Complete Solutions to Exercises 3.5

1. We need to find the row and column vectors of each matrix. Reading these off we have:

(a) For matrix $\mathbf{A}=\begin{pmatrix}1 & 2\\ 3 & 4\end{pmatrix}$ we have two row vectors; $\begin{pmatrix}1\\ 2\end{pmatrix}$ and $\begin{pmatrix}3\\ 4\end{pmatrix}$.

What are the column vectors of the given matrix ***A****?*

$$\begin{pmatrix}1\\ 3\end{pmatrix} \text{ and } \begin{pmatrix}2\\ 4\end{pmatrix}$$

(b) For $\mathbf{B}=\begin{pmatrix}1 & 2 & 3 & 4\\ 5 & 6 & 7 & 8\\ 9 & 10 & 11 & 12\end{pmatrix}$ we have three row vectors;

$$\begin{pmatrix}1\\ 2\\ 3\\ 4\end{pmatrix}, \begin{pmatrix}5\\ 6\\ 7\\ 8\end{pmatrix} \text{ and } \begin{pmatrix}9\\ 10\\ 11\\ 12\end{pmatrix}$$

How many column vectors does the given matrix ***B*** *have?*

4 and these are $\begin{pmatrix}1\\ 5\\ 9\end{pmatrix}, \begin{pmatrix}2\\ 6\\ 10\end{pmatrix}, \begin{pmatrix}3\\ 7\\ 11\end{pmatrix}$ and $\begin{pmatrix}4\\ 8\\ 12\end{pmatrix}$.

(c) *What are the row vectors of the given matrix* $\mathbf{C}=\begin{pmatrix}1 & 2\\ 3 & 4\\ 5 & 6\end{pmatrix}$*?*

There are three row vectors and these are $\begin{pmatrix}1\\ 2\end{pmatrix}, \begin{pmatrix}3\\ 4\end{pmatrix}$ and $\begin{pmatrix}5\\ 6\end{pmatrix}$. The two column vectors of matrix **C** are $\begin{pmatrix}1\\ 3\\ 5\end{pmatrix}$ and $\begin{pmatrix}2\\ 4\\ 6\end{pmatrix}$.

(d) The two row vectors of the given matrix $\mathbf{D}=\begin{pmatrix}1 & 2 & 3\\ 4 & 5 & 6\end{pmatrix}$ are $\begin{pmatrix}1\\ 2\\ 3\end{pmatrix}$ and $\begin{pmatrix}4\\ 5\\ 6\end{pmatrix}$. The three column vectors of this matrix $\mathbf{D}=\begin{pmatrix}1 & 2 & 3\\ 4 & 5 & 6\end{pmatrix}$ are $\begin{pmatrix}1\\ 4\end{pmatrix}, \begin{pmatrix}2\\ 5\end{pmatrix}$ and $\begin{pmatrix}3\\ 6\end{pmatrix}$.

(e) Similarly the row vectors of $\mathbf{E}=\begin{pmatrix}-1 & 2 & 5\\ -3 & 7 & 0\\ -8 & 1 & 3\end{pmatrix}$ are $\begin{pmatrix}-1\\ 2\\ 5\end{pmatrix}, \begin{pmatrix}-3\\ 7\\ 0\end{pmatrix}$ and $\begin{pmatrix}-8\\ 1\\ 3\end{pmatrix}$.

The column vectors of $\mathbf{E}=\begin{pmatrix}-1 & 2 & 5\\ -3 & 7 & 0\\ -8 & 1 & 3\end{pmatrix}$ are $\begin{pmatrix}-1\\ -3\\ -8\end{pmatrix}, \begin{pmatrix}2\\ 7\\ 1\end{pmatrix}$ and $\begin{pmatrix}5\\ 0\\ 3\end{pmatrix}$.

(f) The four row vectors of $\mathbf{F}=\begin{pmatrix}-5 & 2 & 3\\ 7 & 1 & 0\\ -7 & 6 & 1\\ -2 & 5 & 2\end{pmatrix}$ are

$$\begin{pmatrix}-5\\ 2\\ 3\end{pmatrix}, \begin{pmatrix}7\\ 1\\ 0\end{pmatrix}, \begin{pmatrix}-7\\ 6\\ 1\end{pmatrix} \text{ and } \begin{pmatrix}-2\\ 5\\ 2\end{pmatrix}$$

The three column vectors of **F** are $\begin{pmatrix}-5\\ 7\\ -7\\ -2\end{pmatrix}, \begin{pmatrix}2\\ 1\\ 6\\ 5\end{pmatrix}$ and $\begin{pmatrix}3\\ 0\\ 1\\ 2\end{pmatrix}$.

2. We use MATLAB to place each of the given matrices into reduced row echelon form. (You may use other software which you are familiar with.)

(a) The reduced row echelon form of the given matrix $\mathbf{A}=\begin{pmatrix}1 & 2\\ 3 & 4\end{pmatrix}$ is $\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$. Therefore a basis for the row space of matrix **A** is $\left\{\begin{pmatrix}1\\ 0\end{pmatrix}, \begin{pmatrix}0\\ 1\end{pmatrix}\right\}$. *What is* $rank(\mathbf{A})$ *equal to?*

Since we have two vectors in the basis for the row space of **A** therefore $rank(\mathbf{A})=2$.

