Complete Solution to Exercises 3.3

1. (a) We are given the matrices $\mathbf{A}=\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$ and $\mathbf{B}=\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}$ and the matrix **A** is **not** a multiple of matrix **B** therefore the matrices are linearly independent.

(b) *What do you notice about given matrices* $\mathbf{A}=\begin{pmatrix}1 & 1\\ 1 & 1\end{pmatrix}$ *and* $\mathbf{B}=\begin{pmatrix}2 & 2\\ 2 & 2\end{pmatrix}$*?*

Matrix **B** is twice matrix **A**, that is $\mathbf{B}=2\mathbf{A}$ or $\mathbf{B}-2\mathbf{A}=\mathbf{O}$ which means that the matrices are linearly dependent.

(c) Matrices $\mathbf{A}=\begin{pmatrix}1 & 2\\ 3 & 4\end{pmatrix}$ and $\mathbf{B}=\begin{pmatrix}2 & 2\\ 2 & 2\end{pmatrix}$ are **not** multiples of each other therefore they are linearly independent.

(d) *Can you spot a relationship between* $\mathbf{A}=\begin{pmatrix}1 & 2\\ 3 & 4\end{pmatrix}$ *and* $\mathbf{B}=\begin{pmatrix}2/5 & 4/5\\ 6/5 & 8/5\end{pmatrix}$*?*

$\mathbf{B}=\frac{2}{5}\mathbf{A}$ or $\mathbf{B}-\frac{2}{5}\mathbf{A}=\mathbf{O}$. Since we can produce the zero vector with non-zero scalars, 1 and $-\frac{2}{5}$, therefore the given vectors are linearly dependent.

2. (a) We are given the functions $\mathbf{f}=(x+1)^2$ and $\mathbf{g}=x^2+2x+1$. *What do you notice about these functions?*

$$(x+1)^2=x^2+2x+1$$

This means that we have

$$(x+1)^2-(x^2+2x+1)=0$$

$$\mathbf{f}-\mathbf{g}=\mathbf{O}\qquad\left[\text{Because } \mathbf{f}=(x+1)^2 \text{ and } \mathbf{g}=x^2+2x+1\right]$$

Hence **f** and **g** are linearly dependent.

(b) Using scalars k and c we have

$$k\mathbf{f}+c\,\mathbf{g}=k(2)+cx^2=0 \qquad (\$)$$

Substituting $x=0$ into ($) gives $2k+c(0)^2=2k=0 \;\Rightarrow\; k=0$. Substituting $x=1$ into ($) gives $2k+c(1)^2=2k+c=0$ because $k=0$ therefore $c=0$.

Hence $k=0$ and $c=0$ that is **all** (both) scalars are zero therefore we conclude that the given functions $\mathbf{f}=2$ and $\mathbf{g}=x^2$ are linearly independent.

(c) Using scalars k and c we have

$$k\mathbf{f}+c\,\mathbf{g}=k(1)+ce^x=0$$

Substituting $x=0$ and $x=1$ gives the simultaneous equations

$$k+c=0$$
$$k+ce=0$$

Solving these simultaneous equations gives $k=0$ and $c=0$.

All (both) scalars are zero therefore we conclude that the given functions $\mathbf{f}=1$ and $\mathbf{g}=e^x$ are linearly independent.

(d) Using scalars k and c we have

$$k\mathbf{f}+c\,\mathbf{g}=k\cos(x)+c\sin(x)=0 \qquad (*)$$

Substituting $x = 0$ into (*)

$$k \underbrace{\cos(0)}_{=1} + c \underbrace{\sin(0)}_{=0} = 0 \quad \text{gives} \quad k = 0$$

Substituting $x = \frac{\circledR}{2}$ into (*)

$$k \underbrace{\cos\left(\frac{\circledR}{2}\right)}_{=0} + c \underbrace{\sin\left(\frac{\circledR}{2}\right)}_{=1} = 0 \quad \text{gives} \quad c = 0$$

Hence $k = 0$ and $c = 0$. Both scalars are zero therefore the given functions

$$\mathbf{f} = \cos(x) \quad \text{and} \quad \mathbf{g} = \sin(x)$$

are linearly independent.
[Showing $\cos(x)$ and $\sin(x)$ are linearly independent is important in the theory of differential equations].
(e) We need to test $\mathbf{f} = \sin(x)$ and $\mathbf{g} = \sin(2x)$ for linear independence. Since $\sin(2x)$ is not a scalar multiple of $\sin(x)$ so they are linear independent.

