Complete Solutions to Exercises 7.3

1. (a) (i) We are given the matrix $\mathbf{A}=\begin{pmatrix}1&0\\0&2\end{pmatrix}$ for which we need to find the eigenvalues and corresponding eigenvectors. Since we have a diagonal matrix therefore by Proposition (7-3) the eigenvalues are the entries along the leading diagonal, $\lambda_1=1$ and $\lambda_2=2$.

For the eigenvalue $\lambda_1=1$:

$$\begin{pmatrix}1-1&0\\0&2-1\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix}$$

$$\begin{pmatrix}0&0\\0&1\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix}\quad\text{gives } x=1 \text{ and } y=0$$

Our eigenvector belonging to $\lambda_1=1$ is $\mathbf{u}=\begin{pmatrix}1\\0\end{pmatrix}$. The eigenvector for the other eigenvalue $\lambda_2=2$ is

$$\begin{pmatrix}1-2&0\\0&2-2\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix}$$

$$\begin{pmatrix}-1&0\\0&0\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix}\quad\text{gives } x=0 \text{ and } y=1$$

The eigenvector corresponding to $\lambda_2=2$ is $\mathbf{v}=\begin{pmatrix}0\\1\end{pmatrix}$.

(ii) To find the invertible matrix **P** we need to follow the procedure outlined in the main text.

<u>Step 1:</u>

Need to check that the eigenvectors are linearly independent. Since $\mathbf{u}=\begin{pmatrix}1\\0\end{pmatrix}$ and $\mathbf{v}=\begin{pmatrix}0\\1\end{pmatrix}$ are **not** multiplies of each other therefore eigenvectors **u** and **v** are linearly independent.

<u>Step 2:</u>

The matrix **P** is given by

$$\mathbf{P}=(\mathbf{u}\ \vdots\ \mathbf{v})=\begin{pmatrix}1&0\\0&1\end{pmatrix}=\mathbf{I}$$

Note that the matrix **P** is the identity matrix **I**.

<u>Step 3:</u>

The diagonal matrix **D** is given by $\mathbf{D}=\mathbf{P}^{-1}\mathbf{AP}$:

$$\mathbf{D}=\mathbf{I}^{-1}\mathbf{AI}=\mathbf{A}=\begin{pmatrix}1&0\\0&2\end{pmatrix}$$

Note that **D** is a diagonal matrix with **eigenvalues** $\lambda_1=1$ and $\lambda_2=2$ as the entries along the leading diagonal.

(b) (i) We need to find the eigenvalues and corresponding eigenvectors of $\mathbf{A}=\begin{pmatrix}1&1\\1&1\end{pmatrix}$:

Proposition (7-3). If matrix **A** is a diagonal or triangular matrix then the eigenvalues of **A** are the entries along the leading diagonal.

$$\det(\mathbf{A}-\lambda\mathbf{I}) = \det\begin{pmatrix}1-\lambda & 1\\ 1 & 1-\lambda\end{pmatrix}$$
$$= (1-\lambda)(1-\lambda)-1$$
$$= 1-2\lambda+\lambda^2-1$$
$$= \lambda^2-2\lambda = \lambda(\lambda-2) = 0 \text{ gives } \lambda_1 = 0 \text{ and } \lambda_2 = 2$$

For the eigenvalue $\lambda_1 = 0$ we can find the eigenvector by:

$$\begin{pmatrix}1-0 & 1\\ 1 & 1-0\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix} = \begin{pmatrix}1 & 1\\ 1 & 1\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix} = \begin{pmatrix}0\\ 0\end{pmatrix} \text{ gives } x+y=0$$

Thus $x=-y$ and let $y=1$ then $x=-1$. The corresponding eigenvector is $\mathbf{u} = \begin{pmatrix}-1\\ 1\end{pmatrix}$.

Similarly we can find the eigenvector **v** belonging to the other eigenvalue $\lambda_2 = 2$:

$$\begin{pmatrix}1-2 & 1\\ 1 & 1-2\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix} = \begin{pmatrix}-1 & 1\\ 1 & -1\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix} = \begin{pmatrix}0\\ 0\end{pmatrix} \text{ gives } x=1 \text{ and } y=1$$

Thus $\mathbf{v} = \begin{pmatrix}1\\ 1\end{pmatrix}$ belongs to the eigenvalue $\lambda_2 = 2$.

(ii) Step 1:

Need to check that $\mathbf{u} = \begin{pmatrix}-1\\ 1\end{pmatrix}$ and $\mathbf{v} = \begin{pmatrix}1\\ 1\end{pmatrix}$ are linearly independent. Since one vector is **not** a multiple of the other therefore they are linearly independent.

Step 2:

Let $\mathbf{P} = (\mathbf{u} \vdots \mathbf{v}) = \begin{pmatrix}-1 & 1\\ 1 & 1\end{pmatrix}$.

Step 3:

Then taking the inverse of this matrix **P** we get $\mathbf{P}^{-1} = \frac{1}{2}\begin{pmatrix}-1 & 1\\ 1 & 1\end{pmatrix}$. Thus $\mathbf{D} = \mathbf{P}^{-1}\mathbf{AP}$ is

$$\mathbf{D} = \mathbf{P}^{-1}\mathbf{AP} = \frac{1}{2}\begin{pmatrix}-1 & 1\\ 1 & 1\end{pmatrix}\begin{pmatrix}1 & 1\\ 1 & 1\end{pmatrix}\begin{pmatrix}-1 & 1\\ 1 & 1\end{pmatrix}$$
$$= \frac{1}{2}\begin{pmatrix}0 & 0\\ 2 & 2\end{pmatrix}\begin{pmatrix}-1 & 1\\ 1 & 1\end{pmatrix} \qquad \left[\begin{array}{l}\text{Multiplying the}\\ \text{two Left Hand matrices}\end{array}\right]$$
$$= \frac{1}{2}\begin{pmatrix}0 & 0\\ 0 & 4\end{pmatrix} = \begin{pmatrix}0 & 0\\ 0 & 2\end{pmatrix} \qquad \left[\text{Multiplying by scalar } \frac{1}{2}\right]$$

Again the leading diagonal entries are 0 and 2 which are the eigenvalues of the given matrix **A**.

(c) (i) The eigenvalues of $\mathbf{A} = \begin{pmatrix}3 & 0\\ 4 & 4\end{pmatrix}$ are $\lambda_1 = 3$ and $\lambda_2 = 4$ because by Proposition (7-3) the entries along the leading diagonal are the eigenvalues of a triangular matrix and **A** is a triangular matrix.

Proposition (7-3). If matrix **A** is a diagonal or triangular matrix then the eigenvalues of **A** are the entries along the leading diagonal.

The eigenvector **u** belonging to $\lambda_1 = 3$ is given by

$$\begin{pmatrix} 3-3 & 0 \\ 4 & 4-3 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 4 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \text{ gives } x = 1 \text{ and } y = -4$$

Hence $\mathbf{u} = \begin{pmatrix} 1 \\ -4 \end{pmatrix}$. The eigenvector **v** corresponding to $\lambda_2 = 4$ is given by

$$\begin{pmatrix} 3-4 & 0 \\ 4 & 4-4 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 4 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \text{ gives } x = 0 \text{ and } y = 1$$

Thus $\mathbf{v} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ for $\lambda_2 = 4$.

(ii) Step 1:

The eigenvectors $\mathbf{u} = \begin{pmatrix} 1 \\ -4 \end{pmatrix}$ and $\mathbf{v} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ are linearly independent.

Step 2:

Let $\mathbf{P} = (\mathbf{u} \vdots \mathbf{v}) = \begin{pmatrix} 1 & 0 \\ -4 & 1 \end{pmatrix}$.

Step 3:

Then taking the inverse of this matrix **P** we get $\mathbf{P}^{-1} = \begin{pmatrix} 1 & 0 \\ 4 & 1 \end{pmatrix}$. Thus $\mathbf{D} = \mathbf{P}^{-1}\mathbf{A}\mathbf{P}$ is

$$\begin{aligned} \mathbf{D} = \mathbf{P}^{-1}\mathbf{A}\mathbf{P} &= \begin{pmatrix} 1 & 0 \\ 4 & 1 \end{pmatrix}\begin{pmatrix} 3 & 0 \\ 4 & 4 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ -4 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 3 & 0 \\ 16 & 4 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ -4 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 3 & 0 \\ 0 & 4 \end{pmatrix} \end{aligned}$$

Again the leading diagonal entries are 3 and 4 which are the eigenvalues of matrix **A**.

(d) (i) We are given the matrix $\mathbf{A} = \begin{pmatrix} 2 & 2 \\ 1 & 3 \end{pmatrix}$. The eigenvalues are

$$\begin{aligned} \det(\mathbf{A} - \lambda\mathbf{I}) &= \det\begin{pmatrix} 2-\lambda & 2 \\ 1 & 3-\lambda \end{pmatrix} \\ &= (2-\lambda)(3-\lambda) - 2 \\ &= \lambda^2 - 5\lambda + 4 = (\lambda-1)(\lambda-4) = 0 \text{ gives } \lambda_1 = 1 \text{ and } \lambda_2 = 4 \end{aligned}$$

The eigenvector **u** belonging to $\lambda_1 = 1$ is evaluated by

$$\begin{pmatrix} 2-1 & 2 \\ 1 & 3-1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \text{ gives } x = 2 \text{ and } y = -1$$

Hence $\mathbf{u} = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$. The eigenvector **v** corresponding to $\lambda_2 = 4$ is given by

$$\begin{pmatrix} 2-4 & 2 \\ 1 & 3-4 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -2 & 2 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \text{ gives } x = 1 \text{ and } y = 1$$

Thus the eigenvector $\mathbf{v}=\begin{pmatrix}1\\1\end{pmatrix}$ belongs to the eigenvalue $\lambda_2=4$.

(ii) Step 1:

The eigenvectors $\mathbf{u}=\begin{pmatrix}2\\-1\end{pmatrix}$ and $\mathbf{v}=\begin{pmatrix}1\\1\end{pmatrix}$ are linearly independent because they are **not** multiples of each other.

Step 2:

Let $\mathbf{P}=(\mathbf{u}\ \vdots\ \mathbf{v})=\begin{pmatrix}2&1\\-1&1\end{pmatrix}$.

Step 3:

Then taking the inverse of this matrix **P** we get $\mathbf{P}^{-1}=\frac{1}{3}\begin{pmatrix}1&-1\\1&2\end{pmatrix}$. Substituting these into $\mathbf{P}^{-1}\mathbf{AP}$ gives

$$\begin{aligned}\mathbf{P}^{-1}\mathbf{AP}&=\frac{1}{3}\begin{pmatrix}1&-1\\1&2\end{pmatrix}\begin{pmatrix}2&2\\1&3\end{pmatrix}\begin{pmatrix}2&1\\-1&1\end{pmatrix}\\&=\frac{1}{3}\begin{pmatrix}1&-1\\4&8\end{pmatrix}\begin{pmatrix}2&1\\-1&1\end{pmatrix}\\&=\frac{1}{3}\begin{pmatrix}3&0\\0&12\end{pmatrix}=\begin{pmatrix}1&0\\0&4\end{pmatrix}\end{aligned}$$

Again the leading diagonal entries are 1 and 4 which are the eigenvalues of matrix **A**.

2. (a) By Proposition (7-14) we have $\mathbf{A}^m=\mathbf{PD}^m\mathbf{P}^{-1}$ and we use this to find $\mathbf{A}^5$.