(b) *How do we find the reduced row echelon form of the given matrix* $\mathbf{B}=\begin{pmatrix}1 & 2 & 3 & 4\\ 5 & 6 & 7 & 8\\ 9 & 10 & 11 & 12\end{pmatrix}$*?*

By entering the matrix into MATLAB and then using the command rref(B). We have

$$\mathbf{R}=\begin{pmatrix}1 & 0 & -1 & -2\\ 0 & 1 & 2 & 3\\ 0 & 0 & 0 & 0\end{pmatrix}$$

where **R** is the reduced row echelon form matrix. *What is a basis for the row space of **B**?*
It is the non-zero rows of the matrix **R**. The basis vectors for the row space of **B** are

$$\left\{\begin{pmatrix}1\\ 0\\ -1\\ -2\end{pmatrix}, \begin{pmatrix}0\\ 1\\ 2\\ 3\end{pmatrix}\right\}$$

We have two vectors in the basis for the row space of **B** therefore $rank(\mathbf{B})=2$. We can span the row space of matrix **B** by the first rows of matrix **R**, don't need all three rows of matrix **B**.

(c) The reduced row echelon form of $\mathbf{C}=\begin{pmatrix}1 & 2\\ 3 & 4\\ 5 & 6\end{pmatrix}$ is $\begin{pmatrix}1 & 0\\ 0 & 1\\ 0 & 0\end{pmatrix}$. *What are the basis vectors for the row space of matrix **C** equal to?*

A basis for the row space is $\left\{\begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix}\right\}$. Thus $rank(\mathbf{C}) = 2$.

(d) By using MATLAB to find the reduced row echelon form of $\mathbf{D} = \begin{pmatrix}1 & 2 & 3\\4 & 5 & 6\end{pmatrix}$, we have

$$\mathbf{R} = \begin{pmatrix}1 & 0 & -1\\0 & 1 & 2\end{pmatrix}$$

What are the basis vectors for the row space of the given matrix ***D****?*

$$\left\{\begin{pmatrix}1\\0\\-1\end{pmatrix}, \begin{pmatrix}0\\1\\2\end{pmatrix}\right\}$$

We have two vectors in a basis for the row space therefore $rank(\mathbf{D}) = 2$.

(e) Similarly we have the reduced row echelon form matrix **R** of $\mathbf{E} = \begin{pmatrix}-1 & 2 & 5\\-3 & 7 & 0\\-8 & 1 & 3\end{pmatrix}$ is

$$\mathbf{R} = \begin{pmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{pmatrix}$$

A basis for the row space of **E** is $\left\{\begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\end{pmatrix}, \begin{pmatrix}0\\0\\1\end{pmatrix}\right\}$. *What is the rank of matrix* ***E****?*

We have three vectors in the basis for row space of **E** therefore $rank(\mathbf{E}) = 3$.

(f) Repeating the above process we have the reduced row echelon form matrix **R** of $\mathbf{F} = \begin{pmatrix}-5 & 2 & 3\\7 & 1 & 0\\-7 & 6 & 1\\-2 & 5 & 2\end{pmatrix}$ is $\mathbf{R} = \begin{pmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\\0 & 0 & 0\end{pmatrix}$. *What is a basis for the row space of* ***F****?*

It is non-zero rows of the matrix **R**. Thus $\left\{\begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\end{pmatrix}, \begin{pmatrix}0\\0\\1\end{pmatrix}\right\}$ is a basis for the row space of **F**.

Hence $rank(\mathbf{F}) = 3$.

3. To find a basis for the subspaces of $\mathbb{R}^n$ which is spanned by $\{\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3, \cdots, \mathbf{r}_m\}$ we use the stated procedure:

<u>Step 1</u>: Form the matrix $\mathbf{A} = \begin{pmatrix}\mathbf{r}_1\\\vdots\\\mathbf{r}_m\end{pmatrix}$.

<u>Step 2</u>: By using MATLAB or otherwise put this matrix **A** into (reduced) row echelon form, **R** say.

Step 3: The non-zero rows of this reduced row echelon form matrix **R** is a basis for $span(\{\mathbf{r}_1,\ \mathbf{r}_2,\ \mathbf{r}_3,\ \cdots,\ \mathbf{r}_m\})$.

(a) Step 1: We are given the vectors $\mathbf{u}=\begin{pmatrix}1\\5\end{pmatrix}$, $\mathbf{v}=\begin{pmatrix}7\\2\end{pmatrix}$ therefore $\mathbf{A}=\begin{pmatrix}\mathbf{u}\\\mathbf{v}\end{pmatrix}=\begin{pmatrix}1&5\\7&2\end{pmatrix}$.

Step 2: The reduced row echelon form is:

$$\mathbf{R}=\begin{pmatrix}1&0\\0&1\end{pmatrix}$$

Step 3: A basis for $span(\mathbf{u},\ \mathbf{v})$ which is the whole of $\mathbb{R}^2$ is $\left\{\begin{pmatrix}1\\0\end{pmatrix},\ \begin{pmatrix}0\\1\end{pmatrix}\right\}$.

(b) Step 1: We are given the vectors $\mathbf{u}=\begin{pmatrix}-1\\-3\end{pmatrix}$, $\mathbf{v}=\begin{pmatrix}4\\12\end{pmatrix}$ therefore $\mathbf{A}=\begin{pmatrix}-1&-3\\4&12\end{pmatrix}$.

Step 2: The reduced row echelon form is:

$$\mathbf{R}=\begin{pmatrix}1&3\\0&0\end{pmatrix}$$

Step 3: A basis for $span(\mathbf{u},\ \mathbf{v})$ which is a subspace of $\mathbb{R}^2$ is $\left\{\begin{pmatrix}1\\3\end{pmatrix}\right\}$. Remember we only select the non-zero rows of matrix **R**.

(c) The matrix $\mathbf{A}=\begin{pmatrix}\mathbf{u}\\\mathbf{v}\\\mathbf{w}\end{pmatrix}=\begin{pmatrix}3&6&5\\2&1&2\\12&15&15\end{pmatrix}$. Putting this into reduced row echelon form gives $\mathbf{R}=\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}$. Thus a basis for $span(\mathbf{u},\ \mathbf{v},\ \mathbf{w})$ which is the whole of $\mathbb{R}^3$ is $\left\{\begin{pmatrix}1\\0\\0\end{pmatrix},\ \begin{pmatrix}0\\1\\0\end{pmatrix},\ \begin{pmatrix}0\\0\\1\end{pmatrix}\right\}$.

