Complete Solutions to Exercises 6.2

1. We use the formula for the determinant of 3×3 matrix:

(a) Expand along the middle row because it contains a zero:

$$\det\begin{pmatrix}1 & 3 & -1\\ 2 & 0 & 5\\ -6 & 3 & 1\end{pmatrix}=-2\det\begin{pmatrix}3 & -1\\ 3 & 1\end{pmatrix}+0\det\begin{pmatrix}1 & -1\\ -6 & 1\end{pmatrix}-5\det\begin{pmatrix}1 & 3\\ -6 & 3\end{pmatrix}$$

$$\underset{\text{by (6.1)}}{=}-2\left[(3\times1)-(3\times(-1))\right]+0-5\left[(1\times3)-((-6)\times3)\right]=-117$$

(b) Similarly we have

$$\det\begin{pmatrix}2 & -10 & 11\\ 5 & 3 & -4\\ 7 & 9 & 12\end{pmatrix}=2\det\begin{pmatrix}3 & -4\\ 9 & 12\end{pmatrix}+10\det\begin{pmatrix}5 & -4\\ 7 & 12\end{pmatrix}+11\det\begin{pmatrix}5 & 3\\ 7 & 9\end{pmatrix}$$

$$\underset{\text{by (6.1)}}{=}2\left[(3\times12)-(9\times(-4))\right]+10\left[(5\times12)-(7\times(-4))\right]+11\left[(5\times9)-(7\times3)\right]=1288$$

(c) Very similar to parts (a) and (b). Thus $\det(\mathbf{C})=-114$.

2. Using (6.4):

$$\det\begin{pmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ 7 & 3 & -2\\ 4 & 2 & 7\end{pmatrix}=\mathbf{i}\left[\det\begin{pmatrix}3 & -2\\ 2 & 7\end{pmatrix}\right]-\mathbf{j}\left[\det\begin{pmatrix}7 & -2\\ 4 & 7\end{pmatrix}\right]+\mathbf{k}\left[\det\begin{pmatrix}7 & 3\\ 4 & 2\end{pmatrix}\right]$$

$$\underset{\text{by (6.1)}}{=}\mathbf{i}\left[(3\times7)-(2\times(-2))\right]-\mathbf{j}\left[(7\times7)-(4\times(-2))\right]+\mathbf{k}\left[(7\times2)-(4\times3)\right]$$

$$=25\mathbf{i}-57\mathbf{j}+2\mathbf{k}$$

3. Expand the 3 by 3 matrix as normal:

$$\det\begin{pmatrix}1 & 0 & -3\\ 5 & x & -7\\ 3 & 9 & x-1\end{pmatrix}=1\det\begin{pmatrix}x & -7\\ 9 & x-1\end{pmatrix}-0-3\det\begin{pmatrix}5 & x\\ 3 & 9\end{pmatrix}$$

$$\underset{\text{by (6.1)}}{=}\left[x(x-1)-(9\times(-7))\right]-3\left[(5\times9)-(3\times x)\right]$$

$$=\left[x^2-x+63\right]-3\left[45-3x\right]$$

$$=x^2-x+63-135+9x=x^2+8x-72$$

Since we want to find the values of x when the determinant is zero, we have to solve

$$x^2+8x-72=0$$

How do we solve this quadratic equation?

Use the quadratic formula with $a=1,\ b=8$ and $c=-72$

$$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}=\frac{-8\pm\sqrt{8^2-(4\times1\times(-72))}}{2}$$

$$=\frac{-8\pm\sqrt{352}}{2}=\frac{-8\pm18.76}{2}=-13.38,\ 5.38$$

Thus $x=-13.38,\ 5.38$.

4. We need to find the cofactor of each element of the matrix $\mathbf{A}=\begin{pmatrix}1 & 0 & 5\\ -2 & 3 & 7\\ 6 & -1 & 0\end{pmatrix}$.

Cofactor of 1 is

$$\det\begin{pmatrix}3 & 7\\ -1 & 0\end{pmatrix}=(3\times 0)-(-1\times 7)=7$$

Cofactor of 0 is

$$-\det\begin{pmatrix}-2 & 7\\ 6 & 0\end{pmatrix}=-\left[(-2\times 0)-(6\times 7)\right]=42$$

Cofactor of 5 is

$$\det\begin{pmatrix}-2 & 3\\ 6 & -1\end{pmatrix}=\left[(-2\times(-1))-(6\times 3)\right]=-16$$

Cofactor of -2 is

$$-\det\begin{pmatrix}0 & 5\\ -1 & 0\end{pmatrix}=-\left[(0\times 0)-(-1\times 5)\right]=-5$$

Cofactor of 3 is

$$\det\begin{pmatrix}1 & 5\\ 6 & 0\end{pmatrix}=\left[(1\times 0)-(6\times 5)\right]=-30$$

Cofactor of 7 is

$$-\det\begin{pmatrix}1 & 0\\ 6 & -1\end{pmatrix}=-\left[(1\times(-1))-(6\times 0)\right]=1$$