(a) We are given $\mathbf{A}=\begin{pmatrix}1&0\\0&2\end{pmatrix}$ and from question 1 part (a) we have $\mathbf{P}=\mathbf{P}^{-1}=\begin{pmatrix}1&0\\0&1\end{pmatrix}=\mathbf{I}$ and $\mathbf{D}=\mathbf{A}=\begin{pmatrix}1&0\\0&2\end{pmatrix}$. Therefore

$$\mathbf{A}^5=\mathbf{PD}^5\mathbf{P}^{-1}=\mathbf{I}\begin{pmatrix}1^5&0\\0&2^5\end{pmatrix}\mathbf{I}=\begin{pmatrix}1&0\\0&32\end{pmatrix}\qquad\begin{bmatrix}\text{Because } 1^5=1\\ \text{and } 2^5=32\end{bmatrix}$$

(b) We are given $\mathbf{A}=\begin{pmatrix}1&1\\1&1\end{pmatrix}$ and from question 1 part (b) we have $\mathbf{P}=\begin{pmatrix}-1&1\\1&1\end{pmatrix}$, $\mathbf{P}^{-1}=\frac{1}{2}\begin{pmatrix}-1&1\\1&1\end{pmatrix}$ and $\mathbf{D}=\begin{pmatrix}0&0\\0&2\end{pmatrix}$. Thus

$$\begin{aligned}\mathbf{A}^5&=\mathbf{PD}^5\mathbf{P}^{-1}\\&=\begin{pmatrix}-1&1\\1&1\end{pmatrix}\begin{pmatrix}0&0\\0&2\end{pmatrix}^5\frac{1}{2}\begin{pmatrix}-1&1\\1&1\end{pmatrix}\\&=\frac{1}{2}\begin{pmatrix}-1&1\\1&1\end{pmatrix}\begin{pmatrix}0&0\\0&32\end{pmatrix}\begin{pmatrix}-1&1\\1&1\end{pmatrix}\qquad\begin{bmatrix}\text{Taking the scalar}\\ \text{1/2 to the front}\end{bmatrix}\\&=\frac{1}{2}\begin{pmatrix}0&32\\0&32\end{pmatrix}\begin{pmatrix}-1&1\\1&1\end{pmatrix}=\frac{1}{2}\begin{pmatrix}32&32\\32&32\end{pmatrix}=\begin{pmatrix}16&16\\16&16\end{pmatrix}\end{aligned}$$

(c) We need to find $\mathbf{A}^5$ for $\mathbf{A}=\begin{pmatrix}3&0\\4&4\end{pmatrix}$. From question 1 part (c) we have

$\mathbf{P}=\begin{pmatrix}1&0\\-4&1\end{pmatrix}$, $\mathbf{P}^{-1}=\begin{pmatrix}1&0\\4&1\end{pmatrix}$ and $\mathbf{D}=\begin{pmatrix}3&0\\0&4\end{pmatrix}$. Thus

$$\begin{aligned}\mathbf{A}^5&=\mathbf{PD}^5\mathbf{P}^{-1}\\&=\begin{pmatrix}1&0\\-4&1\end{pmatrix}\begin{pmatrix}3&0\\0&4\end{pmatrix}^5\begin{pmatrix}1&0\\4&1\end{pmatrix}\\&=\begin{pmatrix}1&0\\-4&1\end{pmatrix}\begin{pmatrix}243&0\\0&1024\end{pmatrix}\begin{pmatrix}1&0\\4&1\end{pmatrix}&&\left[\begin{matrix}\text{Because } 3^5=243\\\text{and } 4^5=1024\end{matrix}\right]\\&=\begin{pmatrix}243&0\\-972&1024\end{pmatrix}\begin{pmatrix}1&0\\4&1\end{pmatrix}=\begin{pmatrix}243&0\\3124&1024\end{pmatrix}\end{aligned}$$

(d) We need to find $\mathbf{A}^5$ given that $\mathbf{A}=\begin{pmatrix}2&2\\1&3\end{pmatrix}$. We use $\mathbf{A}^m=\mathbf{PD}^m\mathbf{P}^{-1}$ with $m=5$. *What is* ***P**, **D** and $\mathbf{P}^{-1}$ equal to?*

By question 1 part (d) we have $\mathbf{P}=\begin{pmatrix}2&1\\-1&1\end{pmatrix}$, $\mathbf{D}=\begin{pmatrix}1&0\\0&4\end{pmatrix}$ and $\mathbf{P}^{-1}=\frac{1}{3}\begin{pmatrix}1&-1\\1&2\end{pmatrix}$. Thus substituting these into $\mathbf{A}^5=\mathbf{PD}^5\mathbf{P}^{-1}$ gives

$$\begin{aligned}\mathbf{A}^5&=\begin{pmatrix}2&1\\-1&1\end{pmatrix}\begin{pmatrix}1&0\\0&4\end{pmatrix}^5\frac{1}{3}\begin{pmatrix}1&-1\\1&2\end{pmatrix}\\&=\frac{1}{3}\begin{pmatrix}2&1\\-1&1\end{pmatrix}\begin{pmatrix}1&0\\0&1024\end{pmatrix}\begin{pmatrix}1&-1\\1&2\end{pmatrix}&&\left[\begin{matrix}\text{Because } 1^5=1\\\text{and } 4^5=1024\end{matrix}\right]\\&=\frac{1}{3}\begin{pmatrix}2&1024\\-1&1024\end{pmatrix}\begin{pmatrix}1&-1\\1&2\end{pmatrix}\\&=\frac{1}{3}\begin{pmatrix}1026&2046\\1023&2049\end{pmatrix}=\begin{pmatrix}342&682\\341&683\end{pmatrix}\end{aligned}$$

(b) By the above part (c) we have

$$\begin{aligned}\mathbf{A}^{-1/2}&=\mathbf{PD}^{-1/2}\mathbf{P}^{-1}\\&=\begin{pmatrix}1&0\\-4&1\end{pmatrix}\begin{pmatrix}3&0\\0&4\end{pmatrix}^{-1/2}\begin{pmatrix}1&0\\4&1\end{pmatrix}\\&=\begin{pmatrix}1&0\\-4&1\end{pmatrix}\begin{pmatrix}1/\sqrt{3}&0\\0&1/2\end{pmatrix}\begin{pmatrix}1&0\\4&1\end{pmatrix}&&\left[\begin{matrix}\text{Because } 3^{-1/2}=1/\sqrt{3}\\\text{and } 4^{-1/2}=1/2\end{matrix}\right]\\&=\begin{pmatrix}1/\sqrt{3}&0\\-4/\sqrt{3}&1/2\end{pmatrix}\begin{pmatrix}1&0\\4&1\end{pmatrix}=\begin{pmatrix}1/\sqrt{3}&0\\2-4/\sqrt{3}&1/2\end{pmatrix}\end{aligned}$$

3\. (a) (i) *What are the eigenvalues of the diagonal matrix* $\mathbf{A}=\begin{pmatrix}1&0&0\\0&2&0\\0&0&3\end{pmatrix}$*?*

Since we have a diagonal matrix therefore the eigenvalues are the entries on the leading diagonal which is $\lambda_1 = 1,\ \lambda_2 = 2$ and $\lambda_3 = 3$. For $\lambda_1 = 1$.

Let **u** be the eigenvector then

$$(\mathbf{A}-\mathbf{I})\mathbf{u} = \begin{pmatrix} 1-1 & 0 & 0 \\ 0 & 2-1 & 0 \\ 0 & 0 & 3-1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \qquad \text{gives } x = 1,\ y = 0 \text{ and } z = 0$$

Thus $\mathbf{u} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ is the eigenvector belonging to $\lambda_1 = 1$. Let $\mathbf{v}$ be the eigenvector belonging to the second eigenvalue $\lambda_2 = 2$:

$$(\mathbf{A}-2\mathbf{I})\mathbf{u} = \begin{pmatrix} 1-2 & 0 & 0 \\ 0 & 2-2 & 0 \\ 0 & 0 & 3-2 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$= \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \qquad \text{gives } x = 0,\ y = 1 \text{ and } z = 0$$

Hence the eigenvector $\mathbf{v} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ corresponds to the eigenvalue $\lambda_2 = 2$. Let **w** be the eigenvector belonging to $\lambda_3 = 3$:

$$(\mathbf{A}-3\mathbf{I})\mathbf{u} = \begin{pmatrix} 1-3 & 0 & 0 \\ 0 & 2-3 & 0 \\ 0 & 0 & 3-3 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$= \begin{pmatrix} -2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \qquad \text{gives } x = 0,\ y = 0 \text{ and } z = 1$$

The eigenvector $\mathbf{w} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ belongs to the eigenvalue $\lambda_3 = 3$.

<u>Step 1:</u>

The eigenvectors $\mathbf{u} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$, $\mathbf{v} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ and $\mathbf{w} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ are linearly independent.

<u>Step 2:</u>

The invertible (nonsingular) matrix is $\mathbf{P}=\left(\mathbf{u}\ \vdots\ \mathbf{v}\ \vdots\ \mathbf{w}\right)=\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}=\mathbf{I}$.

<u>Step 3:</u>
Hence $\mathbf{P}^{-1}=\mathbf{I}^{-1}=\mathbf{I}$. We have

$$\mathbf{D}=\mathbf{I}^{-1}\mathbf{A}\mathbf{I}=\mathbf{A}=\begin{pmatrix}1&0&0\\0&2&0\\0&0&3\end{pmatrix}$$

Again the eigenvalues $\lambda_1=1,\ \lambda_2=2$ and $\lambda_3=3$ of the matrix $\mathbf{A}$ are along the leading diagonal.

(iii) To find $\mathbf{A}^4$ we need to use $\mathbf{A}^m=\mathbf{P}\mathbf{D}^m\mathbf{P}^{-1}$ with $m=4$:

$$\mathbf{A}^4=\mathbf{I}\mathbf{D}^4\mathbf{I}^{-1}=\mathbf{D}^4=\begin{pmatrix}1^4&0&0\\0&2^4&0\\0&0&3^4\end{pmatrix}=\begin{pmatrix}1&0&0\\0&16&0\\0&0&81\end{pmatrix}$$

(b) (i) *What are the eigenvalues of* $\mathbf{A}=\begin{pmatrix}-1&4&0\\0&4&3\\0&0&5\end{pmatrix}$*?*

Since we have an upper triangular matrix therefore by using Proposition (7-3) we have $\lambda_1=-1,\ \lambda_2=4$ and $\lambda_3=5$. *What else do we need to find?*

The eigenvector for each eigenvalue. Let $\mathbf{u}$ be the eigenvector belonging to $\lambda_1=-1$:

$$\left(\mathbf{A}-(-1)\mathbf{I}\right)\mathbf{u}=\begin{pmatrix}-1-(-1)&4&0\\0&4-(-1)&3\\0&0&5-(-1)\end{pmatrix}\begin{pmatrix}x\\y\\z\end{pmatrix}$$

$$=\begin{pmatrix}0&4&0\\0&5&3\\0&0&6\end{pmatrix}\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix}\quad\text{gives}\quad x=1,\ y=0\ \text{and}\ z=0$$

Thus $\mathbf{u}=\begin{pmatrix}1\\0\\0\end{pmatrix}$ is the eigenvector belonging to $\lambda_1=-1$. Let $\mathbf{v}$ be the eigenvector belonging to the second eigenvalue $\lambda_2=4$. We have

$$\left(\mathbf{A}-4\mathbf{I}\right)\mathbf{v}=\begin{pmatrix}-1-4&4&0\\0&4-4&3\\0&0&5-4\end{pmatrix}\begin{pmatrix}x\\y\\z\end{pmatrix}$$

$$=\begin{pmatrix}-5&4&0\\0&0&3\\0&0&1\end{pmatrix}\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix}\quad\text{gives}\quad x=4,\ y=5\ \text{and}\ z=0$$

Thus $\mathbf{v}=\begin{pmatrix}4\\5\\0\end{pmatrix}$. Let **w** be the eigenvector belonging to $\lambda_3=5$. We have

$$(\mathbf{A}-5\mathbf{I})\mathbf{w}=\begin{pmatrix}-1-5 & 4 & 0\\ 0 & 4-5 & 3\\ 0 & 0 & 5-5\end{pmatrix}\begin{pmatrix}x\\y\\z\end{pmatrix}$$

$$=\begin{pmatrix}-6 & 4 & 0\\ 0 & -1 & 3\\ 0 & 0 & 0\end{pmatrix}\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix} \quad \text{gives} \quad x=2,\ y=3 \text{ and } z=1$$

Thus $\mathbf{w}=\begin{pmatrix}2\\3\\1\end{pmatrix}$. We have the eigenvectors $\mathbf{u}=\begin{pmatrix}1\\0\\0\end{pmatrix}$, $\mathbf{v}=\begin{pmatrix}4\\5\\0\end{pmatrix}$ and $\mathbf{w}=\begin{pmatrix}2\\3\\1\end{pmatrix}$ corresponding to the eigenvalues $\lambda_1=-1,\ \lambda_2=4$ and $\lambda_3=5$ respectively.