(d) The matrix $\mathbf{A}=\begin{pmatrix}\mathbf{u}\\\mathbf{v}\\\mathbf{w}\\\mathbf{x}\end{pmatrix}=\begin{pmatrix}-1&-2&-5\\0&1&5\\2&3&2\\-4&1&-7\end{pmatrix}$. Using MATLAB to place this matrix **A** into reduced row echelon form gives $\mathbf{R}=\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\\0&0&0\end{pmatrix}$. *What is a basis for the subspace* $S=span(\mathbf{u},\ \mathbf{v},\ \mathbf{w},\ \mathbf{x})$*?*

It is the vectors which are the non-zero rows of the matrix **R**, that is $\left\{\begin{pmatrix}1\\0\\0\end{pmatrix},\ \begin{pmatrix}0\\1\\0\end{pmatrix},\ \begin{pmatrix}0\\0\\1\end{pmatrix}\right\}$.

In this case S is a whole of $\mathbb{R}^3$.

(e) The matrix $\mathbf{A} = \begin{pmatrix} \mathbf{u} \\ \mathbf{v} \\ \mathbf{w} \\ \mathbf{x} \end{pmatrix} = \begin{pmatrix} 1 & 2 & 2 & 2 \\ -1 & 3 & 5 & 7 \\ 2 & -1 & -3 & -5 \\ 0 & 5 & 7 & 9 \end{pmatrix}$. The reduced row echelon form of $\mathbf{A}$ is

$$\mathbf{R} = \begin{pmatrix} 1 & 0 & -0.8 & -1.6 \\ 0 & 1 & 1.4 & 1.8 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

The basis vectors of the spanned subspace are the non-zero rows of the matrix $\mathbf{R}$. Thus a basis for $S = span(\mathbf{u},\ \mathbf{v},\ \mathbf{w},\ \mathbf{x})$ is $\left\{ \begin{pmatrix} 1 \\ 0 \\ -0.8 \\ -1.6 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1.4 \\ 1.8 \end{pmatrix} \right\}$ which is a subspace of $\mathbb{R}^4$. S is actually a plane in $\mathbb{R}^4$ because we only need 2 axes or 2 basis vectors to create the subspace S.

4. We have the matrix $\mathbf{A} = \begin{pmatrix} \mathbf{u} \\ \mathbf{v} \\ \mathbf{w} \\ \mathbf{x} \\ \mathbf{y} \end{pmatrix} = \begin{pmatrix} -1 & 1 & -1 & 1 \\ -5 & -9 & -7 & -1 \\ 0 & 7 & 1 & 3 \\ -6 & -1 & -7 & 3 \\ 2 & 5 & 3 & 1 \end{pmatrix}$. We find the reduced row echelon form of this matrix by using the hint in the question which says apply the command rats(rref(A)) in MATLAB. We have

$$\mathbf{R} = \begin{pmatrix} 1 & 0 & 8/7 & -4/7 \\ 0 & 1 & 1/7 & 3/7 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

The basis for the given subspace S is the vectors which are the non-zero rows of the matrix $\mathbf{R}$. We have two vectors which form a basis for S and these are first two rows of $\mathbf{R}$:

$$\mathbf{v}_1 = \begin{pmatrix} 1 \\ 0 \\ 8/7 \\ -4/7 \end{pmatrix} = \frac{1}{7}\begin{pmatrix} 7 \\ 0 \\ 8 \\ -4 \end{pmatrix} \text{ and } \mathbf{v}_2 = \begin{pmatrix} 0 \\ 1 \\ 1/7 \\ 3/7 \end{pmatrix} = \frac{1}{7}\begin{pmatrix} 0 \\ 7 \\ 1 \\ 3 \end{pmatrix}$$

Thus a basis for the subspace S of $\mathbb{R}^4$ is $\{\mathbf{v}_1,\ \mathbf{v}_2\}$ where $\mathbf{v}_1$ and $\mathbf{v}_2$ are as above.

Note that we only need 2 vectors, $\{\mathbf{v}_1,\ \mathbf{v}_2\}$, to span the given subspace S which is a plane in $\mathbb{R}^4$.

5. In each case we transpose each of the given matrices and then use MATLAB. The reason for transposing is that the columns of **A** become the rows of $\mathbf{A}^T$ and we can then use elementary row operations on $\mathbf{A}^T$.
An easier method is to use the reduced row echelon form matrix **R** found in question 2 and then consider the columns with leading ones. See the end of part (b).

(a) *What is the transpose of* $\mathbf{A}=\begin{pmatrix}1 & 2\\ 3 & 4\end{pmatrix}$*?*

$$\mathbf{A}^T=\begin{pmatrix}1 & 2\\ 3 & 4\end{pmatrix}^T=\begin{pmatrix}1 & 3\\ 2 & 4\end{pmatrix}$$

Remember when transposing the columns of **A** which are $\left\{\begin{pmatrix}1\\3\end{pmatrix},\ \begin{pmatrix}2\\4\end{pmatrix}\right\}$ become the rows of $\mathbf{A}^T$.

The reduced row echelon form of $\mathbf{A}^T=\begin{pmatrix}1 & 3\\ 2 & 4\end{pmatrix}$ is $\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$. Therefore a basis for the row space of matrix $\mathbf{A}^T$ is $\left\{\begin{pmatrix}1\\0\end{pmatrix},\ \begin{pmatrix}0\\1\end{pmatrix}\right\}$. *What is a basis for the column space of **A**?*

It is exactly these vectors $\left\{\begin{pmatrix}1\\0\end{pmatrix},\ \begin{pmatrix}0\\1\end{pmatrix}\right\}$ because the rows of $\mathbf{A}^T$ are the columns of **A**.

