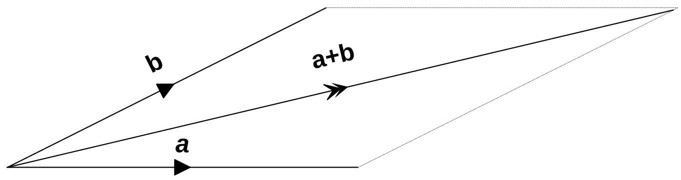
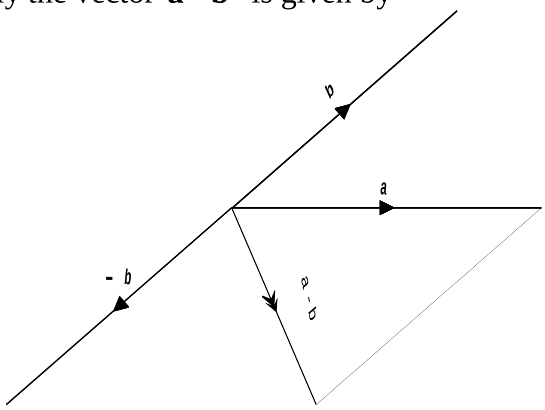
Complete Solutions to Exercises 1.3

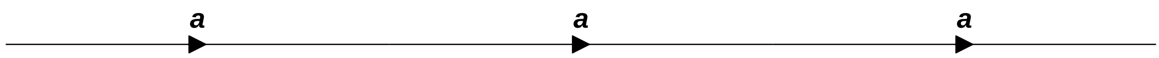
1. (a) By applying the parallelogram rule we have the vector $\mathbf{a}+\mathbf{b}$ given by



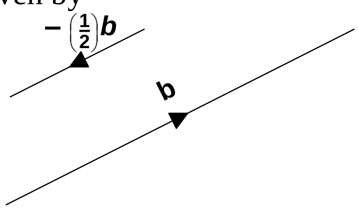
(b) Similarly the vector $\mathbf{a}-\mathbf{b}$ is given by



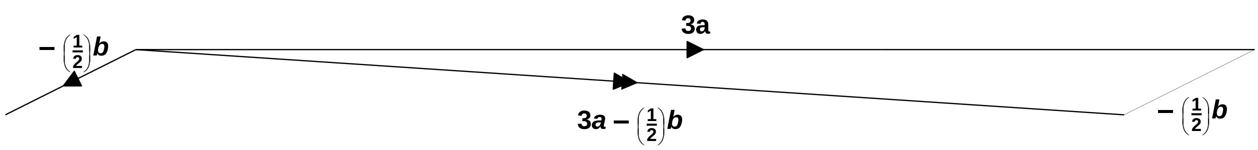
(c) The scalar multiplication $3\mathbf{a}$ is given by



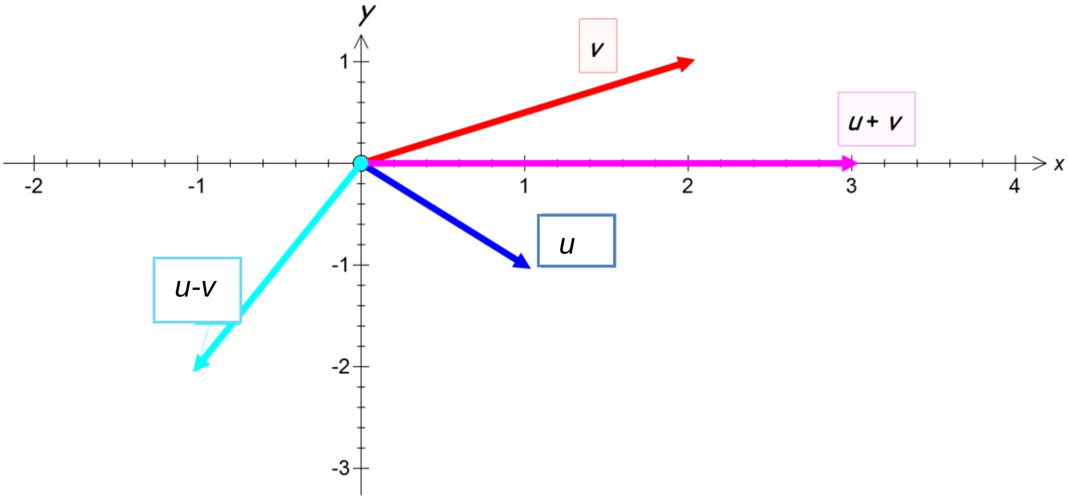
(d) The vector $-\frac{1}{2}\mathbf{b}$ is given by