3. (a) We have the fundamental trigonometric identity

$$\cos^2(x) + \sin^2(x) = 1$$

Multiplying each side by 5 gives

$$5\cos^2(x) + 5\sin^2(x) = 5$$

$$5\cos^2(x) + 5\sin^2(x) - 5 = 0 \qquad (*)$$

Consider the linear combination

$$k_1\mathbf{f} + k_2\mathbf{g} + k_3\mathbf{h} = k_1\cos^2(x) + k_2\sin^2(x) + k_3(5) = 0$$

Comparing this with (*) we have $k_1 = 5,\ k_2 = 5$ and $k_3 = -1$ gives 0. All scalars are **not** zero therefore vectors **f**, **g** and **h** are linearly dependent.
(b) We are given the functions $\mathbf{f} = \cos(2x),\ \mathbf{g} = \sin^2(x)$ and $\mathbf{h} = \cos^2(x)$. *Can you remember any trigonometric identity relating these functions?*

$$\cos(2x) = \cos^2(x) - \sin^2(x)$$

Rearranging this gives

$$\cos(2x) + \sin^2(x) - \cos^2(x) = 0 \qquad (\dagger)$$

The linear combination

$$k_1\mathbf{f} + k_2\mathbf{g} + k_3\mathbf{h} = k_1\cos(2x) + k_2\sin^2(x) + k_3\cos^2(x) = 0$$

Comparing with $(\dagger)$ we have $k_1 = 1,\ k_2 = 1$ and $k_3 = -1$. Hence **f**, **g** and **h** are linearly dependent.
(c) We are given the functions $\mathbf{f} = 1,\ \mathbf{g} = x$ and $\mathbf{h} = x^2$. Writing these as a linear combination:

$$k_1\mathbf{f} + k_2\mathbf{g} + k_3\mathbf{h} = k_1(1) + k_2 x + k_3 x^2 = 0$$

Equating coefficients gives $k_1 = k_2 = k_3 = 0$. The functions **f**, **g** and **h** are linearly independent.
(d) We are given the functions $\mathbf{f} = \sin(2x),\ \mathbf{g} = \sin(x)\cos(x)$ and $\mathbf{h} = \cos(x)$. *Do you remember any trigonometric identity which relates these 3 functions?*

$$\sin(2x) = 2\sin(x)\cos(x)$$

We can write this as

$$\sin(2x) - 2\sin(x)\cos(x) = 0 \qquad (\$)$$

Consider the linear combination

$$k_1\mathbf{f} + k_2\mathbf{g} + k_3\mathbf{h} = k_1\sin(2x) + k_2\sin(x)\cos(x) + k_3\cos(x) = 0$$

Comparing this with ($) gives $k_1 = 1,\ k_2 = -2$ and $k_3 = 0$. Since we have non-zero scalars (k's) therefore **f**, **g** and **h** are linearly dependent.

(e) We need to decide whether the following functions

$$\mathbf{f} = e^x\sin(2x),\ \mathbf{g} = e^x\sin(x)\cos(x) \text{ and } \mathbf{h} = e^x\cos(x)$$

are linearly dependent or independent. Since these are the same functions as part (d) apart from the multiple e^x therefore we have

$$\begin{aligned} k_1\mathbf{f} + k_2\mathbf{g} + k_3\mathbf{h} &= k_1e^x\sin(2x) + k_2e^x\sin(x)\cos(x) + k_3e^x\cos(x) \\ &= e^x\left[k_1\sin(2x) + k_2\sin(x)\cos(x) + k_3\cos(x)\right] = 0 \end{aligned}$$

The square bracket term is 0 for the k values given in part (d) above:

$$e^x\left[\sin(2x) - 2\sin(x)\cos(x) + 0\cos(x)\right] = 0$$

We have $k_1 = 1,\ k_2 = -2$ and $k_3 = 0$. We have non-zero scalars (k's) therefore **f**, **g** and **h** are linearly dependent.