What is $rank(\mathbf{A})$ *equal to?*

By Definition (3-12) we have $\text{rank}(\mathbf{A})=\text{row rank of }\mathbf{A}$ and we have already evaluated the row rank in question 2(a). Also by Proposition (3-18) we have

$$rank(\mathbf{A})=\text{row rank}=\text{column rank}$$

Thus by solution to question 2(a) we have $rank(\mathbf{A})=2$.

We do **not** have to evaluate the rank of the matrix because it was already determined in question 2.

(b) Similarly transposing $\mathbf{B}=\begin{pmatrix}1 & 2 & 3 & 4\\ 5 & 6 & 7 & 8\\ 9 & 10 & 11 & 12\end{pmatrix}$ we have $\mathbf{B}^T=\begin{pmatrix}1 & 5 & 9\\ 2 & 6 & 10\\ 3 & 7 & 11\\ 4 & 8 & 12\end{pmatrix}$. Note that you can transpose a matrix **B** in MATLAB by entering the matrix and then using the command B’.

By entering the matrix into MATLAB and then using the command rref(B) we have

$$\mathbf{R}=\begin{pmatrix}1 & 0 & -1\\ 0 & 1 & 2\\ 0 & 0 & 0\\ 0 & 0 & 0\end{pmatrix}$$

where **R** is the reduced row echelon form. *What is a basis for the row space of* $\mathbf{B}^T$ *?*

It is the non-zero rows of the matrix **R**. The basis vectors for the row space of $\mathbf{B}^T$ are

$$\left\{\begin{pmatrix}1\\0\\-1\end{pmatrix},\begin{pmatrix}0\\1\\2\end{pmatrix}\right\}$$

And these are the basis vectors for the column space of the matrix **B**. Also $rank(\mathbf{B})=2$.
The other approach to finding a basis for matrix **B** is to consider the columns with leading ones.
Since the matrix **R** has leading ones in the first 2 columns so a basis for the column space is the first two columns of matrix **B**. Hence a basis for the column space is

$$\left\{\begin{pmatrix}1\\5\\9\end{pmatrix},\begin{pmatrix}2\\6\\10\end{pmatrix}\right\}$$

We can transfer these to simpler numbers by using row operations:

$$\begin{array}{c} \mathrm{R}_1 \\ \mathrm{R}_2 \end{array}\begin{pmatrix}1&5&9\\2&6&10\end{pmatrix}$$

Executing the following row operations:

$$\begin{array}{c} \mathrm{R}_1 \\ \mathrm{R}_2{*}=\mathrm{R}_2-2\mathrm{R}_1 \end{array}\begin{pmatrix}1&5&9\\0&-4&-8\end{pmatrix}$$

Dividing the bottom row by -4:

$$\begin{array}{c} \mathrm{R}_1 \\ \mathrm{R}_2{*}/-4 \end{array}\begin{pmatrix}1&5&9\\0&1&2\end{pmatrix}$$

Carrying out the row operation:

$$\begin{array}{c} \mathrm{R}_1-5\mathrm{R}_2{**} \\ \mathrm{R}_2{**} \end{array}\begin{pmatrix}1&0&-1\\0&1&2\end{pmatrix}$$

Now we achieve the basis that we had by transposing the matrix:

$$\left\{\begin{pmatrix}1\\0\\-1\end{pmatrix},\begin{pmatrix}0\\1\\2\end{pmatrix}\right\}$$

The advantage of this method is you don't have to put the whole matrix into reduced row echelon form twice – for row space and then for column space. Actually the basis where we copied the first columns of matrix **B** is enough, we don't need to carry out the row operations above. Hence $\left\{\begin{pmatrix}1\\5\\9\end{pmatrix},\begin{pmatrix}2\\6\\10\end{pmatrix}\right\}$ or $\left\{\begin{pmatrix}1\\0\\-1\end{pmatrix},\begin{pmatrix}0\\1\\2\end{pmatrix}\right\}$ are a basis for the column space.

(c) Transposing $\mathbf{C}=\begin{pmatrix}1&2\\3&4\\5&6\end{pmatrix}$ gives $\mathbf{C}^T=\begin{pmatrix}1&3&5\\2&4&6\end{pmatrix}$. Placing $\mathbf{C}^T$ into reduced row echelon form gives $\mathbf{R}=\begin{pmatrix}1&0&-1\\0&1&2\end{pmatrix}$. Hence a basis of the column space of **C** (which is the same as the row space of $\mathbf{C}^T$) is

$$\left\{\begin{pmatrix}1\\0\\-1\end{pmatrix}, \begin{pmatrix}0\\1\\2\end{pmatrix}\right\}$$

Also $rank(\mathbf{C}) = 2$.

(d) Transposing $\mathbf{D} = \begin{pmatrix}1 & 2 & 3\\4 & 5 & 6\end{pmatrix}$ gives $\mathbf{D}^T = \begin{pmatrix}1 & 4\\2 & 5\\3 & 6\end{pmatrix}$ and the reduced row echelon form is

$$\mathbf{R} = \begin{pmatrix}1 & 0\\0 & 1\\0 & 0\end{pmatrix}$$

*What are the basis vectors for the column space of the given matrix **D**?*

$$\left\{\begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix}\right\}$$

We have $rank(\mathbf{D}) = 2$.