(e) The vector $3\mathbf{a}-\frac{1}{2}\mathbf{b}$ is given by



2. The vectors $\mathbf{u}$, $\mathbf{v}$, $\mathbf{u}+\mathbf{v}$ and $\mathbf{u}-\mathbf{v}$ in $\mathbb{R}^2$ are



(e) The dot product is

$$\mathbf{u}\cdot\mathbf{v}=\begin{pmatrix}1\\-1\end{pmatrix}\cdot\begin{pmatrix}2\\1\end{pmatrix}=(1\times 2)+(-1\times 1)=1$$

(f) We have

$$\mathbf{v}\cdot\mathbf{u}=\begin{pmatrix}2\\1\end{pmatrix}\cdot\begin{pmatrix}1\\-1\end{pmatrix}=(2\times 1)+(1\times(-1))=1$$

(g) Similarly we have

$$\mathbf{u}\cdot\mathbf{u}=\begin{pmatrix}1\\-1\end{pmatrix}\cdot\begin{pmatrix}1\\-1\end{pmatrix}=(1\times 1)+(-1\times(-1))=2$$

(h) Also $\mathbf{v}\cdot\mathbf{v}=\begin{pmatrix}2\\1\end{pmatrix}\cdot\begin{pmatrix}2\\1\end{pmatrix}=5$.

3. With vectors in $\mathbb{R}^3$ we evaluate the dot product in the same way.

(a) We have $\mathbf{u}\cdot\mathbf{v}=\begin{pmatrix}2\\3\\-1\end{pmatrix}\cdot\begin{pmatrix}5\\1\\-2\end{pmatrix}=(2\times 5)+(3\times 1)+(-1\times(-2))=15$.

(b) Similarly

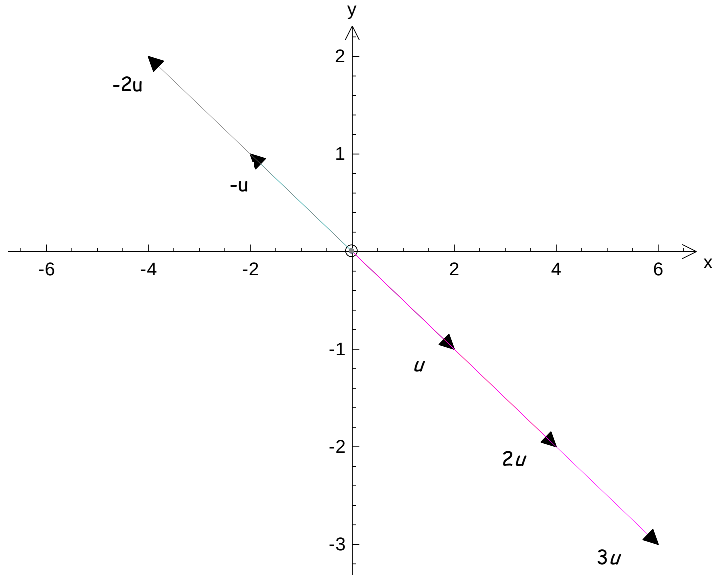
$$\mathbf{v}\cdot\mathbf{u}=\begin{pmatrix}5\\1\\-2\end{pmatrix}\cdot\begin{pmatrix}2\\3\\-1\end{pmatrix}=(5\times 2)+(1\times 3)+(-2\times(-1))=15$$

(c) We have

$$\mathbf{u}\cdot\mathbf{u}=\begin{pmatrix}2\\3\\-1\end{pmatrix}\cdot\begin{pmatrix}2\\3\\-1\end{pmatrix}=2^2+3^2+(-1)^2=14$$

(d) Also $\mathbf{v}\cdot\mathbf{v}=\begin{pmatrix}5\\1\\-2\end{pmatrix}\cdot\begin{pmatrix}5\\1\\-2\end{pmatrix}=5^2+1^2+(-2)^2=30$.