Later on in question 8 we will prove:

A set of vectors $\{\mathbf{v}_1,\ \mathbf{v}_2,\ \mathbf{v}_3,\ \cdots,\ \mathbf{v}_n\}$ is linearly independent then

$$\{k\mathbf{v}_1,\ k\mathbf{v}_2,\ k\mathbf{v}_3,\ \cdots,\ k\mathbf{v}_n\}$$

where k is a non-zero scalar, is also linearly independent.

We can use this result in our case with $k = e^x \neq 0$.

(f) Writing the given functions $\mathbf{f} = 1,\ \mathbf{g} = e^x$ and $\mathbf{h} = e^{-x}$ in a linear combination

$$k_1\mathbf{f} + k_2\mathbf{g} + k_3\mathbf{h} = k_1(1) + k_2e^x + k_3e^{-x} = 0$$

By substituting various values of x we get the only possible solution $k_1 = 0,\ k_2 = 0$ and $k_3 = 0$. Hence the functions **f**, **g** and **h** are linearly independent.

(g) We need to test $\mathbf{f} = e^x$, $\mathbf{g} = e^{2x}$ and $\mathbf{h} = e^{3x}$ for linear independence. Since we cannot write e^{3x} in terms of e^x and e^{2x}:

$$c_1e^x + c_2e^{2x} \neq e^{3x}$$

Similarly

$$k_1e^x + k_2e^{3x} \neq e^{2x}$$

By (3-9):

The vectors in the set $S = \{\mathbf{v}_1,\ \mathbf{v}_2,\ \mathbf{v}_3,\ \cdots,\ \mathbf{v}_n\}$ are linearly dependent $\Leftrightarrow$ one of these vectors, say $\mathbf{v}_k$, is a linear combination of the preceding vectors, that is

$$\mathbf{v}_k = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3 + \cdots + c_{k-1}\mathbf{v}_{k-1}$$

Applying this result to the above we conclude that $\mathbf{f} = e^x$, $\mathbf{g} = e^{2x}$ and $\mathbf{h} = e^{3x}$ linearly independent.

4. Required to show that $\mathbf{f}=\sin(x)$, $\mathbf{g}=\sin(3x)$ and $\mathbf{h}=\sin(5x)$ are linearly independent. From our knowledge of trigonometry we know that $\sin(5x)$ cannot be written in terms of $\sin(x)$ and $\sin(3x)$, that is

$$k\sin(x)+c\sin(3x) \neq \sin(5x)$$

Hence by Proposition (3-9) we conclude that the given vectors

$$\mathbf{f}=\sin(x),\ \mathbf{g}=\sin(3x) \text{ and } \mathbf{h}=\sin(5x)$$

are linearly independent.

5. Need to show that the following matrices form a basis for M_{22}:

$$\mathbf{A}=\begin{pmatrix}1&0\\0&0\end{pmatrix},\ \mathbf{B}=\begin{pmatrix}0&1\\0&0\end{pmatrix},\ \mathbf{C}=\begin{pmatrix}0&0\\1&0\end{pmatrix} \text{ and } \mathbf{D}=\begin{pmatrix}0&0\\0&1\end{pmatrix}$$

For a basis we need to prove that **A**, **B**, **C** and **D** span M_{22} and also these matrices are linearly independent.

Span: Let $\mathbf{X}=\begin{pmatrix}a&b\\c&d\end{pmatrix}$ be an arbitrary matrix and

$$k_1\begin{pmatrix}1&0\\0&0\end{pmatrix}+k_2\begin{pmatrix}0&1\\0&0\end{pmatrix}+k_3\begin{pmatrix}0&0\\1&0\end{pmatrix}+k_4\begin{pmatrix}0&0\\0&1\end{pmatrix}=\begin{pmatrix}k_1&k_2\\k_3&k_4\end{pmatrix}=\begin{pmatrix}a&b\\c&d\end{pmatrix}$$

Equating the entries, we have $k_1=a,\ k_2=b,\ k_3=c$ and $k_4=d$. Since the matrix **X** was arbitrary therefore we can produce any 2 by 2 matrix by a linear combination of matrices **A**, **B**, **C** and **D**. Hence these matrices span M_{22}.