Cofactor of 6 is

$$\det\begin{pmatrix}0 & 5\\ 3 & 7\end{pmatrix}=\left[(0\times 7)-(3\times 5)\right]=-15$$

Cofactor of -1 is

$$-\det\begin{pmatrix}1 & 5\\ -2 & 7\end{pmatrix}=-\left[(1\times 7)-(-2\times 5)\right]=-17$$

Cofactor of 0 is

$$\det\begin{pmatrix}1 & 0\\ -2 & 3\end{pmatrix}=\left[(1\times 3)-(-2\times 0)\right]=3$$

Collecting the cofactors gives the cofactor matrix:

$$\mathbf{C}=\begin{pmatrix}7 & 42 & -16\\ -5 & -30 & 1\\ -15 & -17 & 3\end{pmatrix}$$

Transposing this matrix (interchanging rows and columns) gives $\mathbf{C}^T=\begin{pmatrix}7 & -5 & -15\\ 42 & -30 & -17\\ -16 & 1 & 3\end{pmatrix}$.

The inverse matrix $\mathbf{A}^{-1}=\dfrac{1}{\det(\mathbf{A})}adj(\mathbf{A})$ where $adj(\mathbf{A})=\mathbf{C}^T=\begin{pmatrix}7 & -5 & -15\\ 42 & -30 & -17\\ -16 & 1 & 3\end{pmatrix}$.

*What is the determinant of **A**?*

$$\det(\mathbf{A}) = \det\begin{pmatrix} 1 & 0 & 5 \\ -2 & 3 & 7 \\ 6 & -1 & 0 \end{pmatrix} = \det\begin{pmatrix} 3 & 7 \\ -1 & 0 \end{pmatrix} + 5\det\begin{pmatrix} -2 & 3 \\ 6 & -1 \end{pmatrix}$$

$$= (0+7) + 5(2-18) = -73$$

Substituting $\det(\mathbf{A}) = -73$ and $adj(\mathbf{A}) = \begin{pmatrix} 7 & -5 & -15 \\ 42 & -30 & -17 \\ -16 & 1 & 3 \end{pmatrix}$ into $\mathbf{A}^{-1} = \dfrac{1}{\det(\mathbf{A})} adj(\mathbf{A})$:

$$\mathbf{A}^{-1} = -\frac{1}{73}\begin{pmatrix} 7 & -5 & -15 \\ 42 & -30 & -17 \\ -16 & 1 & 3 \end{pmatrix}$$

5. (a) In this case $\det(\mathbf{A}) = 1$ so we have an invertible matrix and use (6.2) to find the inverse.
Exchanging numbers 3 and 9 and placing a negative sign in front of the other numbers gives:

$$\mathbf{A}^{-1} = \begin{pmatrix} 3 & -2 \\ -13 & 9 \end{pmatrix}$$

(b) Similarly we have $\mathbf{B}^{-1} = \begin{pmatrix} 5 & -7 \\ -12 & 17 \end{pmatrix}$.

(c) By (6.1) we have

$$\det\begin{pmatrix} 5 & 4 \\ 3 & 1 \end{pmatrix} = (5\times 1) - (3\times 4) = -7$$

So using (6.2) we have

$$\mathbf{C}^{-1} = -\frac{1}{7}\begin{pmatrix} 1 & -4 \\ -3 & 5 \end{pmatrix} = \begin{pmatrix} -1/7 & 4/7 \\ 3/7 & -5/7 \end{pmatrix}$$

(d) We are given the matrix

$$\mathbf{D} = \begin{pmatrix} 3 & -5 & 3 \\ 2 & 1 & -7 \\ -10 & 4 & 5 \end{pmatrix}$$

What do we need to find?
The inverse matrix $\mathbf{D}^{-1}$ and to find $\mathbf{D}^{-1}$ we have to evaluate the determinant and the adjoint of **D**.

$$\det\begin{pmatrix} 3 & -5 & 3 \\ 2 & 1 & -7 \\ -10 & 4 & 5 \end{pmatrix} = 3\det\begin{pmatrix} 1 & -7 \\ 4 & 5 \end{pmatrix} - (-5)\det\begin{pmatrix} 2 & -7 \\ -10 & 5 \end{pmatrix} + 3\det\begin{pmatrix} 2 & 1 \\ -10 & 4 \end{pmatrix}$$

$$= 3\left[(1\times 5) - (4\times(-7))\right] + 5\left[(2\times 5) - (10\times 7)\right] + 3\left[(2\times 4) - (-10\times 1)\right]$$

$$= -147$$

Next we find $adj(\mathbf{D})$, which is the cofactor matrix transposed. The cofactor matrix can be obtained using the method described in solution 7. Thus

$$\mathbf{C} = \begin{pmatrix} 33 & 60 & 18 \\ 37 & 45 & 38 \\ 32 & 27 & 13 \end{pmatrix}$$