(e) Transposing $\mathbf{E} = \begin{pmatrix}-1 & 2 & 5\\-3 & 7 & 0\\-8 & 1 & 3\end{pmatrix}$ gives $\mathbf{E}^T = \begin{pmatrix}-1 & -3 & -8\\2 & 7 & 1\\5 & 0 & 3\end{pmatrix}$ and using MATLAB to place this transposed matrix $\mathbf{E}^T$ into reduced row echelon form gives

$$\mathbf{R} = \begin{pmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{pmatrix}$$

A basis for the column space of **E** is $\left\{\begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\end{pmatrix}, \begin{pmatrix}0\\0\\1\end{pmatrix}\right\}$. We have $rank(\mathbf{E}) = 3$ which means matrix **E** is of full rank.

(f) Transposing $\mathbf{F} = \begin{pmatrix}-5 & 2 & 3\\7 & 1 & 0\\-7 & 6 & 1\\-2 & 5 & 2\end{pmatrix}$ gives $\mathbf{F}^T = \begin{pmatrix}-5 & 7 & -7 & -2\\2 & 1 & 6 & 5\\3 & 0 & 1 & 2\end{pmatrix}$. Using MATLAB to convert this into reduced row echelon form gives $\mathbf{R} = \begin{pmatrix}1 & 0 & 0 & 61/128\\0 & 1 & 0 & 5/8\\0 & 0 & 1 & 73/128\end{pmatrix}$.

*What is a basis for the column space of **F**?*

$$\begin{pmatrix}1\\0\\0\\61/128\end{pmatrix} = \frac{1}{128}\begin{pmatrix}128\\0\\0\\61\end{pmatrix}, \begin{pmatrix}0\\1\\0\\5/8\end{pmatrix} = \frac{1}{8}\begin{pmatrix}0\\8\\0\\5\end{pmatrix}, \begin{pmatrix}0\\0\\1\\73/128\end{pmatrix} = \frac{1}{128}\begin{pmatrix}0\\0\\128\\73\end{pmatrix}$$

Multiplying each of these vectors by 128 gives basis vectors for the column space of the given matrix **F**:

$$\left\{\begin{pmatrix}128\\0\\0\\61\end{pmatrix},\begin{pmatrix}0\\128\\0\\80\end{pmatrix},\begin{pmatrix}0\\0\\128\\73\end{pmatrix}\right\}$$

Hence $rank(\mathbf{F})=3$.

6. We need to prove $rank(\mathbf{A})=rank(\mathbf{A}^T)$.

Proof.

The matrix $\mathbf{A}^T$ is the matrix **A** transposed which means that the column vectors of **A** have become the row vectors of $\mathbf{A}^T$. By Proposition (3-18):

(3-18) row rank of $\mathbf{A}=$ column rank of **A**

The column rank of **A** is equal to the row rank of $\mathbf{A}^T$ because they are the same vectors. Hence

$$\text{row rank of } \mathbf{A}=\text{row rank of } \mathbf{A}^T$$

By definition (3-12):

(3-12) $rank(\mathbf{A})=$ row rank

We have $rank(\mathbf{A})=rank(\mathbf{A}^T)$.

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7. We need to prove that $rank(\mathbf{A})=rank(\mathbf{A}^{-1})$ where **A** is an invertible matrix.

Proof.

Let **A** be a *n* by *n* matrix. **A** is an invertible matrix therefore by Proposition (3-19):

(3-19) **A** is invertible $\Leftrightarrow$ $rank(\mathbf{A})=n$.

We have $rank(\mathbf{A})=n$. Matrix **A** is invertible therefore the inverse matrix $\mathbf{A}^{-1}$ is also invertible and is a square *n* by *n* matrix. Again by Proposition (3-19) we have $rank(\mathbf{A}^{-1})=n$.

Hence we have our result $rank(\mathbf{A})=rank(\mathbf{A}^{-1})$.

■

8. (a) We need to prove that **A** has rank *n* $\Leftrightarrow$ the linear system $\mathbf{Ax}=\mathbf{O}$ has the trivial solution $\mathbf{x}=\mathbf{O}$.

To prove this we need to use the following result of Chapter 1:

Theorem (1-15). Let **A** be a *n* by *n* matrix, then the following statements are equivalent: (a) The matrix **A** is invertible (non-singular). (b) The linear system $\mathbf{Ax}=\mathbf{O}$ only has the trivial solution $\mathbf{x}=\mathbf{O}$.

Proof.

By Proposition (3-19):

Proposition (3-19). Let **A** be a *n* by *n* matrix. The matrix **A** is invertible $\Leftrightarrow$ $rank(\mathbf{A})=n$.

We have **A** has rank *n* $\Leftrightarrow$ **A** is invertible. By Theorem (1-15) (b) above we conclude

A is invertible $\Leftrightarrow$ the system $\mathbf{Ax} = \mathbf{O}$ has the trivial solution $\mathbf{x} = \mathbf{O}$.
Hence we have our result, that is **A** has rank n $\Leftrightarrow$ the linear system $\mathbf{Ax} = \mathbf{O}$ has the trivial solution $\mathbf{x} = \mathbf{O}$.

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(b) We need to prove that **A** has rank n $\Leftrightarrow$ the linear system $\mathbf{Ax} = \mathbf{b}$ has a unique solution.
How?
By applying the following result of Chapter 1:

Theorem (1-17). Let **A** be a n by n matrix, then the following are equivalent:
(a) The matrix **A** is invertible (non-singular).
(b) $\mathbf{Ax} = \mathbf{b}$ has a unique solution.

Proof.
By Proposition (3-19) we have **A** has rank n $\Leftrightarrow$ matrix **A** is invertible . By the above theorem we have **A** is invertible $\Leftrightarrow$ the system $\mathbf{Ax} = \mathbf{b}$ has a unique solution.

