Complete Solutions to Exercises 5.3

1. (a) We need to find the kernel, range and dimensions of the given transformation. Let $\mathbf{v}=\begin{pmatrix} x \\ y \end{pmatrix}$.

(i) For $\ker(T)$ we need to find x and y values which produce the zero vector under the linear transformation $T(\mathbf{v})=\mathbf{A}\mathbf{v}$:

$$T(\mathbf{v})=\begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

We have the simultaneous equations

$$x+2y=0 \quad \text{and} \quad x+2y=0$$

This gives $x=-2y$ and let $y=r$ where r is any real number then $x=-2r$.

The vector which gives the zero vector after transformation is $\mathbf{v}=\begin{pmatrix} -2r \\ r \end{pmatrix}=r\begin{pmatrix} -2 \\ 1 \end{pmatrix}$.

Hence $\ker(T)=\left\{ r\begin{pmatrix} -2 \\ 1 \end{pmatrix} \;\middle|\; r \in \mathbb{R} \right\}$.

(ii) *What does* $\text{nullity}(T)$ *mean?*

It is the dimension of $\ker(T)$. *How can we find the dimension of*

$$\ker(T)=\left\{ r\begin{pmatrix} -2 \\ 1 \end{pmatrix} \;\middle|\; r \in \mathbb{R} \right\}?$$

The dimension is the number of vectors in a basis for $\ker(T)$. *What is a basis for* $\ker(T)$ *?*

$\left\{\begin{pmatrix} -2 \\ 1 \end{pmatrix}\right\}$ is a basis for $\ker(T)$. Since there is **only** one vector in the basis therefore

$$\text{nullity}(T)=\dim\big(\ker(T)\big)=1$$

(iii) *How do we find the range of the given linear transformation*

$$T(\mathbf{v})=\begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}?$$

Writing out the equations we have

$$T(\mathbf{v})=\begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}=\begin{pmatrix} x+2y \\ x+2y \end{pmatrix}$$

Both the entries in the vector belonging to the range are identical, that is

$\text{range}(T)=\left\{\begin{pmatrix} a \\ a \end{pmatrix} \;\middle|\; a \in \mathbb{R} \right\}$.

(iv) *What does* $\text{rank}(T)$ *mean?*

It is the dimension of the range of the transformation. Since $\text{range}(T)=\left\{\begin{pmatrix} a \\ a \end{pmatrix} \;\middle|\; a \in \mathbb{R} \right\}$ therefore a basis for this space is $\left\{\begin{pmatrix} 1 \\ 1 \end{pmatrix}\right\}$ which means it has dimension of 1 because we only have one vector in the basis. Thus

$$\text{rank}(T) = \dim(\text{range}(T)) = 1$$

(b) In this case we have $T : \mathbb{R}^3 \to \mathbb{R}^3$ given by $T(\mathbf{v}) = \mathbf{Av}$ where $\mathbf{A} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$.

Let $\mathbf{v} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ because $\mathbf{v}$ is in $\mathbb{R}^3$.

(i) To find the kernel of the given transformation we substitute the above into $T(\mathbf{v}) = \mathbf{Av} = \mathbf{O}$:

$$T\left(\begin{pmatrix} x \\ y \\ z \end{pmatrix}\right) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Opening up this matrix gives $x = 0$, $y = 0$ and $z = 0$. Thus the kernel is the zero vector, that is $\ker(T) = \{\mathbf{O}\}$.

(ii) *What does the* term *nullity*(T) *mean?*

It is the dimension of $\ker(T)$. *What is the dimension of* $\ker(T) = \{\mathbf{O}\}$ *?*

Zero because the definition of dimension of the zero vector is 0. Thus $\text{nullity}(T) = 0$.

(iii) *What is the range of the given linear transformation?*

We have

$$T\left(\begin{pmatrix} x \\ y \\ z \end{pmatrix}\right) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

The arbitrary vector $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$ maps to $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$ under the transformation T therefore $\text{range}(T) = \mathbb{R}^3$.

(iv) The rank is the dimension of $\text{range}(T) = \mathbb{R}^3$. Hence

$$\text{rank}(T) = \dim(\mathbb{R}^3) = 3$$

(c) We are given $T : \mathbb{R}^3 \to \mathbb{R}^3$ defined by $T(\mathbf{v}) = \mathbf{Av}$ where $\mathbf{A} = \begin{pmatrix} 1 & 3 & 5 \\ 2 & 6 & 10 \\ 4 & 12 & 20 \end{pmatrix}$. Let

$\mathbf{v} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$.