4. The vectors $\mathbf{u},\ -\mathbf{u},\ 2\mathbf{u},\ 3\mathbf{u}$ and $-2\mathbf{u}$ in $\mathbb{R}^2$ are



5. We need to find

$$\mathbf{w}=\begin{pmatrix}1\\ \lambda\end{pmatrix}\cdot\begin{pmatrix}\mathbf{u}\\ \mathbf{v}\end{pmatrix}=(1)\mathbf{u}+\lambda\mathbf{v}=\mathbf{u}+\lambda\mathbf{v}$$

where $\mathbf{u}=\begin{pmatrix}-1\\ 1\end{pmatrix}$ and $\mathbf{v}=\begin{pmatrix}3\\ -1\end{pmatrix}$.

(a) For $\lambda=1$ we have

$$\mathbf{w}=\begin{pmatrix}-1\\ 1\end{pmatrix}+1\begin{pmatrix}3\\ -1\end{pmatrix}=\begin{pmatrix}-1+3\\ 1-1\end{pmatrix}=\begin{pmatrix}2\\ 0\end{pmatrix}$$

(b) For $\lambda=-1$ we have

$$\mathbf{w}=\begin{pmatrix}-1\\ 1\end{pmatrix}-1\begin{pmatrix}3\\ -1\end{pmatrix}=\begin{pmatrix}-1-3\\ 1-(-1)\end{pmatrix}=\begin{pmatrix}-4\\ 2\end{pmatrix}$$

(c) For $\lambda=\frac{1}{2}$ we have

$$\mathbf{w}=\begin{pmatrix}-1\\ 1\end{pmatrix}+\frac{1}{2}\begin{pmatrix}3\\ -1\end{pmatrix}=\begin{pmatrix}-1\\ 1\end{pmatrix}+\begin{pmatrix}3/2\\ -1/2\end{pmatrix}$$

$$=\begin{pmatrix}-1+3/2\\ 1-1/2\end{pmatrix}=\begin{pmatrix}1/2\\ 1/2\end{pmatrix}$$

(d) For $\lambda=-\frac{1}{2}$ we have

$$\mathbf{w}=\begin{pmatrix}-1\\ 1\end{pmatrix}-\frac{1}{2}\begin{pmatrix}3\\ -1\end{pmatrix}=\begin{pmatrix}-1\\ 1\end{pmatrix}-\begin{pmatrix}3/2\\ -1/2\end{pmatrix}$$

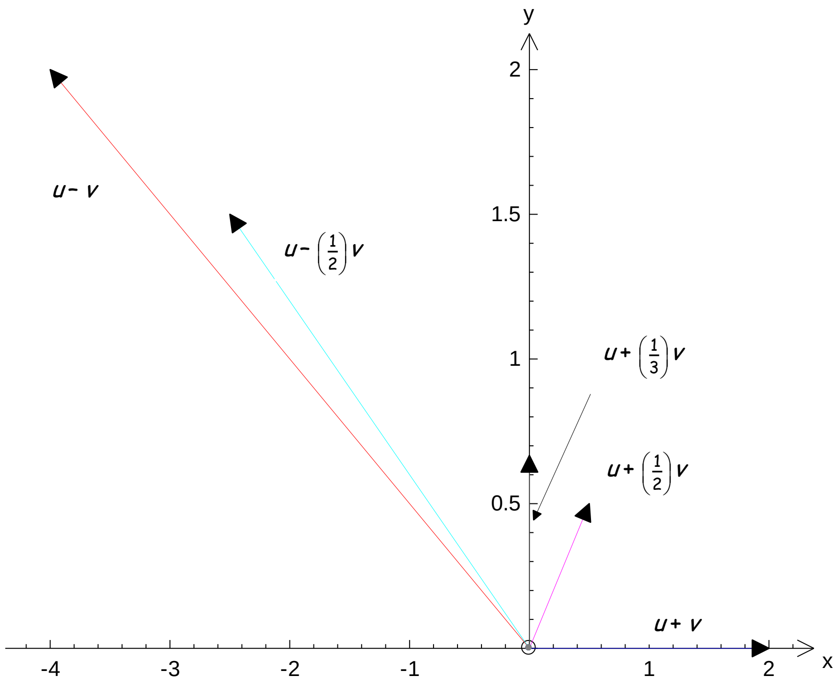
$$=\begin{pmatrix}-1-3/2\\ 1+1/2\end{pmatrix}=\begin{pmatrix}-5/2\\ 3/2\end{pmatrix}$$