(ii) Step 1:

The eigenvectors $\mathbf{u}=\begin{pmatrix}1\\0\\0\end{pmatrix}$, $\mathbf{v}=\begin{pmatrix}4\\5\\0\end{pmatrix}$ and $\mathbf{w}=\begin{pmatrix}2\\3\\1\end{pmatrix}$ are linearly independent.

Step 2:

The invertible (nonsingular) matrix is $\mathbf{P}=(\mathbf{u}\ \vdots\ \mathbf{v}\ \vdots\ \mathbf{w})=\begin{pmatrix}1 & 4 & 2\\ 0 & 5 & 3\\ 0 & 0 & 1\end{pmatrix}$.

Step 3:

We could check $\mathbf{PD}=\mathbf{AP}$ to ensure that we have the correct matrices **P** and **D**. Since we need to find $\mathbf{A}^4$ so we need to find $\mathbf{P}^{-1}$ because $\mathbf{A}^4=\mathbf{PD}^4\mathbf{P}^{-1}$ so we could just check that we have $\mathbf{D}=\mathbf{P}^{-1}\mathbf{AP}$.

The diagonal matrix $\mathbf{D}=\mathbf{P}^{-1}\mathbf{AP}$. We need to find the inverse of **P**. Using MATLAB or our early theory on matrices we have $\mathbf{P}^{-1}=\frac{1}{5}\begin{pmatrix}5 & -4 & 2\\ 0 & 1 & -3\\ 0 & 0 & 5\end{pmatrix}$.

Substituting $\mathbf{P}^{-1}=\frac{1}{5}\begin{pmatrix}5 & -4 & 2\\ 0 & 1 & -3\\ 0 & 0 & 5\end{pmatrix}$, $\mathbf{A}=\begin{pmatrix}-1 & 4 & 0\\ 0 & 4 & 3\\ 0 & 0 & 5\end{pmatrix}$ and $\mathbf{P}=\begin{pmatrix}1 & 4 & 2\\ 0 & 5 & 3\\ 0 & 0 & 1\end{pmatrix}$ into $\mathbf{D}=\mathbf{P}^{-1}\mathbf{AP}$ gives

$$\mathbf{D}=\mathbf{P}^{-1}\mathbf{AP}=\frac{1}{5}\begin{pmatrix}5 & -4 & 2\\ 0 & 1 & -3\\ 0 & 0 & 5\end{pmatrix}\begin{pmatrix}-1 & 4 & 0\\ 0 & 4 & 3\\ 0 & 0 & 5\end{pmatrix}\begin{pmatrix}1 & 4 & 2\\ 0 & 5 & 3\\ 0 & 0 & 1\end{pmatrix}$$

$$=\frac{1}{5}\begin{pmatrix}-5 & 4 & -2\\ 0 & 4 & -12\\ 0 & 0 & 25\end{pmatrix}\begin{pmatrix}1 & 4 & 2\\ 0 & 5 & 3\\ 0 & 0 & 1\end{pmatrix}=\frac{1}{5}\begin{pmatrix}-5 & 0 & 0\\ 0 & 20 & 0\\ 0 & 0 & 25\end{pmatrix}=\begin{pmatrix}-1 & 0 & 0\\ 0 & 4 & 0\\ 0 & 0 & 5\end{pmatrix}$$

(iii) To find $\mathbf{A}^4$ we need to use $\mathbf{A}^m = \mathbf{PD}^m\mathbf{P}^{-1}$ with $m = 4$:

$$\mathbf{A}^4 = \mathbf{PD}^4\mathbf{P}^{-1} = \begin{pmatrix} 1 & 4 & 2 \\ 0 & 5 & 3 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} -1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 5 \end{pmatrix}^4 \frac{1}{5}\begin{pmatrix} 5 & -4 & 2 \\ 0 & 1 & -3 \\ 0 & 0 & 5 \end{pmatrix}$$

$$= \frac{1}{5}\begin{pmatrix} 1 & 4 & 2 \\ 0 & 5 & 3 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 256 & 0 \\ 0 & 0 & 625 \end{pmatrix}\begin{pmatrix} 5 & -4 & 2 \\ 0 & 1 & -3 \\ 0 & 0 & 5 \end{pmatrix} \quad \begin{bmatrix} \text{Because } (-1)^4 = 1 \\ 4^4 = 256 \text{ and } 5^4 = 625 \end{bmatrix}$$

$$= \frac{1}{5}\begin{pmatrix} 1 & 1024 & 1250 \\ 0 & 1280 & 1875 \\ 0 & 0 & 625 \end{pmatrix}\begin{pmatrix} 5 & -4 & 2 \\ 0 & 1 & -3 \\ 0 & 0 & 5 \end{pmatrix}$$

$$= \frac{1}{5}\begin{pmatrix} 5 & 1020 & 3180 \\ 0 & 1280 & 5535 \\ 0 & 0 & 3125 \end{pmatrix} = \begin{pmatrix} 1 & 204 & 636 \\ 0 & 256 & 1107 \\ 0 & 0 & 625 \end{pmatrix} \quad \begin{bmatrix} \text{Multiplying by the} \\ \text{scalar } \frac{1}{5} \end{bmatrix}$$

(c) (i) *What are the eigenvalues of* $\mathbf{A} = \begin{pmatrix} 2 & 0 & 0 \\ 1 & 5 & 0 \\ 1 & 2 & 6 \end{pmatrix}$?

Since we have a lower triangular matrix therefore by using Proposition (7-3) the eigenvalues are $\lambda_1 = 2,\ \lambda_2 = 5$ and $\lambda_3 = 6$. *What else do we need to find?*

The eigenvector for each eigenvalue. Let **u** be the eigenvector belonging to $\lambda_1 = 2$:

$$(\mathbf{A} - \lambda_1\mathbf{I})\mathbf{u} = \begin{pmatrix} 2-2 & 0 & 0 \\ 1 & 5-2 & 0 \\ 1 & 2 & 6-2 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 & 0 \\ 1 & 3 & 0 \\ 1 & 2 & 4 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad \text{gives} \quad x = -12,\ y = 4 \text{ and } z = 1$$

Thus $\mathbf{u} = \begin{pmatrix} -12 \\ 4 \\ 1 \end{pmatrix}$ is the eigenvector belonging to $\lambda_1 = 2$. Let $\mathbf{v}$ be the eigenvector belonging to the second eigenvalue $\lambda_2 = 5$. We have

$$(\mathbf{A} - \lambda_2\mathbf{I})\mathbf{v} = \begin{pmatrix} 2-5 & 0 & 0 \\ 1 & 5-5 & 0 \\ 1 & 2 & 6-5 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$= \begin{pmatrix} -3 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 2 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad \text{gives} \quad x = 0,\ y = 1 \text{ and } z = -2$$

Proposition (7-3). If matrix **A** is a diagonal or triangular matrix then the eigenvalues of **A** are the entries along the leading diagonal.

Thus $\mathbf{v}=\begin{pmatrix}0\\1\\-2\end{pmatrix}$. Let **w** be the eigenvector belonging to $\lambda_3=6$. We have

$$(\mathbf{A}-\lambda_3\mathbf{I})\mathbf{w}=\begin{pmatrix}2-6 & 0 & 0\\1 & 5-6 & 0\\1 & 2 & 6-6\end{pmatrix}\begin{pmatrix}x\\y\\z\end{pmatrix}$$

$$=\begin{pmatrix}-4 & 0 & 0\\1 & -1 & 0\\1 & 2 & 0\end{pmatrix}\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix} \quad \text{gives} \quad x=0,\ y=0 \text{ and } z=1$$

Thus $\mathbf{w}=\begin{pmatrix}0\\0\\1\end{pmatrix}$. We have the eigenvectors $\mathbf{u}=\begin{pmatrix}-12\\4\\1\end{pmatrix}$, $\mathbf{v}=\begin{pmatrix}0\\1\\-2\end{pmatrix}$ and $\mathbf{w}=\begin{pmatrix}0\\0\\1\end{pmatrix}$ corresponding to the eigenvalues $\lambda_1=2,\ \lambda_2=5$ and $\lambda_3=6$ respectively.

(ii) <u>Step 1:</u>

The eigenvectors $\mathbf{u}=\begin{pmatrix}-12\\4\\1\end{pmatrix}$, $\mathbf{v}=\begin{pmatrix}0\\1\\-2\end{pmatrix}$ and $\mathbf{w}=\begin{pmatrix}0\\0\\1\end{pmatrix}$ are linearly independent.

<u>Step 2:</u>

The invertible (nonsingular) matrix is $\mathbf{P}=(\mathbf{u} \vdots \mathbf{v} \vdots \mathbf{w})=\begin{pmatrix}-12 & 0 & 0\\4 & 1 & 0\\1 & -2 & 1\end{pmatrix}$.