(i) For the kernel of the linear transformation we have

$$T(\mathbf{v}) = \begin{pmatrix} 1 & 3 & 5 \\ 2 & 6 & 10 \\ 4 & 12 & 20 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Labelling rows and using row operations we have

$$\begin{array}{c} R_1 \\ R_2 \\ R_3 \end{array}\left(\begin{array}{ccc|c} 1 & 3 & 5 & 0 \\ 2 & 6 & 10 & 0 \\ 4 & 12 & 20 & 0 \end{array}\right) \quad \Leftrightarrow \quad \begin{array}{c} R_1 \\ R_2 - 2R_1 \\ R_3 - 4R_1 \end{array}\left(\begin{array}{ccc|c} 1 & 3 & 5 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \qquad (*)$$

From the first row we have $x + 3y + 5z = 0$ which gives $x = -3y - 5z$. This means we have 2 free variables, *y* and *z*. Let $y = s$ and $z = t$ then $x = -3s - 5t$. Thus the kernel of the transformation is given by

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -3s - 5t \\ s \\ t \end{pmatrix} = s\begin{pmatrix} -3 \\ 1 \\ 0 \end{pmatrix} + t\begin{pmatrix} -5 \\ 0 \\ 1 \end{pmatrix}$$

Hence $\ker(T) = \left\{ s\begin{pmatrix} -3 \\ 1 \\ 0 \end{pmatrix} + t\begin{pmatrix} -5 \\ 0 \\ 1 \end{pmatrix} \;\middle|\; s \in \mathbb{R} \text{ and } t \in \mathbb{R} \right\}$.

(ii) *What is nullity(T) equal to?*

It is dimension of $\ker(T)$ found in part (i). *What is the dimension of this space?*

2 because a basis for $\ker(T)$ is $\left\{ \begin{pmatrix} -3 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -5 \\ 0 \\ 1 \end{pmatrix} \right\}$, that is we need 2 basis vectors for the kernel of the given linear transformation. Thus $\text{nullity}(T) = 2$.

(iii) *What is the range of the given linear transformation?*

By using (*) in the above part (i) we have

$$T(\mathbf{v}) = \begin{pmatrix} 1 & 3 & 5 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x + 3y + 5z \\ 0 \\ 0 \end{pmatrix}$$

where *x*, *y* and *z* are real numbers. Since *x*, *y* and *z* are any real numbers therefore the range is the set of all real numbers $\mathbb{R}$, that is $\text{range}(T) = \mathbb{R}$.

(iv) *What is the rank of the linear transformation?*

Rank is the dimension of the $\text{range}(T) = \mathbb{R}$ which is 1. Thus $\text{rank}(T) = 1$.

(d) We need to find the kernel, range and dimensions of these spaces for the given linear transformation $T(\mathbf{p}) = x\mathbf{p}'$. Let $\mathbf{p} = p(x) = ax^3 + bx^2 + cx + d$.

(i) Remember the kernel of a transformation is the set which is transformed to the zero vector, that is $T(\mathbf{p}) = \mathbf{O}$. We have

$$\begin{aligned} T(\mathbf{p}) &= x\left(ax^3 + bx^2 + cx + d\right)' \\ &= x\left(3ax^2 + 2bx + c\right) \\ &= 3ax^3 + 2bx^2 + cx = 0 \end{aligned}$$

Thus $a = 0,\ b = 0$ and $c = 0$ gives $T(\mathbf{p}) = \mathbf{O}$. This means that

$$\begin{aligned} \ker(T) &= \left\{ ax^3 + bx^2 + cx + d \;\middle|\; a = b = c = 0 \right\} \\ &= \{d\} = \{d \mid d \in \mathbb{R}\} = P_0 \end{aligned}$$

Hence the set of constant polynomials P_0 is the kernel of T.

(ii) The nullity(T) is the dimension of the kernel of T. *What is the dimension of* $\ker(T) = P_0$*?*

A basis for $\ker(T) = P_0$ is $\{1\}$ therefore $\dim(\ker(T)) = 1$ because a basis contains only one vector.