Transposing this gives $adj\mathbf{A}$

$$adj\mathbf{A} = \mathbf{C}^T = \begin{pmatrix} 33 & 37 & 32 \\ 60 & 45 & 27 \\ 18 & 38 & 13 \end{pmatrix}$$

By Theorem (6-4) we have

$$\mathbf{D}^{-1} = -\frac{1}{147}\begin{pmatrix} 33 & 37 & 32 \\ 60 & 45 & 27 \\ 18 & 38 & 13 \end{pmatrix}$$

6. (a) Since there are two zeros in the second row it is easier to expand along this row. Remember the place signs start with + and then alternate.

$$\det\begin{pmatrix} 2 & 3 & 5 \\ 0 & 0 & 6 \\ 1 & 5 & 3 \end{pmatrix} = -0\left[\det\begin{pmatrix} 3 & 5 \\ 5 & 3 \end{pmatrix}\right] + 0\left[\det\begin{pmatrix} 2 & 5 \\ 1 & 3 \end{pmatrix}\right] - 6\left[\det\begin{pmatrix} 2 & 3 \\ 1 & 5 \end{pmatrix}\right]$$

$$= 0 + 0 - 6\left[(2\times 5) - (1\times 3)\right] = -42$$

(b) Similarly since there is a zero along the bottom row, expand along this row.

$$\det\begin{pmatrix} 6 & 7 & 1 \\ 1 & 3 & 2 \\ 0 & 1 & 5 \end{pmatrix} = 0\left[\det\begin{pmatrix} 7 & 1 \\ 3 & 2 \end{pmatrix}\right] - 1\left[\det\begin{pmatrix} 6 & 1 \\ 1 & 2 \end{pmatrix}\right] + 5\left[\det\begin{pmatrix} 6 & 7 \\ 1 & 3 \end{pmatrix}\right]$$

$$= 0 - 1\left[(6\times 2) - 1\right] + 5\left[(6\times 3) - (1\times 7)\right] = 44$$

(c) Expand along the first column since it contains two zeros:

$$\det\begin{pmatrix} 1 & 5 & 1 \\ 0 & 3 & 7 \\ 0 & 2 & 9 \end{pmatrix} = 1\left[\det\begin{pmatrix} 3 & 7 \\ 2 & 9 \end{pmatrix}\right] - 0\left[\det\begin{pmatrix} 5 & 1 \\ 2 & 9 \end{pmatrix}\right] + 0\left[\det\begin{pmatrix} 5 & 1 \\ 3 & 7 \end{pmatrix}\right]$$

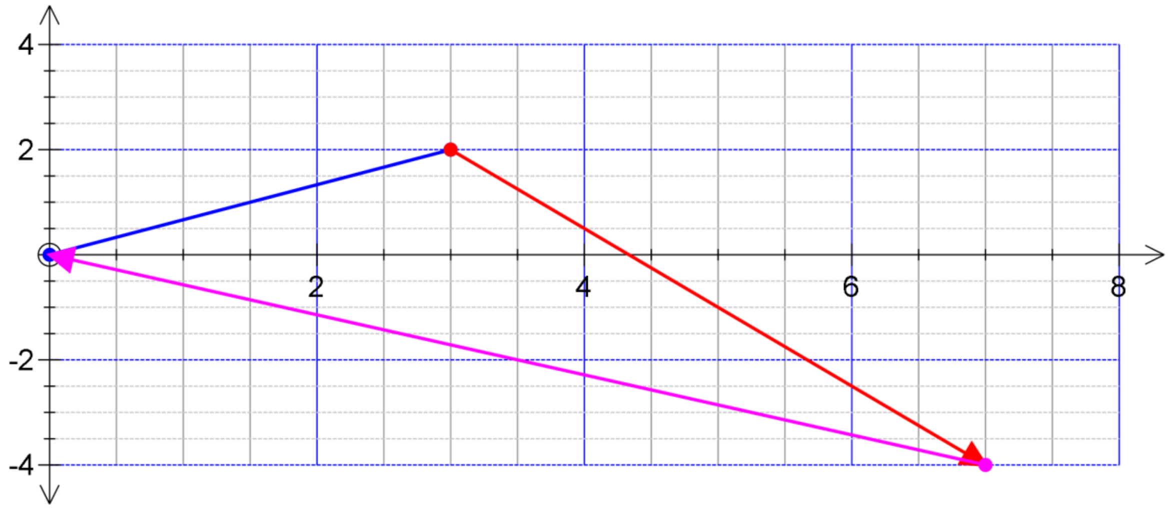
$$= 1\left[(3\times 9) - (2\times 7)\right] - 0 + 0 = 13$$

(d) Expanding along the second column

$$\det\begin{pmatrix} 9 & 5 & 1 \\ 13 & 0 & 2 \\ 11 & 0 & 3 \end{pmatrix} = -5\det\begin{pmatrix} 13 & 2 \\ 11 & 3 \end{pmatrix}$$

$$= -5\left[(13\times 3) - (11\times 2)\right] = -85$$

7. (a) The triangle given by $(0,\ 0)$, $(3,\ 2)$, $(7,\ -4)$ is illustrated below;