<u>Step 3:</u>

The diagonal matrix $\mathbf{D}=\mathbf{P}^{-1}\mathbf{AP}$. We need to find the inverse of **P**. Using MATLAB or our early theory on matrices we have

$$\mathbf{P}^{-1}=-\frac{1}{12}\begin{pmatrix}1 & 0 & 0\\-4 & -12 & 0\\-9 & -24 & -12\end{pmatrix}=\frac{1}{12}\begin{pmatrix}-1 & 0 & 0\\4 & 12 & 0\\9 & 24 & 12\end{pmatrix} \quad \begin{bmatrix}\text{Taking in}\\\text{negative sign}\end{bmatrix}$$

Substituting $\mathbf{P}^{-1}=\frac{1}{12}\begin{pmatrix}-1 & 0 & 0\\4 & 12 & 0\\9 & 24 & 12\end{pmatrix}$, $\mathbf{A}=\begin{pmatrix}2 & 0 & 0\\1 & 5 & 0\\1 & 2 & 6\end{pmatrix}$ and $\mathbf{P}=\begin{pmatrix}-12 & 0 & 0\\4 & 1 & 0\\1 & -2 & 1\end{pmatrix}$ into

$\mathbf{D}=\mathbf{P}^{-1}\mathbf{AP}$ gives

$$\mathbf{D}=\mathbf{P}^{-1}\mathbf{AP}=\frac{1}{12}\begin{pmatrix}-1&0&0\\4&12&0\\9&24&12\end{pmatrix}\begin{pmatrix}2&0&0\\1&5&0\\1&2&6\end{pmatrix}\begin{pmatrix}-12&0&0\\4&1&0\\1&-2&1\end{pmatrix}$$

$$=\frac{1}{12}\begin{pmatrix}-2&0&0\\20&60&0\\54&144&72\end{pmatrix}\begin{pmatrix}-12&0&0\\4&1&0\\1&-2&1\end{pmatrix}\qquad\left[\begin{matrix}\text{Multiplying the}\\\text{two Left Hand}\\\text{matrices}\end{matrix}\right]$$

$$=\frac{1}{12}\begin{pmatrix}24&0&0\\0&60&0\\0&0&72\end{pmatrix}=\begin{pmatrix}2&0&0\\0&5&0\\0&0&6\end{pmatrix}\qquad\left[\begin{matrix}\text{Multiplying by the}\\\text{scalar }\dfrac{1}{2}\end{matrix}\right]$$

Note the entries on the leading diagonal are the eigenvalues of the given matrix **A**.

(iii) To find $\mathbf{A}^4$ we need to use $\mathbf{A}^m=\mathbf{PD}^m\mathbf{P}^{-1}$ with $m=4$:

$$\mathbf{A}^4=\mathbf{PD}^4\mathbf{P}^{-1}=\begin{pmatrix}-12&0&0\\4&1&0\\1&-2&1\end{pmatrix}\begin{pmatrix}2&0&0\\0&5&0\\0&0&6\end{pmatrix}^4\frac{1}{12}\begin{pmatrix}-1&0&0\\4&12&0\\9&24&12\end{pmatrix}$$

$$=\frac{1}{12}\begin{pmatrix}-12&0&0\\4&1&0\\1&-2&1\end{pmatrix}\begin{pmatrix}16&0&0\\0&625&0\\0&0&1296\end{pmatrix}\begin{pmatrix}-1&0&0\\4&12&0\\9&24&12\end{pmatrix}$$

$$=\frac{1}{12}\begin{pmatrix}-192&0&0\\64&625&0\\16&-1250&1296\end{pmatrix}\begin{pmatrix}-1&0&0\\4&12&0\\9&24&12\end{pmatrix}$$

$$=\frac{1}{12}\begin{pmatrix}192&0&0\\2436&7500&0\\6648&16104&15552\end{pmatrix}=\begin{pmatrix}16&0&0\\203&625&0\\554&1342&1296\end{pmatrix}$$

(d)

4. (a) We are given $\mathbf{A}=\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}$ which is the identity matrix **I**. *Is the identity matrix diagonalizable?*

Yes because the invertible matrix $\mathbf{P}=\mathbf{I}$ gives the result $\mathbf{P}^{-1}\mathbf{AP}=\mathbf{III}=\mathbf{I}=\mathbf{D}$ where **D** is the diagonal matrix with 1's as the entries along the leading diagonal.

(b) *Is* $\mathbf{A}=\begin{pmatrix}-1&2&3\\0&2&5\\0&0&8\end{pmatrix}$ *diagonalizable?*

Yes because the entries along the leading diagonal are the eigenvalues of this matrix. Thus $\lambda_1=-1,\ \lambda_2=2$ and $\lambda_3=8$ are the eigenvalues and by Proposition (7-13) we conclude that the given matrix is diagonalizable because we have 3 distinct eigenvalues.

Proposition (7-13). If a n by n matrix **A** has n distinct eigenvalues then the matrix **A** is diagonalizable.

(c) Similar to part (b). Since $\mathbf{A}=\begin{pmatrix}\sqrt{2} & 0 & 0 & 0\\ 1 & \sqrt{3} & 0 & 0\\ 6 & 7 & 1/2 & 0\\ 2 & 9 & 7 & -5\end{pmatrix}$ has distinct eigenvalues $\lambda_1=\sqrt{2},\ \lambda_2=\sqrt{3},\ \lambda_3=1/2$ and $\lambda_4=-5$ therefore by Proposition (7-13) we conclude that the matrix **A** is diagonalizable.

5. From Example 17 we have $\mathbf{u}=\begin{pmatrix}-2\\ 1\end{pmatrix}$ and $\mathbf{v}=\begin{pmatrix}1\\ 1\end{pmatrix}$. Therefore $\mathbf{P}=\begin{pmatrix}1 & -2\\ 1 & 1\end{pmatrix}$ and

$$\mathbf{P}^{-1}=\begin{pmatrix}1 & -2\\ 1 & 1\end{pmatrix}^{-1}=\frac{1}{3}\begin{pmatrix}1 & 2\\ -1 & 1\end{pmatrix}$$

Substituting $\mathbf{P}^{-1}=\frac{1}{3}\begin{pmatrix}1 & 2\\ -1 & 1\end{pmatrix}$, $\mathbf{A}=\begin{pmatrix}1 & 4\\ 2 & 3\end{pmatrix}$ and $\mathbf{P}=\begin{pmatrix}1 & -2\\ 1 & 1\end{pmatrix}$ into $\mathbf{P}^{-1}\mathbf{AP}$:

$$\begin{aligned}\mathbf{P}^{-1}\mathbf{AP}&=\frac{1}{3}\begin{pmatrix}1 & 2\\ -1 & 1\end{pmatrix}\begin{pmatrix}1 & 4\\ 2 & 3\end{pmatrix}\begin{pmatrix}1 & -2\\ 1 & 1\end{pmatrix}\\ &=\frac{1}{3}\begin{pmatrix}5 & 10\\ 1 & -1\end{pmatrix}\begin{pmatrix}1 & -2\\ 1 & 1\end{pmatrix} && \begin{bmatrix}\text{Multiplying the two}\\ \text{Left Hand Matrices}\end{bmatrix}\\ &=\frac{1}{3}\begin{pmatrix}15 & 0\\ 0 & -3\end{pmatrix}=\begin{pmatrix}5 & 0\\ 0 & -1\end{pmatrix} && \begin{bmatrix}\text{Multiplying by the}\\ \text{scalar 1/3}\end{bmatrix}\end{aligned}$$

Note that the eigenvalues have swapped around from the diagonal matrix of Example 17.

6. Since we have distinct eigenvalues $\lambda_1=-2,\ \lambda_2=-5$ and $\lambda_3=-1$ therefore by Proposition (7-13) we conclude that the matrix **A** is diagonalizable. The diagonal matrix will have the eigenvalues along the leading diagonal of the matrix. The eigenvalues on the leading diagonal occur depending on the order of eigenvectors **u**, **v** and **w** in the matrix **P**.

$$\mathbf{D}=\begin{pmatrix}-2 & 0 & 0\\ 0 & -5 & 0\\ 0 & 0 & -1\end{pmatrix}$$

*What is the invertible matrix **P** going to be?*

It will be the eigenvectors **u**, **v** and **w** as the columns of the matrix **P** provided they are linearly independent. Since $\mathbf{u}=\begin{pmatrix}1\\ 2\\ 0\end{pmatrix}$, $\mathbf{v}=\begin{pmatrix}5\\ 4\\ 0\end{pmatrix}$ and $\mathbf{w}=\begin{pmatrix}0\\ 0\\ 1\end{pmatrix}$ are linearly independent therefore

$$\mathbf{P}=\begin{array}{c}\begin{matrix}\mathbf{u} & \mathbf{v} & \mathbf{w}\end{matrix}\\ \begin{pmatrix}1 & 5 & 0\\ 2 & 4 & 0\\ 0 & 0 & 1\end{pmatrix}\end{array}$$

Proposition (7-13). If a n by n matrix **A** has n distinct eigenvalues then the matrix **A** is diagonalizable.

By MATLAB we have $\mathbf{P}^{-1} = -\frac{1}{6}\begin{pmatrix} 4 & -5 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & -6 \end{pmatrix}$. *How do we find* $\mathbf{A}^3$*?*

Substitute $m = 3$ into $\mathbf{A}^m = \mathbf{P}\mathbf{D}^m\mathbf{P}^{-1}$ which gives $\mathbf{A}^3 = \mathbf{P}\mathbf{D}^3\mathbf{P}^{-1}$.

Substituting $\mathbf{P} = \begin{pmatrix} 1 & 5 & 0 \\ 2 & 4 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, $\mathbf{D} = \begin{pmatrix} -2 & 0 & 0 \\ 0 & -5 & 0 \\ 0 & 0 & -1 \end{pmatrix}$ and $\mathbf{P}^{-1} = -\frac{1}{6}\begin{pmatrix} 4 & -5 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & -6 \end{pmatrix}$ into

$\mathbf{A}^3 = \mathbf{P}\mathbf{D}^3\mathbf{P}^{-1}$ gives

$$\begin{aligned}
\mathbf{A}^3 = \mathbf{P}\mathbf{D}^3\mathbf{P}^{-1} &= \begin{pmatrix} 1 & 5 & 0 \\ 2 & 4 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} -2 & 0 & 0 \\ 0 & -5 & 0 \\ 0 & 0 & -1 \end{pmatrix}^3 -\frac{1}{6}\begin{pmatrix} 4 & -5 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & -6 \end{pmatrix} \\
&= -\frac{1}{6}\begin{pmatrix} 1 & 5 & 0 \\ 2 & 4 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} -8 & 0 & 0 \\ 0 & -125 & 0 \\ 0 & 0 & -1 \end{pmatrix}\begin{pmatrix} 4 & -5 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & -6 \end{pmatrix} \\
&= -\frac{1}{6}\begin{pmatrix} -8 & -625 & 0 \\ -16 & -500 & 0 \\ 0 & 0 & -1 \end{pmatrix}\begin{pmatrix} 4 & -5 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & -6 \end{pmatrix} \\
&= -\frac{1}{6}\begin{pmatrix} 1218 & -585 & 0 \\ 936 & -420 & 0 \\ 0 & 0 & 6 \end{pmatrix} = \begin{pmatrix} -203 & 97.5 & 0 \\ -156 & 70 & 0 \\ 0 & 0 & -1 \end{pmatrix}
\end{aligned}$$

7. (a) We are given the matrix $\mathbf{A} = \begin{pmatrix} 2 & -1 \\ 1 & 4 \end{pmatrix}$. The eigenvalues are evaluated by

$$\begin{aligned}
\det(\mathbf{A} - \lambda\mathbf{I}) = \det\begin{pmatrix} 2-\lambda & -1 \\ 1 & 4-\lambda \end{pmatrix} &= (2-\lambda)(4-\lambda)+1 \\
&= 8 - 6\lambda + \lambda^2 + 1 \\
&= \lambda^2 - 6\lambda + 9 = (\lambda - 3)^2 = 0 \text{ gives } \lambda = 3
\end{aligned}$$

The eigenvector corresponding to $\lambda = 3$ is given by substituting $\lambda = 3$ into $(\mathbf{A} - \lambda\mathbf{I})\mathbf{u} = \mathbf{O}$ where $\mathbf{u}$ is the eigenvector:

$$(\mathbf{A} - 3\mathbf{I})\mathbf{u} = \begin{pmatrix} 2-3 & -1 \\ 1 & 4-3 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} -1 & -1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Placing the 2 by 2 matrix into row echelon form gives

$$\begin{pmatrix} -1 & -1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \text{ yields } x = 1 \text{ and } y = -1$$

Thus the eigenvector $\mathbf{u}=\begin{pmatrix}1\\-1\end{pmatrix}$ corresponds to the eigenvalue $\lambda=3$. Since there is only one non-zero equation and 2 unknowns therefore there are $2-1=1$ free variable.
Thus this is the only independent eigenvector of $\lambda=3$ because all the other eigenvectors are multiples of **u**.
Since a 2 by 2 matrix has only one independent eigenvector therefore by Theorem (7-13):

Theorem (7-13). A *n* by *n* matrix **A** is diagonalizable ⇔ it has *n* linearly ind e.vectors.