Linearly Independent: Using the above to produce the zero matrix, $\begin{pmatrix}0&0\\0&0\end{pmatrix}$, we have $k_1=a,\ k_2=b,\ k_3=c$ and $k_4=d$ and because each entry is 0 therefore **all** the k's are zero, that is $k_1=0,\ k_2=0,\ k_3=0$ and $k_4=0$. Hence matrices **A**, **B**, **C** and **D** are linearly independent.

We have both, span and linear independence, therefore the given matrices form a basis for M_{22}.

6. We need to show that $\left\{1,\ t-1,\ (t-1)^2\right\}$ span P_2 and is linearly independent.

Span: Let at^2+bt+c be an arbitrary member of P_2. We have

$$\begin{aligned}k_1(1)+k_2(t-1)+k_3(t-1)^2 &= k_1+k_2t-k_2+k_3\left(t^2-2t+1\right) && [\text{Expanding}]\\ &= k_1+k_2t-k_2+k_3t^2-2k_3t+k_3\\ &= k_3t^2+(k_2-2k_3)t+(k_1-k_2+k_3) && \left[\begin{array}{l}\text{Collecting Like}\\ \text{Terms}\end{array}\right]\\ &= at^2+bt+c\end{aligned}$$

Equating coefficients gives

$$k_3=a,\ k_2-2k_3=b \text{ and } k_1-k_2+k_3=c$$

Substituting the first equation $k_3=a$ into the middle equation $k_2-2k_3=b$ gives

$$k_2-2a=b \quad \Rightarrow \quad k_2=2a+b$$

Substituting $k_2 = 2a + b$ and $k_3 = a$ into the last equation $k_1 - k_2 + k_3 = c$:

$$k_1 - (2a + b) + a = c \quad \text{gives} \quad k_1 = c + (2a + b) - a = c + a + b$$

Hence we have found scalars, $k_1 = c + a + b$, $k_2 = 2a + b$ and $k_3 = a$, which produce the arbitrary polynomial $at^2 + bt + c$ therefore we conclude that the given set of vectors $\left\{1,\ t-1,\ (t-1)^2\right\}$ span P_2.

<u>Linearly Independent</u>: Using the above to produce the zero polynomial, with $a = 0,\ b = 0$ and $c = 0$:

$$k_1 = 0 + 0 + 0 = 0,\ k_2 = 2(0) + 0 = 0 \text{ and } k_3 = 0$$

Since **all** the scalars are zero therefore $\left\{1,\ t-1,\ (t-1)^2\right\}$ is linearly independent.

The set $\left\{1,\ t-1,\ (t-1)^2\right\}$ span P_2 and is linearly independent therefore we can say it forms a basis for P_2.

By the spanning set from above we have

$$at^2 + bt + c = (c + a + b)(1) + (2a + b)(t-1) + a(t-1)^2$$

For our polynomial $\mathbf{p} = t^2 + 1$ we have $a = 1,\ b = 0$ and $c = 1$. Putting these values into the above gives

$$\begin{aligned} t^2 + 1 &= (1+1+0)(1) + (2(1)+0)(t-1) + 1(t-1)^2 \\ &= 2 + 2(t-1) + (t-1)^2 \end{aligned}$$

7. We need to show the following vectors of P_2 do **not** form a basis:

$$\left\{1,\ t^2 - 2t,\ 5(t-1)^2\right\}$$

Easier to show that this set is linearly dependent.

$$\begin{aligned} k_1(1) + k_2\left(t^2 - 2t\right) + 5k_3(t-1)^2 &= k_1 + k_2t^2 - 2tk_2 + 5k_3\left(t^2 - 2t + 1\right) \qquad [\text{Expanding}] \\ &= k_1 + k_2t^2 - 2tk_2 + 5k_3t^2 - 10k_3t + 5k_3 \\ &\underbrace{=}_{\text{Collecting Like Terms}} \left(k_2 + 5k_3\right)t^2 + \left(-2k_2 - 10k_3\right)t + \left(k_1 + 5k_3\right) = 0 \end{aligned}$$

Equating coefficients we have

$$\begin{aligned} t^2:&\qquad k_2 + 5k_3 = 0 \\ t:&\qquad -2k_2 - 10k_3 = 0 \\ const:&\qquad k_1 \qquad + 5k_3 = 0 \end{aligned}$$

From the bottom equation we have $k_1 = -5k_3$. Let $k_3 = 1$ then $k_1 = -5(1) = -5$.