■

9. We need to prove that matrix **A** is invertible $\Leftrightarrow$ $S = \{\mathbf{r}_1,\ \mathbf{r}_2,\ \mathbf{r}_3,\ \cdots,\ \mathbf{r}_n\}$ is a set of linearly independent vectors where these vectors form the rows of matrix **A**.
How do we prove this result?
By applying:

Proposition (2-16). Any n linearly independent vectors in $\mathbb{R}^n$ form a basis for $\mathbb{R}^n$.

And

Proposition (2-17). Any n vectors which span $\mathbb{R}^n$ form a basis for $\mathbb{R}^n$.

Proof.

Let $\mathbf{A} = \begin{pmatrix} \mathbf{r}_1 \\ \vdots \\ \mathbf{r}_n \end{pmatrix}$.

$(\Rightarrow)$. Assuming matrix **A** is invertible so we have $rank(\mathbf{A}) = n$ which means that the row space has n basis vectors. We have the row space is of dimension n. Remember the row space is spanned by the vectors

$$S = \{\mathbf{r}_1,\ \mathbf{r}_2,\ \mathbf{r}_3,\ \cdots,\ \mathbf{r}_n\}$$

By (2-17) we can say that the set S forms a basis for the row space. Since the vectors in set S form a basis therefore they must be linearly independent.
$(\Leftarrow)$. Assuming $S = \{\mathbf{r}_1,\ \mathbf{r}_2,\ \mathbf{r}_3,\ \cdots,\ \mathbf{r}_n\}$ is a set of linearly independent vectors. By (2-16) the set S forms a basis for the row space of **A**. Because we have n basis vectors therefore the row space has dimension n and so $rank(\mathbf{A}) = n$. Hence matrix **A** is invertible.

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10. We need to prove that matrix **A** is invertible $\Leftrightarrow$ $S = \{\mathbf{c}_1,\ \mathbf{c}_2,\ \mathbf{c}_3,\ \cdots,\ \mathbf{c}_n\}$ is a set of linearly independent vectors where these vectors form the columns of matrix **A**.
How do we prove this?
By considering the statement of question 12 of Exercises 2.4:

$\mathbf{A} = (\mathbf{v}_1 \quad \mathbf{v}_2 \quad \cdots \quad \mathbf{v}_n)$ is invertible $\Leftrightarrow$ $\{\mathbf{v}_1, \cdots,\ \mathbf{v}_n\}$ form a basis for $\mathbb{R}^n$.

Proof.
By the above result we have:
Matrix $\mathbf{A} = \begin{pmatrix} \mathbf{c}_1 & \mathbf{c}_2 & \cdots & \mathbf{c}_n \end{pmatrix}$ is invertible $\Leftrightarrow$ $\{\mathbf{c}_1, \cdots, \mathbf{c}_n\}$ form a basis for $\mathbb{R}^n$.
Hence the column vectors $S = \{\mathbf{c}_1, \mathbf{c}_2, \mathbf{c}_3, \cdots, \mathbf{c}_n\}$ is a set of linearly independent vectors.

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11. Required to prove that the row space of **A** is identical to the row space of k**A** where k is a non-zero scalar.
Proof.
Let **A** be a m by n matrix and the row space of **A** be S and the row space of k**A** be T.
Let **v** be an arbitrary vector in S and $\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3, \cdots, \mathbf{r}_m$ be the row vectors of **A**. Thus we can write the vector **v** as a linear combination of these **r** vectors

$$\mathbf{v} = c_1\mathbf{r}_1 + c_2\mathbf{r}_2 + c_3\mathbf{r}_3 + \cdots + c_m\mathbf{r}_m \quad \text{where } c\text{'s are scalars}$$

Since k is non-zero therefore we can rewrite vector **v** as

$$\begin{aligned}\mathbf{v} &= k\left(\frac{c_1}{k}\mathbf{r}_1 + \frac{c_2}{k}\mathbf{r}_2 + \frac{c_3}{k}\mathbf{r}_3 + \cdots + \frac{c_m}{k}\mathbf{r}_m\right) \\ &= \frac{c_1}{k}(k\mathbf{r}_1) + \frac{c_2}{k}(k\mathbf{r}_2) + \frac{c_3}{k}(k\mathbf{r}_3) + \cdots + \frac{c_m}{k}(k\mathbf{r}_m)\end{aligned}$$

The last line shows that the vector **v** is a linear combination of $k\mathbf{r}_1, k\mathbf{r}_2, k\mathbf{r}_3, \cdots, k\mathbf{r}_m$ which are the row vectors of k**A** which means it is in the row space of k**A**.
Similarly we show an arbitrary vector **u** in the row space T of k**A** is in the row space S of **A**.
Let **u** be an arbitrary vector in T which means it is in the row space of k**A**. *What do we need to prove?*
Required to prove that the vector **u** is also in S.
Since **u** is in T therefore we can write the vector **u** as a linear combination of the row vectors of k**A**, that is

$$\begin{aligned}\mathbf{u} &= d_1(k\mathbf{r}_1) + d_2(k\mathbf{r}_2) + d_3(k\mathbf{r}_3) + \cdots + d_m(k\mathbf{r}_m) \\ &= kd_1(\mathbf{r}_1) + kd_2(\mathbf{r}_2) + kd_3(\mathbf{r}_3) + \cdots + kd_m(\mathbf{r}_m)\end{aligned}$$

where d's are scalars. Hence **u** is a vector which is a linear combination of $\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3, \cdots, \mathbf{r}_m$ which means it is in row space S.
Combining these two results together we have $S \subseteq T$ and $T \subseteq S$. Thus $S = T$ which shows our required result that the row space of **A** is identical to the row space of k**A**.