We conclude that the given matrix is **not** diagonalizable.

(b) We are given the matrix $\mathbf{A}=\begin{pmatrix}-2&4\\-1&-6\end{pmatrix}$. The eigenvalues of this matrix are

$$\begin{aligned}\det(\mathbf{A}-\lambda\mathbf{I})&=\det\begin{pmatrix}-2-\lambda&4\\-1&-6-\lambda\end{pmatrix}\\&=(-2-\lambda)(-6-\lambda)+4\\&=12+8\lambda+\lambda^2+4\\&=\lambda^2+8\lambda+16=(\lambda+4)^2=0\ \text{ gives }\ \lambda=-4\end{aligned}$$

Let **u** be the corresponding eigenvector. We have

$$(\mathbf{A}-(-4)\mathbf{I})\mathbf{u}=\begin{pmatrix}-2+4&4\\-1&-6+4\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}2&4\\-1&-2\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix}$$

Putting the 2 by 2 matrix into row echelon form gives

$$\begin{pmatrix}2&4\\0&0\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix}\quad\text{yields }\ x=-2\ \text{ and }\ y=1$$

The eigenvector $\mathbf{u}=\begin{pmatrix}-2\\1\end{pmatrix}$ belongs to the eigenvalue $\lambda=-4$. Since there is 1 non-zero equation with 2 unknowns therefore the number of free variables is $2-1=1$. Hence **u** is the only independent eigenvector.
For the given 2 by 2 matrix we have 1 independent eigenvector which means that the matrix is **not** diagonalizable because the number of independent eigenvectors (1) does **not** equal *n* (2) of the square matrix.

(c) We are given $\mathbf{A}=\begin{pmatrix}1&2&3\\0&1&3\\0&0&1\end{pmatrix}$ which is a triangular matrix. Thus by Proposition (7-3):

Proposition (7-3). If matrix **A** is a diagonal or triangular then the e.values of **A** are the entries along the leading diagonal.

The eigenvalue $\lambda=1$. To find the corresponding eigenvector **u** we substitute $\lambda=1$ into $(\mathbf{A}-\lambda\mathbf{I})\mathbf{u}=\mathbf{O}$:

$$\begin{aligned}(\mathbf{A}-\mathbf{I})\mathbf{u}&=\begin{pmatrix}1-1&2&3\\0&1-1&3\\0&0&1-1\end{pmatrix}\begin{pmatrix}x\\y\\z\end{pmatrix}\\&=\begin{pmatrix}0&2&3\\0&0&3\\0&0&0\end{pmatrix}\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix}\quad\text{gives }\ x=1,\ \ y=0\ \text{ and }\ z=0\end{aligned}$$

The last matrix is already in row echelon form and there are 2 non-zero equations and 3 unknowns which means there are $3-2=1$ free variable. This means that **u** is the only eigenvector of the given matrix **A**.
Since we are given a 3 by 3 matrix therefore $n=3$ but we have only 1 linearly independent eigenvector therefore the given matrix is **not** diagonalizable.

8. (a) *How do we find* $\mathbf{A}^{11}$ *given that* $\mathbf{A}=\begin{pmatrix}-4 & 2\\ -9 & 5\end{pmatrix}$*?*

We need to use Proposition (7-14), that is $\mathbf{A}^m=\mathbf{P}\mathbf{D}^m\mathbf{P}^{-1}$ with $m=11$. To find the matrix **P** we need to find the eigenvectors which means we need the eigenvalues.
Using $\det(\mathbf{A}-\lambda\mathbf{I})=0$ to find λ we have

$$\begin{aligned}\det(\mathbf{A}-\lambda\mathbf{I}) &= \det\begin{pmatrix}-4-\lambda & 2\\ -9 & 5-\lambda\end{pmatrix}\\ &=(-4-\lambda)(5-\lambda)+18\\ &=-20-\lambda+\lambda^2+18\\ &=\lambda^2-\lambda-2\\ &=(\lambda+1)(\lambda-2)=0 \text{ gives } \lambda_1=-1 \text{ and } \lambda_2=2\end{aligned}$$

Since we have two distinct eigenvalues, $\lambda_1=-1$ and $\lambda_2=2$, for a 2 by 2 matrix therefore the given matrix **A** can be diagonalised.
Let **u** be the eigenvector belonging to $\lambda_1=-1$ then substituting this into $(\mathbf{A}-\lambda\mathbf{I})\mathbf{u}$ gives

$$\begin{aligned}(\mathbf{A}+\mathbf{I})\mathbf{u} &= \begin{pmatrix}-4+1 & 2\\ -9 & 5+1\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix}\\ &=\begin{pmatrix}-3 & 2\\ -9 & 6\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix}=\begin{pmatrix}0\\ 0\end{pmatrix} \text{ gives } x=2 \text{ and } y=3\end{aligned}$$

Thus $\mathbf{u}=\begin{pmatrix}2\\ 3\end{pmatrix}$ is the eigenvector belonging to $\lambda_1=-1$. Let $\mathbf{v}=\begin{pmatrix}x\\ y\end{pmatrix}$ be the eigenvector belonging to the other eigenvalue $\lambda_2=2$:

$$\begin{aligned}(\mathbf{A}-2\mathbf{I})\mathbf{v} &= \begin{pmatrix}-4-2 & 2\\ -9 & 5-2\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix}\\ &=\begin{pmatrix}-6 & 2\\ -9 & 3\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix}=\begin{pmatrix}0\\ 0\end{pmatrix} \text{ gives } x=1 \text{ and } y=3\end{aligned}$$

Hence $\mathbf{v}=\begin{pmatrix}1\\ 3\end{pmatrix}$ is the eigenvector belonging to the other eigenvalue $\lambda_2=2$. *What is our matrix **P** equal to?*

$$\mathbf{P}=(\mathbf{u}\ \vdots\ \mathbf{v})=\begin{pmatrix}2 & 1\\ 3 & 3\end{pmatrix} \qquad \left[\text{Because } \mathbf{u}=\begin{pmatrix}2\\ 3\end{pmatrix} \text{ and } \mathbf{v}=\begin{pmatrix}1\\ 3\end{pmatrix}\right]$$

What else do we need to find?
The inverse of **P**. Thus

$$\mathbf{P}^{-1}=\begin{pmatrix}2 & 1\\ 3 & 3\end{pmatrix}^{-1}=\frac{1}{6-3}\begin{pmatrix}3 & -1\\ -3 & 2\end{pmatrix}=\frac{1}{3}\begin{pmatrix}3 & -1\\ -3 & 2\end{pmatrix}$$

We also need to determine the diagonal matrix **D** to use $\mathbf{A}^m = \mathbf{P}\mathbf{D}^m\mathbf{P}^{-1}$. *What is the diagonal matrix **D** equal to in this case?*
It is the diagonal matrix **D** with the eigenvalues, $\lambda_1 = -1$ and $\lambda_2 = 2$, as the entries along the leading diagonal. (You may check that $\mathbf{P}^{-1}\mathbf{A}\mathbf{P} = \mathbf{D}$).

$$\mathbf{D} = \begin{pmatrix} -1 & 0 \\ 0 & 2 \end{pmatrix}$$

Now we have all the ingredients to use the formula $\mathbf{A}^m = \mathbf{P}\mathbf{D}^m\mathbf{P}^{-1}$ with $m = 11$, that is $\mathbf{A}^{11} = \mathbf{P}\mathbf{D}^{11}\mathbf{P}^{-1}$.

Substituting $\mathbf{P} = \begin{pmatrix} 2 & 1 \\ 3 & 3 \end{pmatrix}$, $\mathbf{D} = \begin{pmatrix} -1 & 0 \\ 0 & 2 \end{pmatrix}$ and $\mathbf{P}^{-1} = \frac{1}{3}\begin{pmatrix} 3 & -1 \\ -3 & 2 \end{pmatrix}$ into $\mathbf{A}^{11} = \mathbf{P}\mathbf{D}^{11}\mathbf{P}^{-1}$ gives

$$\begin{aligned}
\mathbf{A}^{11} &= \mathbf{P}\mathbf{D}^{11}\mathbf{P}^{-1} \\
&= \begin{pmatrix} 2 & 1 \\ 3 & 3 \end{pmatrix}\begin{pmatrix} -1 & 0 \\ 0 & 2 \end{pmatrix}^{11}\frac{1}{3}\begin{pmatrix} 3 & -1 \\ -3 & 2 \end{pmatrix} \\
&= \frac{1}{3}\begin{pmatrix} 2 & 1 \\ 3 & 3 \end{pmatrix}\begin{pmatrix} -1 & 0 \\ 0 & 2048 \end{pmatrix}\begin{pmatrix} 3 & -1 \\ -3 & 2 \end{pmatrix} && \begin{bmatrix} \text{Because } (-1)^{11} = -1 \\ \text{and } 2^{11} = 2048 \end{bmatrix} \\
&= \frac{1}{3}\begin{pmatrix} -2 & 2048 \\ -3 & 6144 \end{pmatrix}\begin{pmatrix} 3 & -1 \\ -3 & 2 \end{pmatrix} && \begin{bmatrix} \text{Multiplying the two} \\ \text{Left Hand Matrices} \end{bmatrix} \\
&= \frac{1}{3}\begin{pmatrix} -6150 & 4098 \\ -18441 & 12291 \end{pmatrix} = \begin{pmatrix} -2050 & 1366 \\ -6147 & 4097 \end{pmatrix} && \begin{bmatrix} \text{Multiplying by the} \\ \text{scalar } 1/3 \end{bmatrix}
\end{aligned}$$

Hence $\mathbf{A}^{11} = \begin{pmatrix} -2050 & 1366 \\ -6147 & 4097 \end{pmatrix}$.