(e) For $\lambda=\frac{1}{3}$ we have

$$\mathbf{w} = \begin{pmatrix} -1 \\ 1 \end{pmatrix} + \frac{1}{3}\begin{pmatrix} 3 \\ -1 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \end{pmatrix} + \begin{pmatrix} 3/3 \\ -1/3 \end{pmatrix}$$

$$= \begin{pmatrix} -1+1 \\ 1-1/3 \end{pmatrix} = \begin{pmatrix} 0 \\ 2/3 \end{pmatrix}$$

Plotting these in 2 space, $\mathbb{R}^2$, gives



6. We need to find $\mathbf{w} = \begin{pmatrix} k \\ c \end{pmatrix} \cdot \begin{pmatrix} \mathbf{u} \\ \mathbf{v} \end{pmatrix} = k\mathbf{u} + c\mathbf{v}$ for the following values of k and c:

(a) For $k = 1,\ c = 1$ we have

$$\mathbf{w} = 1\begin{pmatrix} -2 \\ 2 \end{pmatrix} + 1\begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} -2+2 \\ 2+1 \end{pmatrix} = \begin{pmatrix} 0 \\ 3 \end{pmatrix}$$

(b) For $k = \frac{1}{2},\ c = \frac{1}{2}$ we have

$$\mathbf{w} = \frac{1}{2}\begin{pmatrix} -2 \\ 2 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \end{pmatrix} + \begin{pmatrix} 1 \\ 1/2 \end{pmatrix} = \begin{pmatrix} 0 \\ 3/2 \end{pmatrix}$$

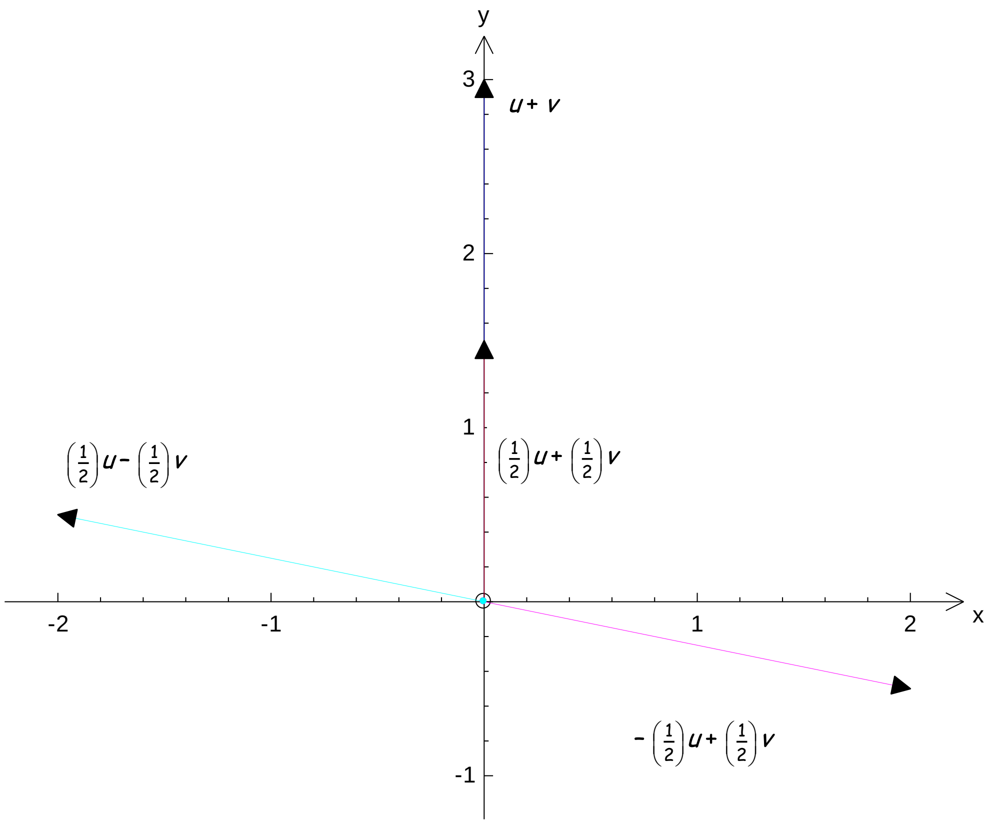
(c) For $k = -\frac{1}{2},\ c = \frac{1}{2}$ we have

$$\mathbf{w} = -\frac{1}{2}\begin{pmatrix} -2 \\ 2 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix} + \begin{pmatrix} 1 \\ 1/2 \end{pmatrix} = \begin{pmatrix} 2 \\ -1/2 \end{pmatrix}$$