(iii) *What is range of the linear transformation?*

The range is the set of vectors given by $T(\mathbf{p})$:

$$\begin{aligned} T(\mathbf{p}) &= x\left(ax^3 + bx^2 + cx + d\right)' \\ &= x\left(3ax^2 + 2bx + c\right) \\ &= 3ax^3 + 2bx^2 + cx \end{aligned}$$

Thus $T(\mathbf{p}) = 3ax^3 + 2bx^2 + cx$ is a cubic polynomial but excludes the constants. We have range(T)$=\left\{dx^3 + ex^2 + fx \mid d \in \mathbb{R},\ e \in \mathbb{R} \text{ and } f \in \mathbb{R}\right\}$.

(iv) Rank(T) is the dimension of range(T) $=\left\{dx^3 + ex^2 + fx\right\}$. Remember from our earlier work on basis, a basis for $=\left\{dx^3 + ex^2 + fx\right\}$ is $\left\{x,\ x^2,\ x^3\right\}$. This means the dimension of range(T) is 3 because we have 3 basis vectors. Thus $\text{Rank}(T) = 3$.

(e) Let $\mathbf{p} = p(x) = ax^3 + bx^2 + cx + d$ because we are given $T:\ P_3 \to P_2$ where P_3 is the set of polynomials of degree 3 or less. The given transformation is defined as $T(\mathbf{p}) = \mathbf{p}'$.

(i) The kernel of the transformation $T(\mathbf{p}) = \mathbf{p}'$ is the set of constant polynomials, that is $\ker(T) = \left\{c \mid c \in \mathbb{R}\right\} = P_0$.

(ii) The dimension of the kernel is 1 because a basis for $\ker(T) = P_0$ is $\{1\}$.

(iii) *What is the range of the given linear transformation?*

$$\begin{aligned} T(\mathbf{p}) &= \left(ax^3 + bx^2 + cx + d\right)' \\ &= 3ax^2 + 2bx + c \end{aligned}$$

The range is the vector space $\left\{3ax^2 + 2bx + c \mid a,\ b,\ c \in \mathbb{R}\right\}$ which is the set of polynomials of degree 2 or less. Thus $\text{range}(T) = P_2$.

(iv) The rank of the given transformation is the dimension of the $\text{range}(T) = P_2$. *What is the dimension of this space?*

3 because $\dim(P_k) = k + 1$ so $\dim(P_2) = 3$. Hence $\text{rank}(T) = 3$.

(f) We are given $T: P_3 \to \mathbb{R}$ where $T(\mathbf{p}) = \int_0^1 p(x)\,dx$. Let

$$\mathbf{p} = p(x) = ax^3 + bx^2 + cx + d \qquad \text{[Cubic Polynomial]}$$

(i) *What is the kernel of this linear transformation?*

The set of vectors $\mathbf{p}$ such that $T(\mathbf{p}) = \mathbf{O}$:

$$T(\mathbf{p}) = \int_0^1 \left[ax^3 + bx^2 + cx + d\right] dx$$

$$= \left[\frac{ax^4}{4} + \frac{bx^3}{3} + \frac{cx^2}{2} + dx\right]_0^1 \qquad \text{[Integrating]}$$

$$= \frac{a}{4} + \frac{b}{3} + \frac{c}{2} + d = 0 \qquad \text{[Substituting limits]}$$

Multiplying the last line by 12 gives

$$3a + 4b + 6c + 12d = 0$$

$$a = -\frac{4}{3}b - 2c - 4d$$

Hence $\ker(T) = \left\{ ax^3 + bx^2 + cx + d \;\middle|\; a = -\frac{4}{3}b - 2c - 4d \right\}$.

(ii) The $\text{nullity}(T)$ is the dimension of $\ker(T)$. *What is the dimension of the kernel?*

Since the variables b, c and d are free in the above derivation:

$$\ker(T) = \left\{ ax^3 + bx^2 + cx + d \;\middle|\; a = -\frac{4}{3}b - 2c - 4d \right\}$$

Therefore $\dim(\ker(T)) = 3$.

[Alternatively if we want to find a basis for this then a lot more work is required as the following shows:

Substituting the above $a = -\frac{4}{3}b - 2c - 4d$ into $p(x) = ax^3 + bx^2 + cx + d$ gives

$$p(x) = ax^3 + bx^2 + cx + d$$

$$= \left(-\frac{4}{3}b - 2c - 4d\right)x^3 + bx^2 + cx + d$$

$$\underset{\substack{\text{Factorising} \\ \text{into } b,\ c \text{ and } d}}{=} \left(-\frac{4}{3}x^3 + x^2\right)b + \left(-2x^3 + x\right)c + \left(-4x^3 + 1\right)d$$

$$= \left(x^2 - \frac{4}{3}x^3\right)b + \left(x - 2x^3\right)c + \left(1 - 4x^3\right)d$$

A basis for $\ker(T)$ is $\left\{ \left(x^2 - \frac{4}{3}x^3\right),\ \left(x - 2x^3\right),\ \left(1 - 4x^3\right) \right\}$. This confirms the above that $\dim(\ker(T)) = 3$].