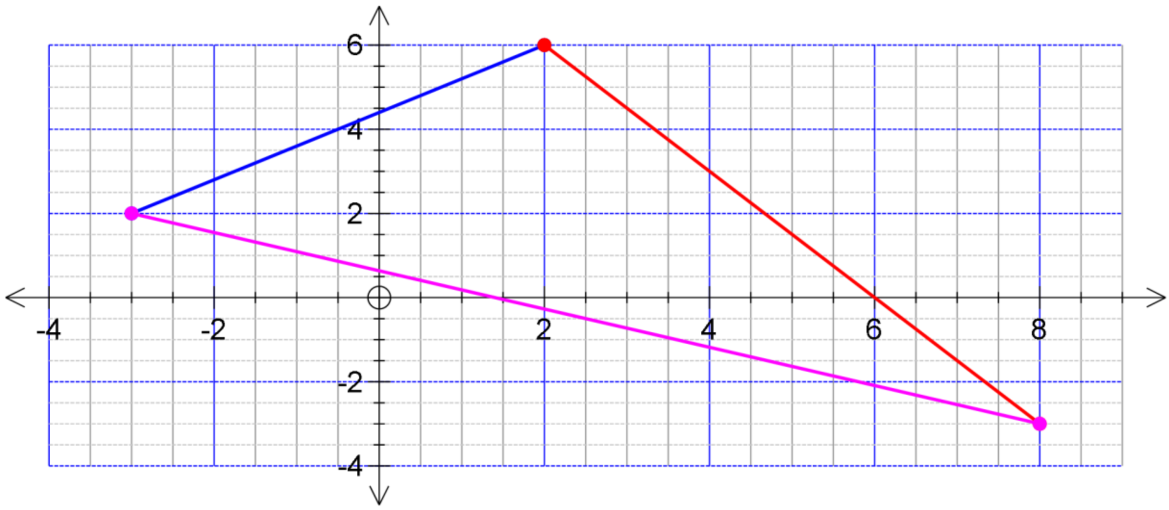


By using the formula in the question we have

$$\frac{1}{2}\det\begin{pmatrix}0 & 0 & 1\\ 3 & 2 & 1\\ 7 & -4 & 1\end{pmatrix}=\frac{1}{2}\left[1\times\det\begin{pmatrix}3 & 2\\ 7 & -4\end{pmatrix}\right]=\frac{1}{2}(-12-14)=-13$$

Area $=|-13|=13$ units^2.

(b) The triangle given by $(-3,\ 2)$, $(2,\ 6)$, $(8,\ -3)$ is illustrated below;



The area is given by

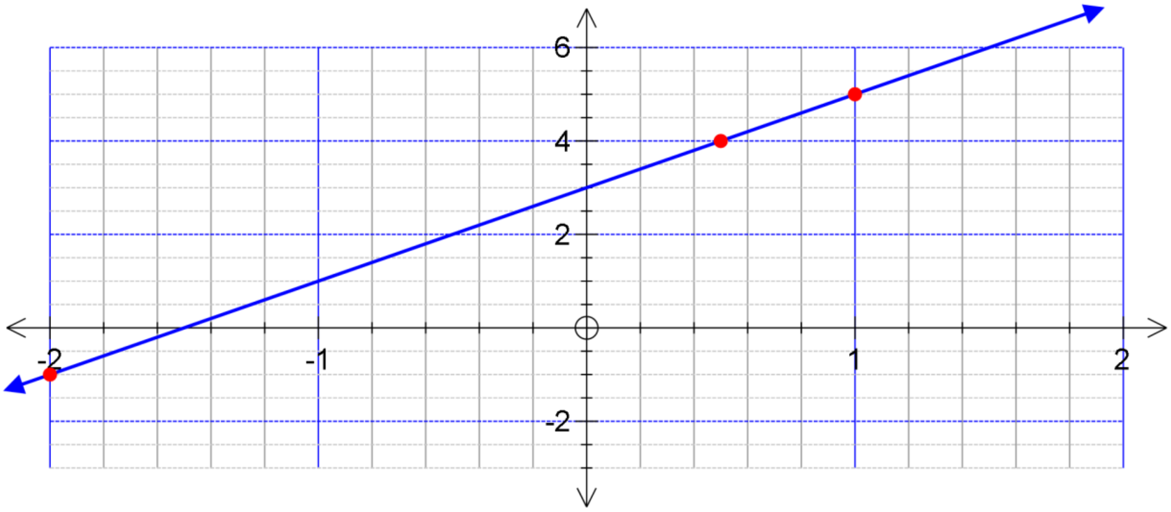
$$\frac{1}{2}\det\begin{pmatrix}-3 & 2 & 1\\ 2 & 6 & 1\\ 8 & -3 & 1\end{pmatrix}=\frac{1}{2}\left[-3\times\det\begin{pmatrix}6 & 1\\ -3 & 1\end{pmatrix}-2\det\begin{pmatrix}2 & 1\\ 8 & 1\end{pmatrix}+\det\begin{pmatrix}2 & 6\\ 8 & -3\end{pmatrix}\right]$$

$$=\frac{1}{2}\left(-3[6+3]-2[2-8]+[-6-48]\right)$$

$$=\frac{1}{2}\left(-3[9]-2[-6]-54\right)=\frac{1}{2}(-69)=-34.5$$

Area $=|-34.5|=34.5$ units^2.