■

12. Need to prove that $rank(k\mathbf{A}) = rank(\mathbf{A})$ where k is a non-zero scalar.
How do we prove this?
By using the result of question 11.
Proof.
By the statement of question 11 we have the row space of **A** is identical to the row space of k**A**. This means that row rank of **A** is equal to the row rank of k**A**. By

(3-18) $$rank(\mathbf{A}) = \text{row rank of } \mathbf{A}.$$

we have our result $rank(k\mathbf{A}) = rank(\mathbf{A})$.

■

13. *What does the proposition claim?*
Proposition (3-14). Let **A** be a m by n matrix. Remember this means that matrix **A** has m rows and n columns. Then

(i) The **row space** of matrix **A** is a subspace of $\mathbb{R}^n$.

(ii) The **column space** of matrix **A** is a subspace of $\mathbb{R}^m$.

How do we prove these?
By using:

Proposition (3-5). A non - empty subset S containing vectors **u** and **v** is a subspace of a vector space V $\Leftrightarrow$ any linear combination $k\mathbf{u}+c\mathbf{v}$ is also in S (k and c are scalars).

Proof of (i).
Since we have a m by n matrix therefore we can view this as

$$\overbrace{\begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{pmatrix}}^{n\text{ columns}} \left.\vphantom{\begin{pmatrix} a_{11} \\ \vdots \\ a_{m1} \end{pmatrix}}\right\} m \text{ rows}$$

The row vectors of matrix **A** are $\mathbf{r}_1 = \begin{pmatrix} a_{11} \\ \vdots \\ a_{1n} \end{pmatrix}, \ \ldots\ , \mathbf{r}_m = \begin{pmatrix} a_{m1} \\ \vdots \\ a_{mn} \end{pmatrix}$ which means each of these row vectors are members of $\mathbb{R}^n$ because it has n real entries.

The row space, call it S, is the space spanned by these vectors which means that it consists of vectors of the form

$$c_1\mathbf{r}_1 + c_2\mathbf{r}_2 + c_3\mathbf{r}_3 + \cdots + c_m\mathbf{r}_m \quad \text{where } c\text{'s are scalars}$$

Any linear combination of the **r**'s is in S. By Proposition (3-5) we have S is a subspace of $\mathbb{R}^n$.

Proof of (ii). (Very similar to proof of part (i)).
We have a m by n matrix therefore

$$\overbrace{\begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{pmatrix}}^{n\text{ columns}} \left.\vphantom{\begin{pmatrix} a_{11} \\ \vdots \\ a_{m1} \end{pmatrix}}\right\} m \text{ rows}$$

The column vectors of matrix **A** are $\mathbf{c}_1 = \begin{pmatrix} a_{11} \\ \vdots \\ a_{m1} \end{pmatrix}, \ \ldots\ , \mathbf{c}_n = \begin{pmatrix} a_{1n} \\ \vdots \\ a_{mn} \end{pmatrix}$ which means each of these column vectors are members of $\mathbb{R}^m$ because it has m real entries.

The column space, call it T, is the space spanned by these vectors which means that it consists of vectors of the form

$$k_1\mathbf{c}_1 + k_2\mathbf{c}_2 + k_3\mathbf{c}_3 + \cdots + k_n\mathbf{c}_n \quad \text{where } k\text{'s are scalars}$$

Any linear combination of the **c**'s is in T. By Proposition (3-5) we have T is a subspace of $\mathbb{R}^m$.

■

14. Need to prove that if **R** is a reduced row echelon matrix then the non-zero rows of **R** are linearly independent.
Proof.

Let $\{\mathbf{r}_1,\ \mathbf{r}_2,\ \mathbf{r}_3,\ \cdots,\ \mathbf{r}_n\}$ be the non-zero rows of matrix **R**, that is $\mathbf{R}=\begin{pmatrix}\mathbf{r}_1\\ \vdots\\ \mathbf{r}_n\\ \mathbf{O}\\ \vdots\end{pmatrix}$.

Note that the zero row vectors come after the nth row because the matrix **R** is in reduced row echelon form.
Suppose $\{\mathbf{r}_1,\ \mathbf{r}_2,\ \mathbf{r}_3,\ \cdots,\ \mathbf{r}_n\}$ are linearly **dependent**. By Proposition (3-8) we have one of these vectors, $\mathbf{r}_k$ say, is a linear combination of the preceding vectors, that is

$$\mathbf{r}_k=c_1\mathbf{r}_1+c_2\mathbf{r}_2+c_3\mathbf{r}_3+\cdots+c_{k-1}\mathbf{r}_{k-1}$$

where $1\le k\le n$ and the c's are scalars.
We can rewrite this as

$$\mathbf{r}_k-\left(c_1\mathbf{r}_1+c_2\mathbf{r}_2+c_3\mathbf{r}_3+\cdots+c_{k-1}\mathbf{r}_{k-1}\right)=\mathbf{O} \qquad (*)$$

If we carry out elementary row operations highlighted in (*) then we result in the zero vector. This is impossible because the zero row vector is at the bottom, that means it comes after the nth row which implies that $k>n$. However we have $1\le k\le n$ which means that we have a contradiction so the vectors $\{\mathbf{r}_1,\ \mathbf{r}_2,\ \mathbf{r}_3,\ \cdots,\ \mathbf{r}_n\}$ are linearly **independent**.