(d) For $k = \frac{1}{2},\ c = -\frac{1}{2}$ we have

$$\mathbf{w} = \frac{1}{2}\begin{pmatrix} -2 \\ 2 \end{pmatrix} - \frac{1}{2}\begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \end{pmatrix} - \begin{pmatrix} 1 \\ 1/2 \end{pmatrix} = \begin{pmatrix} -2 \\ 1/2 \end{pmatrix}$$

Plotting these in $\mathbb{R}^2$ gives



7. We have $\mathbf{u}+\mathbf{v}=\mathbf{O}$ where $\mathbf{u}=\begin{pmatrix} x+3 \\ y-2 \end{pmatrix}$ and $\mathbf{v}=\begin{pmatrix} x-2 \\ y+11 \end{pmatrix}$. Substituting these in gives

$$\mathbf{u}+\mathbf{v}=\begin{pmatrix} x+3 \\ y-2 \end{pmatrix}+\begin{pmatrix} x-2 \\ y+11 \end{pmatrix}$$
$$=\begin{pmatrix} x+3+x-2 \\ y-2+y+11 \end{pmatrix}=\begin{pmatrix} 2x+1 \\ 2y+9 \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

From this we have

$$2x+1=0 \text{ which gives } x=-\frac{1}{2}$$
$$2y+9=0 \text{ which gives } y=-\frac{9}{2}$$

8. Substituting $\mathbf{u}=\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\mathbf{v}=\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ into $x\mathbf{u}+y\mathbf{v}$ gives

$$x\mathbf{u}+y\mathbf{v}=x\begin{pmatrix} 1 \\ 0 \end{pmatrix}+y\begin{pmatrix} 0 \\ 1 \end{pmatrix}$$
$$=\begin{pmatrix} x\times 1 \\ x\times 0 \end{pmatrix}+\begin{pmatrix} y\times 0 \\ y\times 1 \end{pmatrix}$$
$$=\begin{pmatrix} x \\ 0 \end{pmatrix}+\begin{pmatrix} 0 \\ y \end{pmatrix}=\begin{pmatrix} x+0 \\ 0+y \end{pmatrix}=\begin{pmatrix} x \\ y \end{pmatrix}=\mathbf{w}$$

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9. We are given that $\mathbf{u}=\begin{pmatrix}-3\\5\\8\end{pmatrix}$ and $\mathbf{v}=\begin{pmatrix}7\\-1\\2\end{pmatrix}$.

(a) The vector addition $\mathbf{u}+\mathbf{v}$ is given by

$$\mathbf{u}+\mathbf{v}=\begin{pmatrix}-3\\5\\8\end{pmatrix}+\begin{pmatrix}7\\-1\\2\end{pmatrix}=\begin{pmatrix}-3+7\\5-1\\8+2\end{pmatrix}=\begin{pmatrix}4\\4\\10\end{pmatrix}$$

(b) The scalar multiplication $5\mathbf{u}$ is given by

$$5\mathbf{u}=5\begin{pmatrix}-3\\5\\8\end{pmatrix}=\begin{pmatrix}-3\times 5\\5\times 5\\8\times 5\end{pmatrix}=\begin{pmatrix}-15\\25\\40\end{pmatrix}$$

(c) The vector addition and scalar multiplication is given by

$$2\mathbf{u}+6\mathbf{v}=2\begin{pmatrix}-3\\5\\8\end{pmatrix}+6\begin{pmatrix}7\\-1\\2\end{pmatrix}$$

$$=\begin{pmatrix}-3\times 2\\5\times 2\\8\times 2\end{pmatrix}+\begin{pmatrix}7\times 6\\-1\times 6\\2\times 6\end{pmatrix}=\begin{pmatrix}-6\\10\\16\end{pmatrix}+\begin{pmatrix}42\\-6\\12\end{pmatrix}=\begin{pmatrix}-6+42\\10-6\\16+12\end{pmatrix}=\begin{pmatrix}36\\4\\28\end{pmatrix}$$