(b) Similarly $\mathbf{A}^{-1} = \mathbf{P}\mathbf{D}^{-1}\mathbf{P}^{-1}$. Evaluating this gives

$$\begin{aligned}
\mathbf{A}^{-1} &= \mathbf{P}\mathbf{D}^{-1}\mathbf{P}^{-1} \\
&= \begin{pmatrix} 2 & 1 \\ 3 & 3 \end{pmatrix}\begin{pmatrix} -1 & 0 \\ 0 & 2 \end{pmatrix}^{-1}\frac{1}{3}\begin{pmatrix} 3 & -1 \\ -3 & 2 \end{pmatrix} \\
&= \frac{1}{3}\begin{pmatrix} 2 & 1 \\ 3 & 3 \end{pmatrix}\begin{pmatrix} -1 & 0 \\ 0 & 1/2 \end{pmatrix}\begin{pmatrix} 3 & -1 \\ -3 & 2 \end{pmatrix} && \begin{bmatrix} \text{Because } (-1)^{-1} = -1 \\ \text{and } 2^{-1} = 1/2 \end{bmatrix} \\
&= \frac{1}{3}\frac{1}{2}\begin{pmatrix} 2 & 1 \\ 3 & 3 \end{pmatrix}\begin{pmatrix} -2 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 3 & -1 \\ -3 & 2 \end{pmatrix} && \begin{bmatrix} \text{Taking out } \frac{1}{2} \end{bmatrix} \\
&= \frac{1}{6}\begin{pmatrix} -4 & 1 \\ -6 & 3 \end{pmatrix}\begin{pmatrix} 3 & -1 \\ -3 & 2 \end{pmatrix} = \frac{1}{6}\begin{pmatrix} -15 & 6 \\ -27 & 12 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} -5 & 2 \\ -9 & 4 \end{pmatrix} && \begin{bmatrix} \text{Taking in } \frac{1}{3} \end{bmatrix}
\end{aligned}$$

9. We prove this result by induction.
Proof.

Let $\mathbf{D} = \begin{pmatrix} d_1 & & 0 \\ & \ddots & \\ 0 & & d_n \end{pmatrix}$ be a diagonal matrix. Then $\mathbf{D}^1 = \mathbf{D}$ which means all the entries on the leading diagonal are to the index 1.

Assume the result is true for $m = k$ that is

$$\mathbf{D}^k = \begin{pmatrix} d_1^{\,k} & & 0 \\ & \ddots & \\ 0 & & d_n^{\,k} \end{pmatrix} \qquad (*)$$

Required to prove the result for $m = k+1$, that is we need to show

$$\mathbf{D}^{k+1} = \begin{pmatrix} d_1^{\,k+1} & & 0 \\ & \ddots & \\ 0 & & d_n^{\,k+1} \end{pmatrix}$$

We have

$$\mathbf{D}^{k+1} = \mathbf{D}^k\mathbf{D} = \underbrace{\begin{pmatrix} d_1^{\,k} & & 0 \\ & \ddots & \\ 0 & & d_n^{\,k} \end{pmatrix}}_{\text{By }(*)}\begin{pmatrix} d_1 & & 0 \\ & \ddots & \\ 0 & & d_n \end{pmatrix} = \begin{pmatrix} d_1^{\,k+1} & & 0 \\ & \ddots & \\ 0 & & d_n^{\,k+1} \end{pmatrix} = \mathbf{D}^{k+1}$$

Hence we have our result.

■

10. We need to prove proposition (7-9) which is the following:
Let **A**, **B** and **C** be square matrices. Then we have the following:

I. Matrix **A** is similar to matrix **A**.
II. If matrix **B** is similar to matrix **A** then the other way round is also true, that is matrix **A** is similar to matrix **B**.
III. If matrix **A** is similar to **B** and **B** is similar to matrix **C** then matrix **A** is similar to matrix **C**.

How do we prove these results?
Use the definition of similar matrices:

Definition (7-2). A square matrix **B** is **similar** to a matrix **A** if there exists an invertible matrix **P** such that $\mathbf{P}^{-1}\mathbf{AP} = \mathbf{B}$.

Proof of I.
Matrix **A** is similar to matrix **A** because we have

$$\mathbf{I}^{-1}\mathbf{AI} = \mathbf{A}$$

where **I** is the identity matrix.

■

Proof of II.
Assume matrix **B** is similar to matrix **A** then there is an invertible (nonsingular) matrix **P** such that $\mathbf{B} = \mathbf{P}^{-1}\mathbf{AP}$. Pre-multiply this by **P** and post multiply this by $\mathbf{P}^{-1}$:

$$\begin{aligned}\mathbf{PBP}^{-1} &= \mathbf{P}\left(\mathbf{P}^{-1}\mathbf{AP}\right)\mathbf{P}^{-1} \\ &= \left(\mathbf{PP}^{-1}\right)\mathbf{A}\left(\mathbf{PP}^{-1}\right) \\ &= \mathbf{IAI} = \mathbf{A}\end{aligned}$$

We have $\mathbf{A} = \mathbf{PBP}^{-1} = \left(\mathbf{P}^{-1}\right)^{-1}\mathbf{BP}^{-1}$ because $\left(\mathbf{P}^{-1}\right)^{-1} = \mathbf{P}$. Thus we have

$$\mathbf{A} = \left(\mathbf{P}^{-1}\right)^{-1}\mathbf{BP}^{-1}$$

By Definition (7-2):

(7-2). **B** is similar to **A** if there exists an invertible matrix **P** such that $\mathbf{P}^{-1}\mathbf{AP}=\mathbf{B}$.

We conclude that matrix **A** is similar to matrix **B**. This is our required result.

■

Proof of III.

Assume matrix **A** is similar to matrix **B**. This means that there is an invertible (nonsingular) matrix **P** such that

$$\mathbf{A}=\mathbf{P}^{-1}\mathbf{BP} \qquad (*)$$

We are also given that matrix **B** is similar to matrix **C**. This means that there is an invertible (nonsingular) matrix **Q** such that

$$\mathbf{B}=\mathbf{Q}^{-1}\mathbf{CQ}$$

Substituting this $\mathbf{B}=\mathbf{Q}^{-1}\mathbf{CQ}$ into (*) gives

$$\begin{aligned}\mathbf{A}&=\mathbf{P}^{-1}\left(\mathbf{Q}^{-1}\mathbf{CQ}\right)\mathbf{P}\\ &=\left(\mathbf{P}^{-1}\mathbf{Q}^{-1}\right)\mathbf{C}\left(\mathbf{QP}\right) && \left[\text{Using rules of matrices } \mathbf{A}(\mathbf{BC})=\mathbf{AB}(\mathbf{C})\right]\\ &=\left(\mathbf{QP}\right)^{-1}\mathbf{C}\left(\mathbf{QP}\right) && \left[\text{Because } \mathbf{B}^{-1}\mathbf{A}^{-1}=\left(\mathbf{AB}\right)^{-1}\right]\end{aligned}$$

By Definition (7-2) we conclude that matrix **A** is similar to matrix **C**.

■

11. Need to prove that is **A** is diagonalizable then $\mathbf{A}^T$ is diagonalizable.

Proof.

We assume that **A** is diagonalizable. Thus there exists an invertible matrix **P** such that $\mathbf{P}^{-1}\mathbf{AP}=\mathbf{D}$ where **D** is a diagonal matrix.

Taking the transpose of both sides of $\mathbf{P}^{-1}\mathbf{AP}=\mathbf{D}$ gives

$$\left(\mathbf{P}^{-1}\mathbf{AP}\right)^T=\mathbf{D}^T=\mathbf{D} \qquad \left[\text{Because } \mathbf{D} \text{ is a diagonal matrix}\right]$$

What is the Left Hand Side $\left(\mathbf{P}^{-1}\mathbf{AP}\right)^T$ *equal to?*

Using our rules on matrix transpose $\left(\mathbf{ABC}\right)^T=\mathbf{C}^T\mathbf{B}^T\mathbf{A}^T$ we have

$$\begin{aligned}\left(\mathbf{P}^{-1}\mathbf{AP}\right)^T&=\mathbf{P}^T\mathbf{A}^T\left(\mathbf{P}^{-1}\right)^T\\ &=\mathbf{P}^T\mathbf{A}^T\left(\mathbf{P}^T\right)^{-1} && \left[\text{Because } \left(\mathbf{P}^{-1}\right)^T=\left(\mathbf{P}^T\right)^{-1}\right]\end{aligned}$$

Since $\mathbf{P}^T=\left[\left(\mathbf{P}^T\right)^{-1}\right]^{-1}$ therefore

$$\mathbf{P}^T\mathbf{A}^T\left(\mathbf{P}^T\right)^{-1}=\left[\left(\mathbf{P}^T\right)^{-1}\right]^{-1}\mathbf{A}^T\left(\mathbf{P}^T\right)^{-1}=\mathbf{D} \quad \left[\text{Because } \left(\mathbf{P}^{-1}\mathbf{AP}\right)^T=\mathbf{D}\right]$$

By Definition (7-3):

(7-3). A matrix **A** is diagonalizable if it is similar to a diagonal matrix.

We conclude that the matrix $\mathbf{A}^T$ is diagonalizable.

■

12. We are given the matrix $\mathbf{A}=\begin{pmatrix}3&5\\0&2\end{pmatrix}$. The eigenvalues and eigenvectors are

$$\lambda_1=3,\ \mathbf{u}=\begin{pmatrix}1\\0\end{pmatrix} \text{ and } \lambda_2=2,\ \mathbf{v}=\begin{pmatrix}-5\\1\end{pmatrix}$$

The diagonalizing matrix is the eigenvector matrix $\mathbf{P}=\begin{pmatrix}1 & -5\\ 0 & 1\end{pmatrix}$ and $\mathbf{D}=\begin{pmatrix}3 & 0\\ 0 & 2\end{pmatrix}$. The formula for finding higher powers of matrices is:

$$\mathbf{A}^m=\mathbf{P}\mathbf{D}^m\mathbf{P}^{-1}$$

We need to find inverse matrix $\mathbf{P}^{-1}$:

$$\mathbf{P}^{-1}=\begin{pmatrix}1 & 5\\ 0 & 1\end{pmatrix}$$

Substituting $m=2,\ 3$ and 4 and the matrices into this formula $\mathbf{A}^m=\mathbf{P}\mathbf{D}^m\mathbf{P}^{-1}$ gives

$$\mathbf{A}^2=\mathbf{P}\mathbf{D}^2\mathbf{P}^{-1}=\begin{pmatrix}1 & -5\\ 0 & 1\end{pmatrix}\begin{pmatrix}3^2 & 0\\ 0 & 2^2\end{pmatrix}\begin{pmatrix}1 & 5\\ 0 & 1\end{pmatrix}=\begin{pmatrix}1 & -5\\ 0 & 1\end{pmatrix}\begin{pmatrix}9 & 0\\ 0 & 4\end{pmatrix}\begin{pmatrix}1 & 5\\ 0 & 1\end{pmatrix}=\begin{pmatrix}9 & 25\\ 0 & 4\end{pmatrix}$$

Similarly we have

$$\mathbf{A}^3=\begin{pmatrix}1 & -5\\ 0 & 1\end{pmatrix}\begin{pmatrix}27 & 0\\ 0 & 8\end{pmatrix}\begin{pmatrix}1 & 5\\ 0 & 1\end{pmatrix}=\begin{pmatrix}27 & 95\\ 0 & 8\end{pmatrix}$$

$$\mathbf{A}^4=\begin{pmatrix}1 & -5\\ 0 & 1\end{pmatrix}\begin{pmatrix}81 & 0\\ 0 & 16\end{pmatrix}\begin{pmatrix}1 & 5\\ 0 & 1\end{pmatrix}=\begin{pmatrix}81 & 325\\ 0 & 16\end{pmatrix}$$

Substituting $\mathbf{A}=\begin{pmatrix}3 & 5\\ 0 & 2\end{pmatrix}$, $\mathbf{A}^2=\begin{pmatrix}9 & 25\\ 0 & 4\end{pmatrix}$, $\mathbf{A}^3=\begin{pmatrix}27 & 95\\ 0 & 8\end{pmatrix}$ and $\mathbf{A}^4=\begin{pmatrix}81 & 325\\ 0 & 16\end{pmatrix}$ into

$$\begin{aligned}\exp(\mathbf{A}t) &= \mathbf{I}+\mathbf{A}t+\mathbf{A}^2\frac{t^2}{2!}+\mathbf{A}^3\frac{t^3}{3!}+\mathbf{A}^4\frac{t^4}{4!}+\cdots\\ &= \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}+\begin{pmatrix}3 & 5\\ 0 & 2\end{pmatrix}t+\begin{pmatrix}9 & 25\\ 0 & 4\end{pmatrix}\frac{t^2}{2!}+\begin{pmatrix}27 & 95\\ 0 & 8\end{pmatrix}\frac{t^3}{3!}+\begin{pmatrix}81 & 325\\ 0 & 16\end{pmatrix}\frac{t^4}{4!}+\cdots\end{aligned}$$