Substituting $k_3 = 1$ into the top equation $k_2 + 5k_3 = 0$ gives

$$k_2 + 5(1) = 0 \quad \text{gives} \quad k_2 = -5$$

We have non-zero scalars, $k_1 = -5$, $k_2 = -5$ and $k_3 = 1$, therefore the given set of vectors $\left\{1,\ t^2 - 2t,\ 5(t-1)^2\right\}$ is linearly dependent.

This means that $\left\{1,\ t^2 - 2t,\ 5(t-1)^2\right\}$ **cannot** form a basis for P_2.

8. We need to prove that if a set of vectors $\{\mathbf{v}_1,\ \mathbf{v}_2,\ \mathbf{v}_3,\ \cdots,\ \mathbf{v}_n\}$ is linearly independent then $\{k\mathbf{v}_1,\ k\mathbf{v}_2,\ k\mathbf{v}_3,\ \cdots,\ k\mathbf{v}_n\}$, where k is a non-zero scalar, is also linear independent.

Proof.

Consider the linear combination

$$c_1(k\mathbf{v}_1)+c_2(k\mathbf{v}_2)+c_3(k\mathbf{v}_3)+\cdots+c_n(k\mathbf{v}_n)=\mathbf{O} \qquad (*)$$

where the c's are scalars.

What do we need to prove?

Required to prove that the only scalars which satisfy (*) is when they are **all** zero, that is $c_1=c_2=c_3=\cdots=c_n=0$. We have

$$c_1(k\mathbf{v}_1)+c_2(k\mathbf{v}_2)+c_3(k\mathbf{v}_3)+\cdots+c_n(k\mathbf{v}_n)=\mathbf{O}$$
$$kc_1\mathbf{v}_1+kc_2\mathbf{v}_2+kc_3\mathbf{v}_3+\cdots+kc_n\mathbf{v}_n=\mathbf{O}$$
$$k(c_1\mathbf{v}_1+c_2\mathbf{v}_2+c_3\mathbf{v}_3+\cdots+c_n\mathbf{v}_n)=\mathbf{O} \qquad [\text{Taking Out a Factor of } k]$$

$k\neq 0$ [Not Zero] because the proposition states this.

We are also given that $\{\mathbf{v}_1,\ \mathbf{v}_2,\ \mathbf{v}_3,\ \cdots,\ \mathbf{v}_n\}$ are linearly independent therefore they satisfy

$$c_1\mathbf{v}_1+c_2\mathbf{v}_2+c_3\mathbf{v}_3+\cdots+c_n\mathbf{v}_n=0 \quad\Rightarrow\quad c_1=c_2=c_3=\cdots=c_n=0$$

Hence $\{k\mathbf{v}_1,\ k\mathbf{v}_2,\ k\mathbf{v}_3,\ \cdots,\ k\mathbf{v}_n\}$ are linearly independent.

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9. We need to prove a non-zero vector **v** is linearly independent.

Proof.

Let **v** be a non-zero vector in a vector space V. Consider the linear combination

$$k\mathbf{v}=\mathbf{O} \quad\Rightarrow\quad k=0 \ \ or\ \ \mathbf{v}=\mathbf{O} \qquad \text{[By (3-1) part (d)]}$$

The only way this scalar multiplication $k\mathbf{v}$ is zero is if $k=0$ because **v** is non-zero. Hence the vector **v** is linearly independent.

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10. We are required to prove the zero, **O**, vector is dependent.

Proof.

$k\mathbf{O}=\mathbf{O}$ for any non-zero scalar k therefore **O** is linearly dependent.

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11. Need to prove that if any two vectors are equal, $\mathbf{v}_j=\mathbf{v}_m$ where $j\neq m$, then the set is linearly dependent.

Proof.