(d) Similarly we have

$$\mathbf{u}-3\mathbf{v}=\begin{pmatrix}-3\\5\\8\end{pmatrix}-3\begin{pmatrix}7\\-1\\2\end{pmatrix}=\begin{pmatrix}-3\\5\\8\end{pmatrix}-\begin{pmatrix}7\times 3\\-1\times 3\\2\times 3\end{pmatrix}$$

$$=\begin{pmatrix}-3\\5\\8\end{pmatrix}-\begin{pmatrix}21\\-3\\6\end{pmatrix}=\begin{pmatrix}-3-21\\5-(-3)\\8-6\end{pmatrix}=\begin{pmatrix}-24\\8\\2\end{pmatrix}$$

(e) Similarly

$$-5\mathbf{u}-4\mathbf{v}=-5\begin{pmatrix}-3\\5\\8\end{pmatrix}-4\begin{pmatrix}7\\-1\\2\end{pmatrix}$$

$$=\begin{pmatrix}15\\-25\\-40\end{pmatrix}-\begin{pmatrix}28\\-4\\8\end{pmatrix}=\begin{pmatrix}15-28\\-25+4\\-40-8\end{pmatrix}=\begin{pmatrix}-13\\-21\\-48\end{pmatrix}$$

10. We are given $\mathbf{u}=\begin{pmatrix}-9\\2\\4\end{pmatrix}$, $\mathbf{v}=\begin{pmatrix}-2\\1\\3\end{pmatrix}$ and $\mathbf{w}=\begin{pmatrix}1\\-2\\5\end{pmatrix}$.

(a) The vector addition $\mathbf{u}+\mathbf{v}+\mathbf{w}$ is given by

$$\mathbf{u}+\mathbf{v}+\mathbf{w}=\begin{pmatrix}-9\\2\\4\end{pmatrix}+\begin{pmatrix}-2\\1\\3\end{pmatrix}+\begin{pmatrix}1\\-2\\5\end{pmatrix}=\begin{pmatrix}-9-2+1\\2+1-2\\4+3+5\end{pmatrix}=\begin{pmatrix}-10\\1\\12\end{pmatrix}$$

(b) The vector addition $\mathbf{u}-\mathbf{v}-\mathbf{w}$ is given by

$$\mathbf{u}-\mathbf{v}-\mathbf{w}=\begin{pmatrix}-9\\2\\4\end{pmatrix}-\begin{pmatrix}-2\\1\\3\end{pmatrix}-\begin{pmatrix}1\\-2\\5\end{pmatrix}=\begin{pmatrix}-9-(-2)-1\\2-1-(-2)\\4-3-5\end{pmatrix}=\begin{pmatrix}-8\\3\\-4\end{pmatrix}$$

(c) Similarly we have

$$2\mathbf{u}+\mathbf{v}-\mathbf{w}=2\begin{pmatrix}-9\\2\\4\end{pmatrix}+\begin{pmatrix}-2\\1\\3\end{pmatrix}-\begin{pmatrix}1\\-2\\5\end{pmatrix}$$

$$=\begin{pmatrix}(-9\times 2)-2-1\\(2\times 2)+1+2\\(4\times 2)+3-5\end{pmatrix}=\begin{pmatrix}-21\\7\\6\end{pmatrix}$$

(d) Also we have

$$-2\mathbf{u}+3\mathbf{v}+5\mathbf{w}=-2\begin{pmatrix}-9\\2\\4\end{pmatrix}+3\begin{pmatrix}-2\\1\\3\end{pmatrix}+5\begin{pmatrix}1\\-2\\5\end{pmatrix}$$

$$=\begin{pmatrix}-2\times(-9)\\-2\times 2\\-2\times 4\end{pmatrix}+\begin{pmatrix}-2\times 3\\1\times 3\\3\times 3\end{pmatrix}+\begin{pmatrix}1\times 5\\-2\times 5\\5\times 5\end{pmatrix}$$

$$=\begin{pmatrix}18\\-4\\-8\end{pmatrix}+\begin{pmatrix}-6\\3\\9\end{pmatrix}+\begin{pmatrix}5\\-10\\25\end{pmatrix}=\begin{pmatrix}18-6+5\\-4+3-10\\-8+9+25\end{pmatrix}=\begin{pmatrix}17\\-11\\26\end{pmatrix}$$