13. We are given that $\mathbf{F}=\begin{pmatrix}1 & 1\\ 1 & 0\end{pmatrix}$. The eigenvalues are given by

$$\begin{aligned}\det(\mathbf{F}-\lambda\mathbf{I}) &= \det\begin{pmatrix}1-\lambda & 1\\ 1 & 0-\lambda\end{pmatrix}\\ &= (1-\lambda)(0-\lambda)-1\\ &= \lambda^2-\lambda-1=0\end{aligned}$$

Solving the quadratic by using the formula $\lambda=\dfrac{-b\pm\sqrt{b^2-4ac}}{2a}$ with $a=1,\ b=-1$ and $c=-1$ gives

$$\lambda=\frac{-(-1)\pm\sqrt{(-1)^2-\left[4\times 1\times(-1)\right]}}{2}=\frac{1\pm\sqrt{5}}{2}$$

The eigevalues are $\lambda_1=\dfrac{1+\sqrt{5}}{2}$ and $\lambda_2=\dfrac{1-\sqrt{5}}{2}$. *What are the eigenvectors?*

Let **u** be the eigenvector belonging to $\lambda_1=\dfrac{1+\sqrt{5}}{2}$.

$$\left[\mathbf{F}-\left(\frac{1+\sqrt{5}}{2}\right)\mathbf{I}\right]\mathbf{u}=\begin{pmatrix}1-\frac{1+\sqrt{5}}{2} & 1\\ 1 & 0-\frac{1+\sqrt{5}}{2}\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix}$$

$$=\begin{pmatrix}\frac{1-\sqrt{5}}{2} & 1\\ 1 & -\frac{1+\sqrt{5}}{2}\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix}$$

$$=\begin{pmatrix}\lambda_2 & 1\\ 1 & -\lambda_1\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix}=\begin{pmatrix}0\\ 0\end{pmatrix}\qquad\left[\text{Because } \lambda_2=\frac{1-\sqrt{5}}{2} \text{ and } \lambda_1=\frac{1+\sqrt{5}}{2}\right]$$

Expanding matrices we have

$$\left.\begin{aligned}\lambda_2 x+y=0\\ x-\lambda_1 y=0\end{aligned}\right\}\text{ gives } x=\lambda_1 \text{ and } y=1$$

Note that this is the solution because $\lambda_1\lambda_2=\left(\frac{1+\sqrt{5}}{2}\right)\left(\frac{1-\sqrt{5}}{2}\right)=-1$. Thus $\mathbf{u}=\begin{pmatrix}\lambda_1\\ 1\end{pmatrix}$ is the eigenvector belonging to $\lambda_1=\frac{1+\sqrt{5}}{2}$. Let **v** be the eigenvector belonging to $\lambda_2=\frac{1-\sqrt{5}}{2}$

$$\left[\mathbf{F}-\left(\frac{1-\sqrt{5}}{2}\right)\mathbf{I}\right]\mathbf{v}=\begin{pmatrix}1-\frac{1-\sqrt{5}}{2} & 1\\ 1 & 0-\frac{1-\sqrt{5}}{2}\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix}$$

$$=\begin{pmatrix}\frac{1+\sqrt{5}}{2} & 1\\ 1 & -\frac{1-\sqrt{5}}{2}\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix}$$

$$=\begin{pmatrix}\lambda_1 & 1\\ 1 & -\lambda_2\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix}=\begin{pmatrix}0\\ 0\end{pmatrix}$$

Expanding the matrices we have

$$\left.\begin{aligned}\lambda_1 x+y=0\\ x-\lambda_2 y=0\end{aligned}\right\}\text{ gives } x=\lambda_2 \text{ and } y=1$$

Thus $\mathbf{v}=\begin{pmatrix}\lambda_2\\ 1\end{pmatrix}$ is the eigenvector belonging to $\lambda_2=\frac{1-\sqrt{5}}{2}$.

Since the eigenvectors **u** and **v** are linearly independent therefore the eigenvector matrix **P** is given by $\mathbf{P}=(\mathbf{u}\ \vdots\ \mathbf{v})=\begin{pmatrix}\lambda_1 & \lambda_2\\ 1 & 1\end{pmatrix}$.

The eigenvalue matrix $\mathbf{D}=\begin{pmatrix}\lambda_1 & 0\\ 0 & \lambda_2\end{pmatrix}$.

To ensure that we have the correct matrices **P** and **D** we check that $\mathbf{PD}=\mathbf{FP}$:

$$\mathbf{PD} = \begin{pmatrix} \lambda_1 & \lambda_2 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} = \begin{pmatrix} \lambda_1^2 & \lambda_2^2 \\ \lambda_1 & \lambda_2 \end{pmatrix}$$

$$\mathbf{FP} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} \lambda_1 & \lambda_2 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} \lambda_1 + 1 & \lambda_2 + 1 \\ \lambda_1 & \lambda_2 \end{pmatrix}$$

The bottom rows of **PD** and **FP** are equal but the top rows seem to be different. *Can we show that these are equivalent?*

$$\lambda_1^2 = \left(\frac{1+\sqrt{5}}{2}\right)^2 = \frac{1+2\sqrt{5}+5}{4}$$

$$= \frac{6+2\sqrt{5}}{4} = \frac{3+\sqrt{5}}{2} = 1 + \frac{1+\sqrt{5}}{2} = 1 + \lambda_1$$

Similarly we have $\lambda_2^2 = 1 + \lambda_2$.

$\mathbf{PD} = \mathbf{FP}$ so we have the correct eigenvector and eigenvalues matrices **P** and **D** respectively.

14. By Question 3 of the last Exercises 7.2 the characteristic polynomial $p(\lambda)$ is given by

$$p(\lambda) = \lambda^2 - tr(\mathbf{A})\lambda + \det(\mathbf{A})$$

We find λ by equating this to zero and solving the resulting quadratic equation by using the formula with $a = 1$, $b = -tr(\mathbf{A})$ and $c = \det(\mathbf{A})$

$$\lambda^2 - tr(\mathbf{A})\lambda + \det(\mathbf{A}) = 0$$

$$\lambda = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{tr(\mathbf{A}) \pm \sqrt{[tr(\mathbf{A})]^2 - 4\det(\mathbf{A})}}{2}$$

Since we are given that $[tr(\mathbf{A})]^2 > 4\det(\mathbf{A})$ therefore we have 2 distinct roots of the quadratic, say $\lambda_1 = \dfrac{tr(\mathbf{A}) + \sqrt{[tr(\mathbf{A})]^2 - 4\det(\mathbf{A})}}{2}$ and $\lambda_2 = \dfrac{tr(\mathbf{A}) - \sqrt{[tr(\mathbf{A})]^2 - 4\det(\mathbf{A})}}{2}$.

Since we have a 2 by 2 matrix and 2 distinct eigenvalues therefore by Proposition (7-13):

Proposition (7-13). If a square *n* by *n* matrix **A** has *n* distinct eigenvalues then the matrix **A** is diagonalizable.

We conclude that the matrix **A** is diagonalizable.

■

15. Required to prove that if **A** is an invertible and diagonalizable matrix then $\mathbf{A}^{-1}$ is also diagonalizable.

Proof.

Assuming **A** is diagonalizable means that there exists matrices **P** and **D** where **D** is a diagonal matrix such that

$$\mathbf{P}^{-1}\mathbf{A}\mathbf{P} = \mathbf{D}$$

Using our rules of matrices we have

$$\mathbf{P}^{-1}\mathbf{A}\mathbf{P} = \mathbf{D}$$

$$\mathbf{A} = \mathbf{P}\mathbf{D}\mathbf{P}^{-1} \qquad \left[\text{Pre-multiplying by } \mathbf{P} \text{ and Post multiplying by } \mathbf{P}^{-1}\right]$$

$$\mathbf{A}^{-1} = \left(\mathbf{P}\mathbf{D}\mathbf{P}^{-1}\right)^{-1}$$

$$= \left(\mathbf{P}^{-1}\right)^{-1}\mathbf{D}^{-1}\mathbf{P}^{-1} \qquad \left[\text{Applying } \left(\mathbf{ABC}\right)^{-1} = \mathbf{C}^{-1}\mathbf{B}^{-1}\mathbf{A}^{-1}\right]$$

This last result $\mathbf{A}^{-1} = \left(\mathbf{P}^{-1}\right)^{-1}\mathbf{D}^{-1}\mathbf{P}^{-1}$ means that $\mathbf{D}^{-1}$ and $\mathbf{A}^{-1}$ are similar matrices because:

Definition (7-2). A square matrix **B** is **similar** to a matrix **A** if there exists an invertible matrix **P** such that $\mathbf{P}^{-1}\mathbf{A}\mathbf{P} = \mathbf{B}$.

$\mathbf{D}^{-1}$ is a diagonal matrix therefore $\mathbf{A}^{-1}$ is diagonalizable.

■

16. We need to prove that if **A** is diagonalizable then $\mathbf{A}^m$ (where $m \in \mathbb{N}$) is diagonalizable.
Proof.
A is diagonalizable therefore by Proposition (7-14):

Proposition (7-14). If a *n* by *n* matrix **A** is diagonalizable with $\mathbf{P}^{-1}\mathbf{A}\mathbf{P} = \mathbf{D}$ where **D** is a diagonal matrix then $\mathbf{A}^m = \mathbf{P}\mathbf{D}^m\mathbf{P}^{-1}$

We have

$$\mathbf{A}^m = \mathbf{P}\mathbf{D}^m\mathbf{P}^{-1} = \left(\mathbf{P}^{-1}\right)^{-1}\mathbf{D}^m\mathbf{P}^{-1} \qquad \left[\text{Remember } \left(\mathbf{P}^{-1}\right)^{-1} = \mathbf{P}\right]$$

Thus $\mathbf{A}^m$ is similar to $\mathbf{D}^m$ and the result of question 9 we have $\mathbf{D}^m$ is a diagonal matrix. By Definition (7-3):

Definition (7-3). A *n* by *n* matrix **A** is diagonalizable if it is similar to a diagonal matrix.

We conclude that the matrix **A** is diagonalizable.

■

17. Required to prove that **AB** is similar to **BA** provided **A** and **B** are invertible matrices.
Proof.
Let the invertible matrix $\mathbf{P} = \mathbf{A}$ then $\mathbf{P}^{-1} = \mathbf{A}^{-1}$ and

$$\mathbf{P}^{-1}\left(\mathbf{AB}\right)\mathbf{P} = \mathbf{A}^{-1}\left(\mathbf{AB}\right)\mathbf{A}$$

$$= \left(\mathbf{A}^{-1}\mathbf{A}\right)\mathbf{BA} \qquad \left[\text{Because } \mathbf{A}\left(\mathbf{BC}\right) = \left(\mathbf{AB}\right)\mathbf{C}\right]$$

$$= \mathbf{IBA} = \mathbf{BA}$$

Since we have $\mathbf{P}^{-1}\left(\mathbf{AB}\right)\mathbf{P} = \mathbf{BA}$ therefore by Definition (7-2) :

Definition (7-2). A square matrix **B** is **similar** to a matrix **A** if there exists an invertible matrix **P** such that $\mathbf{P}^{-1}\mathbf{A}\mathbf{P} = \mathbf{B}$.

