\mathbf{f}+c\mathbf{g}=k\cos(2x)+c\sin(2x)=\sin(2x)\ \Rightarrow\ k=0 \text{ and } c=1$$

Therefore $\sin(2x)$ is in the span of $\{\mathbf{f},\ \mathbf{g}\}$.

(c) From trigonometry we have the identity $\cos^2(x)-\sin^2(x)=\cos(2x)$ therefore $\cos^2(x)-\sin^2(x)$ is in the span of $\{\mathbf{f},\ \mathbf{g}\}$.

(d) From trigonometry we have $\cos^2(x)+\sin^2(x)=1$ and by part (c)

$$\cos^2(x)-\sin^2(x)=\cos(2x)$$

Putting this into the linear combination:

$$k\mathbf{f}+c\mathbf{g}=k\cos(2x)+c\sin(2x)$$
$$=k\left[\cos^2(x)-\sin^2(x)\right]+c\sin(2x)=\cos^2(x)+\sin^2(x)$$

Equating coefficients $\cos^2(x)$, $\sin^2(x)$ and $\sin(2x)$ gives

$$k=1,\ k=-1,\ c=0$$

This is inconsistent because we have two different values for k, that is $k=1,\ k=-1$.

Hence 1 is **not** in the span of $\{\mathbf{f},\ \mathbf{g}\}$.

19. We need to see if $x+1$ and $(x+1)^2$ span P_2. Let $\mathbf{p}=ax^2+bx+c$ be a polynomial in P_2. Let k_1 and k_2 be scalars such that

$$k_1(x+1)+k_2(x+1)^2=k_1x+k_1+k_2(x^2+2x+1)$$
$$=k_2x^2+(k_1+2k_2)x+(k_1+k_2)=ax^2+bx+c$$

Equating coefficients

$$\begin{aligned} x^2: &\quad k_2 = a \\ x: &\quad k_1 + 2k_2 = b \\ Const: &\quad k_1 + k_2 = c \end{aligned}$$

From the first equation we have $k_2 = a$. Substituting this into the middle equation gives:

$$k_1 + 2a = b \quad \Rightarrow \quad k_1 = b - 2a$$

Putting $k_1 = b - 2a$ and $k_2 = a$ into the bottom equation yields

$$b - 2a + a = b - a = c$$

We are forced to have $c = b - a$. Hence $x+1$ and $(x+1)^2$ cannot span a polynomial $ax^2 + bx + c$ where $c \neq b - a$. Therefore $x+1$ and $(x+1)^2$ does not span P_2.

20. We need to prove that span$\{S\}$ is a subspace of the vector space *V* where *S* is a non - empty subset of *V*.

Proof. We are given that *S* is non – empty so let $\mathbf{v}_1,\ \mathbf{v}_2,\ \cdots,\ \mathbf{v}_n$ be vectors in *S*. By

Definition (3-3). If *every* vector in *V* can be produced by a linear combination of vectors $\mathbf{v}_1,\ \mathbf{v}_2, \mathbf{v}_3,\ \cdots$ and $\mathbf{v}_n$ then these vectors **span** or **generate** the vector space *V*. We write this as span$\{\mathbf{v}_1,\ \mathbf{v}_2, \mathbf{v}_3,\ \cdots,\ \mathbf{v}_n\}$.

we have span$\{\mathbf{v}_1,\ \mathbf{v}_2, \mathbf{v}_3,\ \cdots,\ \mathbf{v}_n\}$ is a linear combination of the vectors $\mathbf{v}_1,\ \mathbf{v}_2,\ \cdots,\ \mathbf{v}_n$. Since any linear combination of $\{\mathbf{v}_1,\ \mathbf{v}_2, \mathbf{v}_3,\ \cdots,\ \mathbf{v}_n\}$ is in span$\{\mathbf{v}_1,\ \mathbf{v}_2, \mathbf{v}_3,\ \cdots,\ \mathbf{v}_n\}$ so by

Proposition (3-5). A non - empty subset *S* containing vectors **u** and **v** is a subspace of a vector space *V* $\Leftrightarrow$ any linear combination $k\mathbf{u} + c\mathbf{v}$ is also in *S* (k and c are scalars).

We conclude that span$\{\mathbf{v}_1,\ \mathbf{v}_2, \mathbf{v}_3,\ \cdots,\ \mathbf{v}_n\} = span\{S\}$ is subspace of *V*.

■

21. Take *T* to be the subset of *S* which does not contain the zero vector. Then clearly *T* is a subset of *S* but *T* is not subspace of *S* because it must have the zero vector in order to be a subspace.

22. The question should say prove or disprove.

(a) Required to prove that $S \cap T$ is a subspace of *V* provided *S* and *T* are subspaces of *V*.

Proof.

Let vectors **u** and **v** be vectors in $S \cap T$. This means that **u** and **v** is in *S* and *T*. As *S* and *T* are subspaces of *V* therefore the linear combination of **u** and **v** is also in both *S* and *T*. Hence linear combination is in $S \cap T$. By

Proposition (3-5). A non - empty subset *S* containing vectors **u** and **v** is a subspace of a vector space *V* $\Leftrightarrow$any linear combination $k\mathbf{u} + c\mathbf{v}$ is also in *S* (k and c are scalars).

We can say that $S \cap T$ is a subspace of *V*.

■

(b) The union of subspaces is *not* a subspace. For example, consider the following:

$$S = Span\left\{\begin{pmatrix}1\\0\end{pmatrix}\right\} \text{ and } T = Span\left\{\begin{pmatrix}0\\1\end{pmatrix}\right\}$$

are subspaces of $\mathbb{R}^2$. Let $\mathbf{u} = \begin{pmatrix}1\\0\end{pmatrix}$ be in *S* and $\mathbf{v} = \begin{pmatrix}0\\1\end{pmatrix}$ be in *T*. Then

$$\mathbf{u} + \mathbf{v} = \begin{pmatrix}1\\1\end{pmatrix} \text{ which is } \textit{not} \text{ in } S \cup T$$

Hence $S \cup T$ is *not* a subspace.