11. Substituting the given vectors $\mathbf{u}=\begin{pmatrix}1\\0\\0\end{pmatrix}$, $\mathbf{v}=\begin{pmatrix}0\\1\\0\end{pmatrix}$ and $\mathbf{w}=\begin{pmatrix}0\\0\\1\end{pmatrix}$ into $x\mathbf{u}+y\mathbf{v}+z\mathbf{w}$ gives

$$x\mathbf{u}+y\mathbf{v}+z\mathbf{w}=x\begin{pmatrix}1\\0\\0\end{pmatrix}+y\begin{pmatrix}0\\1\\0\end{pmatrix}+z\begin{pmatrix}0\\0\\1\end{pmatrix}$$

$$=\begin{pmatrix}x\\0\\0\end{pmatrix}+\begin{pmatrix}0\\y\\0\end{pmatrix}+\begin{pmatrix}0\\0\\z\end{pmatrix}=\begin{pmatrix}x\\y\\z\end{pmatrix}=\mathbf{x}$$

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12. We have

$$x\begin{pmatrix}1\\2\\0\end{pmatrix}+y\begin{pmatrix}0\\1\\-1\end{pmatrix}+z\begin{pmatrix}-2\\0\\6\end{pmatrix}=\begin{pmatrix}x\\2x\\0\end{pmatrix}+\begin{pmatrix}0\\y\\-y\end{pmatrix}+\begin{pmatrix}-2z\\0\\6z\end{pmatrix}$$
$$=\begin{pmatrix}x+0-2z\\2x+y+0\\0-y+6z\end{pmatrix}=\begin{pmatrix}5\\3\\17\end{pmatrix}$$

We need to solve the linear equations:

$$\begin{aligned}x-2z&=5\\2x+\ \ y&=3\\-y+6z&=17\end{aligned}$$

From the first equation we have $x=5+2z$ and substituting this into the second equation gives:

$$\begin{aligned}2(5+2z)+y&=3\\10+4z+y&=3\\4z+y&=-7\end{aligned}$$

Adding $-y+6z=17$ and $4z+y=-7$ gives

$$\begin{aligned}-y+6z&=17\\y+4z&=-7\\\hline 0+10z&=10\end{aligned}$$

Hence $z=1$ and substituting this into the last given equation, $-y+6z=17$, gives

$$\begin{aligned}-y+6&=17\\-y&=17-6=11\quad\text{gives }y=-11\end{aligned}$$

Also substituting $z=1$ into the first given equation we have

$$\begin{aligned}x-2&=5\\x&=7\end{aligned}$$

Hence the solution is $x=7$, $y=-11$ and $z=1$.

13. We are given $\mathbf{u}=\begin{pmatrix}-1\\3\\2\\0\end{pmatrix}$, $\mathbf{v}=\begin{pmatrix}3\\-2\\5\\1\end{pmatrix}$, $\mathbf{w}=\begin{pmatrix}0\\-1\\1\\2\end{pmatrix}$ and $\mathbf{x}=\begin{pmatrix}x\\y\\z\\a\end{pmatrix}$.

(a) Vector addition

$$\mathbf{u}+\mathbf{v}+\mathbf{w}=\begin{pmatrix}-1\\3\\2\\0\end{pmatrix}+\begin{pmatrix}3\\-2\\5\\1\end{pmatrix}+\begin{pmatrix}0\\-1\\1\\2\end{pmatrix}=\begin{pmatrix}-1+3+0\\3-2-1\\2+5+1\\0+1+2\end{pmatrix}=\begin{pmatrix}2\\0\\8\\3\end{pmatrix}$$

(b) We have

$$\mathbf{u}-\mathbf{v}-\mathbf{w}=\begin{pmatrix}-1\\3\\2\\0\end{pmatrix}-\begin{pmatrix}3\\-2\\5\\1\end{pmatrix}-\begin{pmatrix}0\\-1\\1\\2\end{pmatrix}=\begin{pmatrix}-1-3-0\\3-(-2)-(-1)\\2-5-1\\0-1-2\end{pmatrix}=\begin{pmatrix}-4\\6\\-4\\-3\end{pmatrix}$$

(c) We have

$$\mathbf{u}-2\mathbf{v}+3\mathbf{w}=\begin{pmatrix}-1\\3\\2\\0\end{pmatrix}-2\begin{pmatrix}3\\-2\\5\\1\end{pmatrix}+3\begin{pmatrix}0\\-1\\1\\2\end{pmatrix}$$

$$=\begin{pmatrix}-1\\3\\2\\0\end{pmatrix}-\begin{pmatrix}6\\-4\\10\\2\end{pmatrix}+\begin{pmatrix}0\\-3\\3\\6\end{pmatrix}=\begin{pmatrix}-1-6\\3-(-4)-3\\2-10+3\\0-2+6\end{pmatrix}=\begin{pmatrix}-7\\4\\-5\\4\end{pmatrix}$$