(c) We are given the points $(-2,\ -1)$, $(1,\ 5)$ and $(0.5,\ 4)$ which are illustrated below:



What is the area in this case?

Area is zero because we have **no** triangle. Check that $\frac{1}{2}\det\begin{pmatrix} -2 & -1 & 1 \\ 1 & 5 & 1 \\ 0.5 & 4 & 1 \end{pmatrix} = 0$.

All the points lie on a line or we say the three points are **collinear** and we can use the determinate to test if given points are **collinear** (lie on one line). We conclude that if

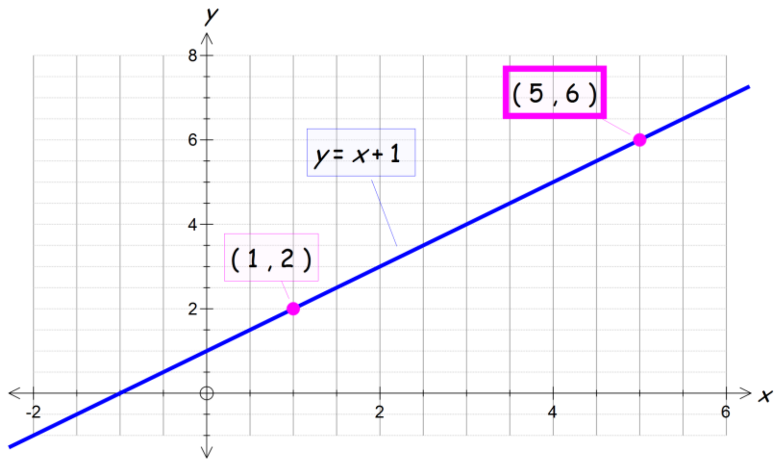
$$\det\begin{pmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{pmatrix} = 0$$

then $(x_1,\ y_1)$, $(x_2,\ y_2)$, $(x_3,\ y_3)$ are *collinear*.

8. (a) We need to find the equation through the points $(1,\ \ 2)$ and $(5,\ \ 6)$:
Substituting $x_1 = 1,\ y_1 = 2$ and $x_2 = 5,\ y_2 = 6$ into the given formula:

$$\det\begin{pmatrix} x & y & 1 \\ 1 & 2 & 1 \\ 5 & 6 & 1 \end{pmatrix} = -4 - 4x + 4y = 0 \quad \Rightarrow \quad y = x + 1$$

We can plot this:

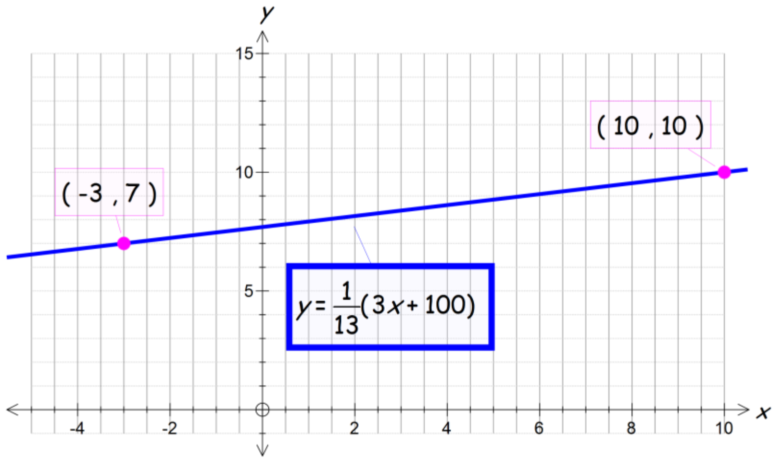


(b) Similarly we have to find the equation through the points $(-3,\ \ 7)$ and $(10,\ \ 10)$:

Substituting $x_1 = -3,\ y_1 = 7$ and $x_2 = 10,\ y_2 = 10$ into the given formula:

$$\det\begin{pmatrix} x & y & 1 \\ -3 & 7 & 1 \\ 10 & 10 & 1 \end{pmatrix} = 13y - 3x - 100 = 0 \quad \Rightarrow \quad y = \frac{1}{13}(3x+100)$$

We can plot this:

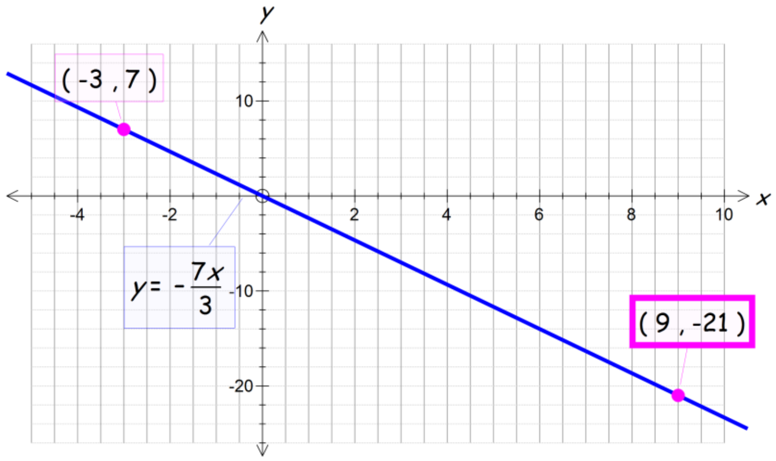


(c) Need to find the equation through the points $(-3,\ \ 7)$ and $(9,\ \ -21)$:

Substituting $x_1 = -3,\ y_1 = 7$ and $x_2 = 9,\ y_2 = -21$ into the given formula:

$$\det\begin{pmatrix} x & y & 1 \\ -3 & 7 & 1 \\ 9 & -21 & 1 \end{pmatrix} = 28x + 12y = 0 \quad \Rightarrow \quad y = -\frac{28}{12}x = -\frac{7}{3}x$$

We can plot this:



9. The place sign of a_{31} is $(-1)^{3+1} = (-1)^4 = 1$. The place sign of a_{56} is

$$(-1)^{5+6} = (-1)^{11} = -1$$

The place sign of a_{62} is $(-1)^{6+2} = (-1)^8 = 1$. The place sign of a_{65} is $(-1)^{5+6} = -1$.

Since we have a 6 by 6 matrix therefore there is **no** a_{71} entry in matrix **A**.

10. The place sign of a_{nm} is equal to $(-1)^{n+m} = (-1)^{m+n}$ and $(-1)^{m+n}$ is the place sign of a_{mn}.

Hence we have our result.

11. To find $\det\begin{pmatrix} a & b & c & d \\ 0 & 0 & 0 & 0 \\ e & f & g & h \\ i & j & k & l \end{pmatrix}$ we expand along the second row. *Why?*

Because all the entries along the second row is zero therefore $\det(\mathbf{A})=0$.

12. We need to find the value of k where $\det(\mathbf{A})\neq 0$ [Not Zero]. Easier to find the values of k where $\det(\mathbf{A})=0$:

$$\det\begin{pmatrix} k & 1 & 2 \\ 0 & k & 2 \\ 5 & -5 & k \end{pmatrix} = 0 + k\det\begin{pmatrix} k & 2 \\ 5 & k \end{pmatrix} - 2\det\begin{pmatrix} k & 1 \\ 5 & -5 \end{pmatrix}$$

$$= k\left(k^2-10\right)-2(-5k-5)$$

$$= k^3-10k+10k+10 = k^3+10 = 0$$

The matrix in invertible for **all** real values of k apart from where $k^3+10=0$ or $k^3=-10$.

13. Expanding along the first row:

$$\det\begin{pmatrix} 1 & 1 & 1 \\ x & y & z \\ x^2 & y^2 & z^2 \end{pmatrix} = \det\begin{pmatrix} y & z \\ y^2 & z^2 \end{pmatrix} - \det\begin{pmatrix} x & z \\ x^2 & z^2 \end{pmatrix} + \det\begin{pmatrix} x & y \\ x^2 & y^2 \end{pmatrix}$$

$$= \left(yz^2-y^2z\right)-\left(xz^2-x^2z\right)+\left(xy^2-x^2y\right)$$

$$= yz^2-y^2z-xz^2+x^2z+xy^2-x^2y$$

Expanding the Right-Hand Side of the given result, which is $(x-y)(y-z)(z-x)$, yields

$$(x-y)(y-z)(z-x) = \left(xy-xz-y^2+yz\right)(z-x)$$

$$= xyz-x^2y-xz^2+x^2z-y^2z+xy^2+yz^2-xyz$$

$$= yz^2-y^2z-xz^2+x^2z+xy^2-x^2y$$

Comparing our answers gives our required result.

14. We need to find the absolute value of the following determinant:

$$\det\begin{pmatrix}\mathbf{u} & \mathbf{v} & \mathbf{w}\end{pmatrix} = \det\begin{pmatrix} 1 & 2 & 7 \\ 2 & 3 & 10 \\ 1 & 5 & -1 \end{pmatrix} = -7$$

The volume is given by $|-7| = 7\,\text{unit}^3$.