We conclude that **AB** is similar to **BA**.

■

18. (a) Need to prove that if matrices **A** and **B** are similar then $tr\left(\mathbf{A}\right) = tr\left(\mathbf{B}\right)$ where *tr* is trace.
Proof.
Assume matrices **A** and **B** are similar. Then by Proposition (7-10):

(7-10). Let **A** and **B** be similar matrices. The eigenvalues of these matrices are identical.

We have matrices **A** and **B** have the same eigenvalues, call them $\lambda_1,\ \lambda_2,\ \lambda_3,\ \cdots$ and λ_n. Then by Proposition (7-6) part (b):

(7-6) (b) Trace of a matrix **A** is given by sum of eigenvalues, $tr(\mathbf{A}) = \lambda_1 + \lambda_2 + \lambda_3 + \cdots + \lambda_n$

We have

$$tr(\mathbf{A}) = \lambda_1 + \lambda_2 + \lambda_3 + \cdots + \lambda_n = tr(\mathbf{B})$$

Hence we have our required result, that is $tr(\mathbf{A}) = tr(\mathbf{B})$.

■

(b) Required to prove that if matrices **A** and **B** are similar then $\det(\mathbf{A}) = \det(\mathbf{B})$.
Proof.
Suppose matrices **A** and **B** are similar. By Proposition (7-10) matrices **A** and **B** have the same eigenvalues and by (7-6) part (a):

(7-6) (a) The determinant of the matrix **A** is given by $\det(\mathbf{A}) = \lambda_1 \times \lambda_2 \times \lambda_3 \times \cdots \times \lambda_n$.

Hence $\det(\mathbf{A}) = \det(\mathbf{B})$.

■

19. Let $\lambda_1,\ \lambda_2,\ \lambda_3,\ \cdots$ and λ_n be the eigenvalues of matrix **A**. We are given that

$$|\lambda_1| < 1,\ |\lambda_2| < 1,\ |\lambda_3| < 1,\ \cdots \text{ and } |\lambda_n| < 1$$

We are also told that the matrix **A** is diagonalizable which means that by Proposition (7-14) we have

$$\mathbf{A}^m = \mathbf{P}\mathbf{D}^m\mathbf{P}^{-1}$$

What is $\mathbf{D}^m$ *equal to?*

$$\mathbf{D}^m = \begin{pmatrix} (\lambda_1)^m & 0 & \mathbf{0} \\ 0 & \ddots & 0 \\ \mathbf{0} & 0 & (\lambda_n)^m \end{pmatrix}$$

By using the hint in the question we know that if $|x| < 1$ then $x^m \to 0$ as $m \to \infty$. Applying this to $(\lambda_1)^m,\ (\lambda_2)^m,\ (\lambda_3)^m,\ \cdots,\ (\lambda_n)^m$ we have $\lim\limits_{m\to\infty}(\lambda_1)^m = \lim\limits_{m\to\infty}(\lambda_2)^m,\ \cdots,\ \lim\limits_{m\to\infty}(\lambda_n)^m = 0$.

$$\lim_{m\to\infty}(\mathbf{D}^m) = \begin{pmatrix} (\lambda_1)^m & 0 & \mathbf{0} \\ 0 & \ddots & 0 \\ \mathbf{0} & 0 & (\lambda_n)^m \end{pmatrix} = \begin{pmatrix} 0 & 0 & \mathbf{0} \\ 0 & \ddots & 0 \\ \mathbf{0} & 0 & 0 \end{pmatrix} = \mathbf{O}$$

Thus as $m \to \infty$ the matrix $\mathbf{D}^m$ is the zero matrix. Hence as $m \to \infty$ we have

$$\mathbf{A}^m = \mathbf{P}\mathbf{D}^m\mathbf{P}^{-1} = \mathbf{P}\mathbf{O}\mathbf{P}^{-1} = \mathbf{O}$$

This means that as $m \to \infty$ the matrix $\mathbf{A}^m$ is the zero matrix.

20. We need to prove that the eigevalues of $\mathbf{A}^m$ are $(\lambda_1)^m,\ (\lambda_2)^m,\ \cdots,\ (\lambda_n)^m$ providing **A** is diagonalizable with eigenvalues $\lambda_1,\ \lambda_2,\ \cdots,\ \lambda_n$.
Proof.
We are told that **A** is diagonalizable so by Proposition (7-14):

Proposition (7-14). If **A** is diagonalizable then $\mathbf{A}^m = \mathbf{P}\mathbf{D}^m\mathbf{P}^{-1}$

We have

$$\mathbf{A}^m = \mathbf{P}\mathbf{D}^m\mathbf{P}^{-1} \qquad (*)$$

where **D** is the diagonal matrix with eigenvalues along the leading diagonal:

$$\mathbf{D} = \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix}$$

What is $\mathbf{D}^m$ *equal to?*

$$\mathbf{D}^m = \begin{pmatrix} (\lambda_1)^m & & 0 \\ & \ddots & \\ 0 & & (\lambda_n)^m \end{pmatrix}$$

From (*) we have $\mathbf{A}^m$ is similar to $\mathbf{D}^m$ and by Proposition (7-9) part (b) the diagonal matrix $\mathbf{D}^m$ is similar to $\mathbf{A}^m$. Therefore

$$\mathbf{P}^{-1}\mathbf{A}^m\mathbf{P} = \mathbf{D}^m$$

The eigenvalues of $\mathbf{A}^m$ are the entries on the leading diagonal in $\mathbf{D}^m$. Thus $(\lambda_1)^m,\ (\lambda_2)^m,\ \cdots,\ (\lambda_n)^m$ are the eigenvalues of $\mathbf{A}^m$.

21. Theorem (7-11) is the following:

A *n* by *n* matrix **A** is diagonalizable $\Leftrightarrow$ it has *n* linearly independent eigenvectors.

How do we prove this theorem?

We have the arrows pointing in both directions therefore we need to prove

1) **A** is diagonalizable $\Rightarrow$ it has *n* linearly independent eigenvectors.
2) **A** has *n* linearly independent eigenvectors $\Rightarrow$ **A** is diagonalizable.

Proof.

$(\Rightarrow)$. Assume the matrix **A** is diagonalizable and the eigenvectors of **A** are

$$\{\mathbf{v}_1,\ \mathbf{v}_2,\ \mathbf{v}_3,\ \cdots,\ \mathbf{v}_n\}$$

This means there is an invertible matrix **P** such that $\mathbf{P}^{-1}\mathbf{AP} = \mathbf{D}$ where **D** is a diagonal matrix. Remember eigenvector matrix **P** contains the eigenvectors of matrix **A**:

$$\mathbf{P} = \begin{pmatrix}\mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3 & \cdots & \mathbf{v}_n\end{pmatrix}$$

Matrix **P** is invertible and from chapter 2 result (2-14):

> Proposition (2-14) . Let **A** be the *n* by *n* matrix whose columns are given by the vectors $\mathbf{v}_1,\ \mathbf{v}_2,\ \mathbf{v}_3,\ \cdots$ and $\mathbf{v}_n$:
>
> $$\mathbf{A} = \begin{pmatrix}\mathbf{v}_1 & \mathbf{v}_2 & \cdots & \mathbf{v}_n\end{pmatrix}$$
>
> Then vectors $\mathbf{v}_1,\ \mathbf{v}_2,\ \cdots,\ \mathbf{v}_n$ are linearly independent $\Leftrightarrow$ matrix **A** is invertible.

Hence the eigenvectors are linearly independent.

$(\Leftarrow)$. Assume matrix **A** has *n* linearly independent eigenvectors belonging to the eigenvalues $\lambda_1,\ \lambda_2,\ \lambda_3,\ \cdots,\ \lambda_n$ respectively and

$$\mathbf{P} = \begin{pmatrix}\mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3 & \cdots & \mathbf{v}_n\end{pmatrix}$$

Consider the matrix multiplication $\mathbf{P}^{-1}\mathbf{AP}$. We can carry out this matrix multiplication as $\mathbf{P}^{-1}\mathbf{AP} = \mathbf{P}^{-1}(\mathbf{AP})$:

$$\begin{aligned}\mathbf{AP} &= \mathbf{A}\begin{pmatrix}\mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3 & \cdots & \mathbf{v}_n\end{pmatrix} \\ &= \begin{pmatrix}\mathbf{Av}_1 & \mathbf{Av}_2 & \mathbf{Av}_3 & \cdots & \mathbf{Av}_n\end{pmatrix}\end{aligned}$$

Since $\mathbf{v}_1,\ \mathbf{v}_2,\ \cdots,\ \mathbf{v}_n$ are eigenvectors of matrix **A** so by the definition (7.1):

(7.1)	$\mathbf{Au} = \lambda\mathbf{u}$

We have $\mathbf{Av}_1 = \lambda_1\mathbf{v}_1$, $\mathbf{Av}_2 = \lambda_2\mathbf{v}_2$, $\mathbf{Av}_3 = \lambda_3\mathbf{v}_3$, $\cdots$, $\mathbf{Av}_n = \lambda_n\mathbf{v}_n$. Therefore

$$\mathbf{AP} = \begin{pmatrix}\lambda_1\mathbf{v}_1 & \lambda_2\mathbf{v}_2 & \lambda_3\mathbf{v}_3 & \cdots & \lambda_n\mathbf{v}_n\end{pmatrix}$$

We can also evaluate **PD** where **D** is a diagonal matrix. Let $\mathbf{D} = \begin{pmatrix} k_1 & 0 & \mathbf{0} \\ 0 & \ddots & 0 \\ \mathbf{0} & 0 & k_n \end{pmatrix}$ then

$$\mathbf{PD} = \begin{pmatrix}\mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3 & \cdots & \mathbf{v}_n\end{pmatrix}\begin{pmatrix} k_1 & 0 & \mathbf{0} \\ 0 & \ddots & 0 \\ \mathbf{0} & 0 & k_n \end{pmatrix} = \begin{pmatrix}k_1\mathbf{v}_1 & k_2\mathbf{v}_2 & k_3\mathbf{v}_3 & \cdots & k_n\mathbf{v}_n\end{pmatrix}$$

Let $k_1 = \lambda_1$, $k_2 = \lambda_2$, $\cdots$, $k_n = \lambda_n$ then we have

$$\mathbf{PD} = \begin{pmatrix}k_1\mathbf{v}_1 & k_2\mathbf{v}_2 & k_3\mathbf{v}_3 & \cdots & k_n\mathbf{v}_n\end{pmatrix} = \mathbf{AP}$$

Left multiplying both sides of $\mathbf{PD} = \mathbf{AP}$ by $\mathbf{P}^{-1}$ gives

$$\mathbf{P}^{-1}(\mathbf{PD}) = \left(\mathbf{P}^{-1}\mathbf{P}\right)\mathbf{D} = \mathbf{ID} = \mathbf{D} = \mathbf{P}^{-1}\mathbf{AP}$$

Hence $\mathbf{P}^{-1}\mathbf{AP} = \mathbf{D}$ means the eigenvector matrix **P** diagonalizes the given matrix **A**. Hence matrix **A** is diagonalizable.

■