(d) We have

$$\mathbf{u}-3\mathbf{w}+\mathbf{x}=\begin{pmatrix}-1\\3\\2\\0\end{pmatrix}-3\begin{pmatrix}0\\-1\\1\\2\end{pmatrix}+\begin{pmatrix}x\\y\\z\\a\end{pmatrix}$$

$$=\begin{pmatrix}-1\\3\\2\\0\end{pmatrix}-\begin{pmatrix}0\\-3\\3\\6\end{pmatrix}+\begin{pmatrix}x\\y\\z\\a\end{pmatrix}=\begin{pmatrix}-1+x\\3-(-3)+y\\2-3+z\\0-6+a\end{pmatrix}=\begin{pmatrix}x-1\\y+6\\z-1\\a-6\end{pmatrix}$$

Next we need to find the values of x, y, z and a in the vector $\mathbf{x}$ so that it satisfies $\mathbf{u}+\mathbf{v}+\mathbf{w}+\mathbf{x}=\mathbf{O}$:

$$\mathbf{u}+\mathbf{v}+\mathbf{w}+\mathbf{x}=\begin{pmatrix}-1\\3\\2\\0\end{pmatrix}+\begin{pmatrix}3\\-2\\5\\1\end{pmatrix}+\begin{pmatrix}0\\-1\\1\\2\end{pmatrix}+\begin{pmatrix}x\\y\\z\\a\end{pmatrix}$$

$$=\begin{pmatrix}-1+3+0+x\\3-2-1+y\\2+5+1+z\\0+1+2+a\end{pmatrix}=\begin{pmatrix}x+2\\y\\z+8\\a+3\end{pmatrix}=\begin{pmatrix}0\\0\\0\\0\end{pmatrix}$$

Equating this we have

$$\begin{aligned}x+2=0 &\quad \text{gives} \quad x=-2\\ y=0 &\\ z+8=0 &\quad \text{gives} \quad z=-8\\ a+3=0 &\quad \text{gives} \quad a=-3\end{aligned}$$

Our solution is $x=-2$, $y=0$, $z=-8$ and $a=-3$.

14. Expanding $x_1\mathbf{e}_1+x_2\mathbf{e}_2+\cdots x_k\mathbf{e}_k+\cdots+x_n\mathbf{e}_n$ gives

$$x_1\mathbf{e}_1 + x_2\mathbf{e}_2 + \cdots + x_k\mathbf{e}_k + \cdots + x_n\mathbf{e}_n = x_1\begin{pmatrix}1\\0\\0\\\vdots\\0\end{pmatrix} + x_2\begin{pmatrix}0\\1\\0\\\vdots\\0\end{pmatrix} + \cdots + x_k\begin{pmatrix}0\\\vdots\\1\\0\\\vdots\end{pmatrix} + \cdots + x_n\begin{pmatrix}0\\0\\0\\\vdots\\1\end{pmatrix}$$

$$= \begin{pmatrix}x_1\\0\\0\\\vdots\\0\end{pmatrix} + \begin{pmatrix}0\\x_2\\0\\\vdots\\0\end{pmatrix} + \cdots + x_k\begin{pmatrix}0\\\vdots\\1\\0\\\vdots\end{pmatrix} + \cdots + \begin{pmatrix}0\\0\\0\\\vdots\\x_n\end{pmatrix} = \begin{pmatrix}x_1\\\vdots\\x_k\\\vdots\\x_n\end{pmatrix} = \mathbf{u}$$

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15. Let $\mathbf{u} = \begin{pmatrix}-2\\1\end{pmatrix}$ and $\mathbf{v} = \begin{pmatrix}1\\2\end{pmatrix}$ be vectors in $\mathbb{R}^2$. Then

$$\mathbf{u}\cdot\mathbf{v} = \begin{pmatrix}-2\\1\end{pmatrix}\cdot\begin{pmatrix}1\\2\end{pmatrix} = -2 + 2 = 0$$

Here we have two **non**-zero vectors $\mathbf{u} = \begin{pmatrix}-2\\1\end{pmatrix}$ and $\mathbf{v} = \begin{pmatrix}1\\2\end{pmatrix}$ but $\mathbf{u}\cdot\mathbf{v} = 0$.