Without Loss of Generality we can assume $j<m$. Consider the linear combination

$$k_1\mathbf{v}_1+k_2\mathbf{v}_2+k_3\mathbf{v}_3+\cdots+k_j\mathbf{v}_j+\cdots+k_m\mathbf{v}_m+\cdots+k_n\mathbf{v}_n=\mathbf{O} \qquad (*)$$

Take all the k's to equal zero apart from k_j and k_m. Let $k_j=1$ and $k_m=-1$ then (*) becomes

$$0\mathbf{v}_1+0\mathbf{v}_2+0\mathbf{v}_3+\cdots+\mathbf{v}_j+\cdots+(-1)\mathbf{v}_m+\cdots+0\mathbf{v}_n=\mathbf{v}_j-\mathbf{v}_m$$

Since we are given $\mathbf{v}_j=\mathbf{v}_m$ therefore $\mathbf{v}_j-\mathbf{v}_m=\mathbf{O}$. We have non-zero scalars, $k_j=1$ and $k_m=-1$, which produce the zero vector therefore the given set is linearly dependent.

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12. We need to prove that any non-empty subset of a linearly independent set of $S = \{\mathbf{v}_1,\ \mathbf{v}_2,\ \mathbf{v}_3,\ \cdots,\ \mathbf{v}_n\}$ is also independent.

Proof.

Any non-empty subset of S will contain vectors from this list $\mathbf{v}_1,\ \mathbf{v}_2,\ \mathbf{v}_3,\ \cdots,\ \mathbf{v}_n$. Let these vectors be $\mathbf{v}_j,\ \mathbf{v}_{j+1},\ \cdots, \mathbf{v}_{m-1},\ \mathbf{v}_m$ where $1 \le j \le m$ and $m \le n$.

Suppose these are linearly dependent. Consider the linear combination

$$k_j\mathbf{v}_j + k_{j+1}\mathbf{v}_{j+1} + \cdots + k_{m-1}\mathbf{v}_{m-1} + k_m\mathbf{v}_m = \mathbf{O} \qquad (\dagger)$$

Then all the scalars k's are not zero.

Take $k_1 = k_2 = k_{j-1} = k_{m+1} = \cdots = k_n = 0$. The linear combination

$$k_1\mathbf{v}_1 + k_2\mathbf{v}_2 + k_3\mathbf{v}_3 + \cdots + k_n\mathbf{v}_n = \mathbf{O} \qquad (\dagger\dagger)$$

By (†) all the scalars are not zero in (††) which means the vectors in

$$S = \{\mathbf{v}_1,\ \mathbf{v}_2,\ \cdots,\ \mathbf{v}_n\}$$

are linearly dependent. This cannot be the case because we are given that these vectors are independent. Hence our supposition – the vectors $\mathbf{v}_j,\ \mathbf{v}_{j+1},\ \cdots, \mathbf{v}_{m-1},\ \mathbf{v}_m$ are dependent must be wrong so they are linearly independent.

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13. We need to prove that $\{\mathbf{v}_1,\ \mathbf{v}_2,\ \mathbf{v}_3,\ \cdots,\ \mathbf{v}_n,\ \mathbf{w}\}$ spans *V* but is linearly dependent provided that $\{\mathbf{v}_1,\ \mathbf{v}_2,\ \mathbf{v}_3,\ \cdots,\ \mathbf{v}_n\}$ spans *V*.

Proof.

Vectors $\{\mathbf{v}_1,\ \mathbf{v}_2,\ \mathbf{v}_3,\ \cdots,\ \mathbf{v}_n\}$ span V therefore we can write the vector $\mathbf{w} \in V$ as a linear combination of the vectors in $\{\mathbf{v}_1,\ \mathbf{v}_2,\ \mathbf{v}_3,\ \cdots,\ \mathbf{v}_n\}$.

$$k_1\mathbf{v}_1 + k_2\mathbf{v}_2 + k_3\mathbf{v}_3 + \cdots + k_n\mathbf{v}_n = \mathbf{w}$$

Rearranging this gives

$$k_1\mathbf{v}_1 + k_2\mathbf{v}_2 + k_3\mathbf{v}_3 + \cdots + k_n\mathbf{v}_n - \mathbf{w} = \mathbf{O}$$

We can produce the zero vector with **non-zero** scalars (the scalar associated with **w** is -1) therefore $\{\mathbf{v}_1,\ \mathbf{v}_2,\ \mathbf{v}_3,\ \cdots,\ \mathbf{v}_n,\ \mathbf{w}\}$ is linearly dependent.