15. We need to show that the determinant of the rotational matrix **R** is equal to 1:

$$\det(\mathbf{R}) = \det\begin{pmatrix} \cos(\theta) & \sin(\theta) & 0 \\ -\sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{pmatrix} = \cos^2(\theta)+\sin^2(\theta)=1$$

The determinant of one means that the volume of the transformed object has not changed as we would expect when an object has been rotated.

16. We have

$$\det(\mathbf{J}) = \det\begin{pmatrix} \cos(\theta) & -r\sin(\theta) \\ \sin(\theta) & r\cos(\theta) \end{pmatrix}$$

$$= r\cos^2(\theta) + r\sin^2(\theta) = r^2\underbrace{\left[\cos^2(\theta)+\sin^2(\theta)\right]}_{=1} = r(1) = r$$

17. We are given that $\det\begin{pmatrix} \cos(\theta)\sin(\phi) & -\rho\sin(\theta)\sin(\phi) & \rho\cos(\theta)\cos(\phi) \\ \sin(\theta)\sin(\phi) & \rho\cos(\theta)\sin(\phi) & \rho\sin(\theta)\cos(\phi) \\ \cos(\phi) & 0 & -\rho\sin(\phi) \end{pmatrix}$:

$$\det\begin{pmatrix} \cos(\theta)\sin(\phi) & -\rho\sin(\theta)\sin(\phi) & \rho\cos(\theta)\cos(\phi) \\ \sin(\theta)\sin(\phi) & \rho\cos(\theta)\sin(\phi) & \rho\sin(\theta)\cos(\phi) \\ \cos(\phi) & 0 & -\rho\sin(\phi) \end{pmatrix}$$

$$= \left[\cos(\phi)\right]\det\begin{pmatrix} -\rho\sin(\theta)\sin(\phi) & \rho\cos(\theta)\cos(\phi) \\ \rho\cos(\theta)\sin(\phi) & \rho\sin(\theta)\cos(\phi) \end{pmatrix} + \left[-\rho\sin(\phi)\right]\det\begin{pmatrix} \cos(\theta)\sin(\phi) & -\rho\sin(\theta)\sin(\phi) \\ \sin(\theta)\sin(\phi) & \rho\cos(\theta)\sin(\phi) \end{pmatrix}$$

$$= \left[\cos(\phi)\right]\left[-\rho^2\sin^2(\theta)\sin(\phi)\cos(\phi) - \rho^2\cos^2(\theta)\cos(\phi)\sin(\phi)\right]$$

$$-\left[\rho\sin(\phi)\right]\left[\rho\cos^2(\theta)\sin^2(\phi) + \rho\sin^2(\theta)\sin^2(\phi)\right]$$

$$= -\rho^2\left\{\left[\cos^2(\phi)\sin(\phi)\right]\underbrace{\left[\sin^2(\theta)+\cos^2(\theta)\right]}_{=1} + \sin(\phi)\sin^2(\phi)\underbrace{\left[\sin^2(\theta)+\cos^2(\theta)\right]}_{=1}\right\}$$

$$= -\rho^2\sin(\phi)\left\{\cos^2(\phi)+\sin^2(\phi)\right\} = -\rho^2\sin(\phi)$$

Taking the modulus of this gives $\rho^2\sin(\phi)$.

18. We have

$$W(1,\quad \cos(x),\quad \sin(x)) = \det\begin{pmatrix} 1 & \cos(x) & \sin(x) \\ 0 & -\sin(x) & \cos(x) \\ 0 & -\cos(x) & -\sin(x) \end{pmatrix}$$

$$= \sin^2(x) + \cos^2(x) = 1 \qquad \left[\text{Expanding along the first column}\right]$$

19. We need to prove for every natural number n that $\det(\mathbf{I}_n) = 1$. Remember $\mathbf{I}_n$ is the $n \times n$ identity matrix. We use proof by induction. *What is the procedure for proof by induction?*

(i) Prove the result for a base case n_0.

(ii) Assume the result is true for $n = k$.

(iii) Prove the result for $n = k + 1$.

Proof.

<u>Step (i)</u>:

For $n = 2$ we have the identity $\mathbf{I}_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and clearly

$$\det(\mathbf{I}_2)=\det\begin{pmatrix}1&0\\0&1\end{pmatrix}=1$$

Hence the result is true for $n=2$.

<u>Step (ii)</u>:

Assume the result is true for $n=k$ that is $\det(\mathbf{I}_k)=1$.

<u>Step (iii)</u>:

Need to prove it for $n=k+1$ that is required to prove $\det(\mathbf{I}_{k+1})=1$.

$\mathbf{I}_{k+1}$ is the $k+1$ by $k+1$ identity matrix:

$$\mathbf{I}_{k+1}=\begin{pmatrix}1&0&0&0\\0&1&0&0\\\vdots&\cdots&\ddots&\vdots\\0&0&0&1\end{pmatrix}$$

To find the determinant of $\mathbf{I}_{k+1}$ we can expand along the first row which is 1 times the determinant of the remaining matrix after deleting the first row and column. The remaining matrix is $\mathbf{I}_k$ and by assumption we have $\det(\mathbf{I}_k)=1$ therefore

$$\det(\mathbf{I}_{k+1})=1\times1=1$$

Hence we have proven our result by induction.