■

15. We need to prove that if **A** is a matrix and the reduced row echelon form matrix **R** of **A** contains zero rows then the rows of **A** are linearly dependent.
Proof.
Assume matrix **A** has m rows where $m>n$. If any of the rows of matrix **A** are zero then they are linearly dependent. *Why?*
Suppose row j, that is $\mathbf{a}_j=\mathbf{O}$, of matrix **A**. Consider the linear combination of rows of matrix **A** which give the zero vector:

$$k_1\mathbf{a}_1+k_2\mathbf{a}_2+\cdots+k_j\mathbf{a}_j+\cdots+k_m\mathbf{a}_m=\mathbf{O}$$

For the following scalars $k_1=k_2=\cdots=k_{j-1}=k_{j+1}=\cdots=k_m=0$ and $k_j\ne 0$ we obtain the zero vector. Since there is a non-zero scalar, $k_j\ne 0$, so the rows of matrix **A** are linearly dependent.
Consider the case where none of the rows of matrix **A** are zero rows.
By applying elementary row operations to matrix **A** and given that matrix **R** has zero rows, we have:

$$\mathbf{A}=\begin{pmatrix}\mathbf{a}_1\\ \vdots\\ \mathbf{a}_n\\ \mathbf{a}_{n+1}\\ \vdots\end{pmatrix} \Rightarrow \mathbf{R}=\begin{pmatrix}\mathbf{r}_1\\ \vdots\\ \mathbf{r}_n\\ \mathbf{O}\\ \vdots\end{pmatrix} \quad (n+1)\text{ th row}$$

Examining the $(n+1)$ th row, we have that the linear combination of the row vectors of matrix **A** gives the zero vector. This means that we have

$$k_1\mathbf{a}_1 + k_2\mathbf{a}_2 + k_3\mathbf{a}_3 + \cdots + k_m\mathbf{a}_m = \mathbf{O}$$

where all the scalars *k*'s are not zero. Remember for an elementary row operation we can only multiply by non-zero constants.
Hence the row vectors $\mathbf{a}_1,\ \mathbf{a}_2,\ \cdots,\ \mathbf{a}_m$ are linearly dependent.

■

16. *What is Proposition (3-16)?*
If a matrix **R** is in reduced row echelon form then its non-zero rows form a basis for the row space of matrix **R**. *How do we prove this?*
Need to show that the non-zero rows are linearly independent and span the row space.
By using the result of question 14 we have already shown that these non-zero rows of **R** are linearly independent so we only need to show that they span the row space.
Proof.

R is in reduced row echelon form therefore we have $\mathbf{R} = \begin{pmatrix} \mathbf{r}_1 \\ \vdots \\ \mathbf{r}_n \\ \mathbf{O} \\ \vdots \end{pmatrix}$ where the **r**'s are the non-zero rows of **R** and **O** is the zero vector.
By definition (3-11) part (a) we have the row space *S* is the space spanned by the row vectors of **R**, that is $\{\mathbf{r}_1,\ \mathbf{r}_2,\ \mathbf{r}_3,\ \cdots,\ \mathbf{r}_n,\ \mathbf{O}\}$. We can generate the zero vector **O** with **all** the scalars equal to zero, that is

$$0\mathbf{r}_1 + 0\mathbf{r}_2 + 0\mathbf{r}_3 + \cdots + 0\mathbf{r}_n = \mathbf{O}$$

Thus the zero vector **O** is a linear combination of $\{\mathbf{r}_1,\ \mathbf{r}_2,\ \mathbf{r}_3,\ \cdots,\ \mathbf{r}_n\}$ which means that the row space of **R** can be spanned by these vectors $\{\mathbf{r}_1,\ \mathbf{r}_2,\ \mathbf{r}_3,\ \cdots,\ \mathbf{r}_n\}$.
Thus the non-zero rows of **R**, $\{\mathbf{r}_1,\ \mathbf{r}_2,\ \mathbf{r}_3,\ \cdots,\ \mathbf{r}_n\}$, span the row space and are linearly independent which means that they form a basis for the row space of **R**.

■

17. We need to prove that if **A** is a matrix whose rows are given by the set of linear independent vectors $S = \{\mathbf{r}_1,\ \mathbf{r}_2,\ \mathbf{r}_3,\ \cdots,\ \mathbf{r}_n\}$ then $rank(\mathbf{A}) = n$.
Proof.

We are given that $\mathbf{A} = \begin{pmatrix} \mathbf{r}_1 \\ \mathbf{r}_2 \\ \vdots \\ \mathbf{r}_n \end{pmatrix}$ and we need to show that $rank(\mathbf{A}) = n$. By result of question 9:

> Let **A** be a square *n* by *n* matrix whose row vectors are given by the set of vectors $S = \{\mathbf{r}_1,\ \mathbf{r}_2,\ \mathbf{r}_3,\ \cdots,\ \mathbf{r}_n\}$. Then matrix **A** is invertible $\Leftrightarrow$ *S* is a set of linearly independent vectors.

We have matrix **A** is invertible. By (3-19):

Proposition (3-19). Let **A** be a n by n matrix. The matrix **A** is invertible $\Leftrightarrow$ $rank(\mathbf{A}) = n$.

Hence $rank(\mathbf{A}) = n$.

■

18. We need to prove $\text{row rank} = \text{column rank}$ of a matrix **A**.
Proof.
Let **R** be the reduced row echelon form of the matrix **A** with n columns. Let row rank equal r. This means there are r non-zero rows because the row rank is the number of non-zero rows in reduced row echelon form:

$$\mathbf{R} = \begin{pmatrix} 1 & 0 & & & & \\ 0 & \ddots & & & & \\ \vdots & & 1 & a_1 & \cdots & a_{n-r} \\ 0 & 0 & 0 & \cdots & \cdots & 0 \\ 0 & 0 & 0 & \cdots & \cdots & 0 \end{pmatrix} \begin{array}{l} \left.\begin{array}{l} \\ \\ \\ \end{array}\right\} r \text{ non} - \text{zero rows} \\ \\ \\ \end{array}$$

This is identical to the number of leading ones. By the assumption in the question the basis for the column space are these columns with a leading one which means the dimension of the column space is r. Hence $\text{row rank} = \text{column rank}$.

■