Let **u** be an arbitrary vector in *V*. Since we are given that $\{\mathbf{v}_1,\ \mathbf{v}_2,\ \mathbf{v}_3,\ \cdots,\ \mathbf{v}_n\}$ spans *V* therefore

$$\begin{aligned}\mathbf{u} &= k_1\mathbf{v}_1 + k_2\mathbf{v}_2 + k_3\mathbf{v}_3 + \cdots + k_n\mathbf{v}_n \\ &= k_1\mathbf{v}_1 + k_2\mathbf{v}_2 + k_3\mathbf{v}_3 + \cdots + k_n\mathbf{v}_n + (0)\mathbf{w}\end{aligned}$$

Hence $\{\mathbf{v}_1,\ \mathbf{v}_2,\ \mathbf{v}_3,\ \cdots,\ \mathbf{v}_n,\ \mathbf{w}\}$ also spans *V*.

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14. We are required to prove that if $B = \{\mathbf{v}_1,\ \mathbf{v}_2,\ \mathbf{v}_3,\ \cdots,\ \mathbf{v}_n\}$ is a basis for V and $S = \{\mathbf{v}_1,\ \mathbf{v}_2,\ \mathbf{v}_3,\ \cdots,\ \mathbf{v}_m\}$ is a set of linearly independent vectors in *V* then $m \le n$.

Proof.

Suppose $m > n$ then by the result of question 13 the set $S = \{\mathbf{v}_1,\ \mathbf{v}_2,\ \mathbf{v}_3,\ \cdots,\ \mathbf{v}_m\}$ is linearly dependent which contradicts that *S* is linearly independent. Hence $m \le n$.

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15. We need to prove that if $B_1 = \{\mathbf{v}_1,\ \mathbf{v}_2,\ \mathbf{v}_3,\ \cdots,\ \mathbf{v}_n\}$ and $B_2 = \{\mathbf{u}_1,\ \mathbf{u}_2,\ \mathbf{u}_3,\ \cdots,\ \mathbf{u}_m\}$ are bases for a vector space *V* then $n = m$.

Proof.
By the result (proposition) of question 13 we have $n \le m$ and $m \le n$ which means that $n = m$.

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16. We need to prove that if the largest number of linearly independent vectors in a vector space *V* is *n* then any *n* linearly independent vectors forms a basis for *V*.
Proof.
Let $\{\mathbf{v}_1, \ \mathbf{v}_2, \ \mathbf{v}_3, \ \cdots, \ \mathbf{v}_n\}$ be a set of *n* linearly independent vectors in *V*.
Let **w** be an arbitrary vector in *V*. Since *n* is the largest number of independent vectors in *V* therefore $\{\mathbf{v}_1, \ \mathbf{v}_2, \ \mathbf{v}_3, \ \cdots, \ \mathbf{v}_n, \ \mathbf{w}\}$ is a set of linearly dependent vectors which means that the vector **w** is a linear combination of $\{\mathbf{v}_1, \ \mathbf{v}_2, \ \mathbf{v}_3, \ \cdots, \ \mathbf{v}_n\}$. Hence the set $\{\mathbf{v}_1, \ \mathbf{v}_2, \ \mathbf{v}_3, \ \cdots, \ \mathbf{v}_n\}$ span *V*. Since we know these **v**'s are linearly independent therefore $\{\mathbf{v}_1, \ \mathbf{v}_2, \ \mathbf{v}_3, \ \cdots, \ \mathbf{v}_n\}$ are a basis for *V*.

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17. We need to prove that if *S* and *V* have the same basis then $S = V$.
Proof.
Let $\{\mathbf{v}_1, \ \mathbf{v}_2, \ \mathbf{v}_3, \ \cdots, \ \mathbf{v}_n\}$ be the basis vectors of *S* and *V*. Let a vector **u** be in *S*. This vector must also be a member of *V*. *Why?*
Because $\mathbf{u} = k_1\mathbf{v}_1 + k_2\mathbf{v}_2 + k_3\mathbf{v}_3 + \ \cdots + k_n\mathbf{v}_n$ and *V* is a vector space. Similarly a vector in *V* must be in *S*. Since every vector in *S* is in *V* and every vector in *V* is in *S* so they must be equal, $S = V$.

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