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20. *Proof*. Since **A** is invertible we have $\mathbf{AA}^{-1}=\mathbf{I}$ therefore $\det(\mathbf{AA}^{-1})=\det(\mathbf{I})\underset{\text{By Question 19}}{=}1$

■

21. *Proof*.

Expanding along the zero row or column gives

$$0\det(\;)+0\det(\;)+0\det(\;)+\cdots+0\det(\;)=0$$

Hence our result.

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22. *Proof*.

Let $\mathbf{A}=\begin{pmatrix}a_{11}&a_{12}&\cdots&a_{1n}\\a_{21}&a_{22}&\cdots&a_{2n}\\\vdots&\vdots&\cdots&\vdots\\a_{n1}&a_{n2}&\cdots&a_{nn}\end{pmatrix}$ then by expanding along the first row and using

(6.5) $\det(\mathbf{A})=a_{11}C_{11}+a_{12}C_{12}+a_{13}C_{13}+\cdots+a_{1n}C_{1n}$

we have

$$\det(\mathbf{A})=a_{11}C_{11}+a_{12}C_{12}+a_{13}C_{13}+\cdots+a_{1n}C_{1n}$$

Taking the transpose of matrix **A** we have $\mathbf{A}^T=\begin{pmatrix}a_{11}&a_{21}&\cdots&a_{n1}\\a_{12}&a_{22}&\cdots&a_{n2}\\\vdots&\vdots&\cdots&\vdots\\a_{1n}&a_{2n}&\cdots&a_{nn}\end{pmatrix}$. Expanding along the first column of the transposed matrix gives

$$\det\left(\mathbf{A}^T\right) = a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13} + \cdots + a_{1n}C_{1n}$$

Note that the cofactors are identical because we delete the same elements of the matrix whether we expand along the first row of matrix **A** or the first column of the transposed matrix $\mathbf{A}^T$.

Hence $\det\left(\mathbf{A}^T\right) = \det(\mathbf{A})$.

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23. *Proof.*

Consider the matrix **B** obtained from matrix **A** by multiplying the ith row by a scalar k.

$$\begin{array}{c} \\ \\ \text{ith row} \\ \\ \\ \end{array} \mathbf{A} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{i1} & a_{i2} & \cdots & a_{in} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix} \quad \text{and} \quad \mathbf{B} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & \vdots & \vdots & \vdots \\ ka_{i1} & ka_{i2} & \cdots & ka_{in} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix} \begin{array}{c} \\ \\ \text{ith row} \\ \\ \\ \end{array}$$

We can find the determinant of matrix **B** by expanding along the ith row and using (6.6):

$$\begin{aligned} \det(\mathbf{B}) &= ka_{i1}C_{i1} + ka_{i2}C_{i2} + ka_{i3}C_{i3} + \cdots + ka_{in}C_{in} \\ &= k\underbrace{\left(a_{i1}C_{i1} + a_{i2}C_{i2} + a_{i3}C_{i3} + \cdots + a_{in}C_{in}\right)}_{\det(\mathbf{A})} \\ &= k\det(\mathbf{A}) \end{aligned}$$

Hence we have our required result, that is $\det(\mathbf{B}) = k\det(\mathbf{A})$.

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24. Need to prove $\det(k\mathbf{A}) = k^n \det(\mathbf{A})$.

Proof.

Let $\mathbf{A}_1$ be the matrix obtained from **A** by multiplying a row by k. Then by result of question 23 we have $\det(\mathbf{A}_1) = k\det(\mathbf{A})$.

Let $\mathbf{A}_2$ be the matrix obtained from $\mathbf{A}_1$ by multiplying a non k row by k. Then by result of question 23 we have

$$\det(\mathbf{A}_2) = k\underbrace{\det(\mathbf{A}_1)}_{=k\det(\mathbf{A})} = kk\det(\mathbf{A}) = k^2\det(\mathbf{A})$$

Continuing in this manner we obtain a matrix $\mathbf{A}_n$ from $\mathbf{A}_{n-1}$ by multiplying the last non k row by k. Note that to obtain the matrix $\mathbf{A}_n$ we need to multiply each of the n rows by the scalar k. We have

$$\begin{aligned} \det(\mathbf{A}_n) &= k\underbrace{\det(\mathbf{A}_{n-1})}_{=k\det(\mathbf{A}_{n-2})} \\ &= kk\det(\mathbf{A}_{n-2}) \\ &= \underbrace{kk\cdots k}_{n\text{ copies}}\det(\mathbf{A}) = k^n\det(\mathbf{A}) \end{aligned}$$

We have proven that $\det(k\mathbf{A}) = k^n\det(\mathbf{A})$ where **A** is a n by n matrix.

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