(iii) *What is the range of the given linear transformation* $T(\mathbf{p}) = \int_0^1 p(x)\,dx$ *?*

$$\text{Range}(T) = T(\mathbf{p}) = \int_0^1 p(x)\,dx$$

$$= \frac{a}{4} + \frac{b}{3} + \frac{c}{2} + d \qquad \text{[By part (i)]}$$

Since a, b, c and d are arbitrary real numbers therefore $\text{range}(T)$ is the set of **all** real numbers $\mathbb{R}$. We have $\text{range}(T) = \mathbb{R}$.

(iv) *What does the term rank of a linear transformation mean?*
It is the dimension of the range which in this case is $\text{range}(T) = □$. *What is the dimension of* □ *?*
1. Thus $\text{rank}(T) = 1$.
(g) The given linear transformation $T : M_{22} \to P_1$ is

$$T\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) = (a+c)x + (b+d).$$

(i) *What is the kernel of this transformation?*
Let $\mathbf{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ then the kernel is the set of matrices M_{22} which are transformed to the zero vector, that is $T(\mathbf{A}) = \mathbf{O}$.

$$T\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) = (a+c)x + (b+d) = 0$$

This gives $a + c = 0$ and $b + d = 0$. We have $c = -a$ and $d = -b$. Hence the kernel of T is given by

$$\ker(T) = \left\{\begin{pmatrix} a & b \\ -a & -b \end{pmatrix}\right\}$$

(ii) The nullity of T is the dimension of the kernel. *How can we evaluate the dimension of the kernel?*
By using row operations we have

$$\begin{array}{c} R_1 \\ R_2 \end{array}\begin{pmatrix} a & b \\ -a & -b \end{pmatrix} \quad \Leftrightarrow \quad \begin{array}{c} R_1 \\ R_2 + R_1 \end{array}\begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix}$$

A basis for $\begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix}$ is the set $\left\{\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\right\}$. *What is the dimension of the kernel,* $\ker(T) = \left\{\begin{pmatrix} a & b \\ -a & -b \end{pmatrix}\right\}$?
2 because we only need 2 matrices in a basis for $\ker(T)$. Thus $\mathit{nullity}(T) = 2$.

(iii) *What is the range of the given transformation* $T\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) = (a+c)x + (b+d)$*?*

It is vectors $T(\mathbf{v})$ which is same as the above

$$T\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) = (a+c)x + (b+d)$$

The range is $\{(a+c)x + (b+d)\}$ which is the set of linear polynomials, that is $\mathit{range}(T) = P_1$.
(iv) *What is* $\mathit{rank}(T)$ *equal to?*
It is the dimension of the $\mathit{range}(T) = P_1$. Thus $\mathit{rank}(T) = 2$ because $\dim(P_k) = k + 1$.

2. (a) We are given $T:\mathbb{R}^2\to\mathbb{R}^2$ defined by $T(\mathbf{v})=\mathbf{Av}$ where $\mathbf{A}=\begin{pmatrix}1&2\\1&2\end{pmatrix}$. *What is the dimension of the domain* $\mathbb{R}^2$ *?*

The dimension of $\mathbb{R}^2$ is 2. We have $n=2$. By solution to question 1(a) parts (ii) and (iv) we have

$$\text{nullity}(T)+\text{rank}(T)=1+1=2$$

This verifies $\text{nullity}(T)+\text{rank}(T)=n$.

(b) For $T:\mathbb{R}^3\to\mathbb{R}^3$ given by $T(\mathbf{v})=\mathbf{Av}$ where $\mathbf{A}=\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}$ we have the dimension of the domain $\mathbb{R}^3$ is 3, that is $n=3$. *What is the* $\textit{nullity}(T)$ *and* $\textit{rank}(T)$ *equal to in this case?*

By solution to question 1(b) parts (ii) and (iv) we have

$$\text{nullity}(T)=0 \text{ and } \text{rank}(T)=3$$

Thus

$$\text{nullity}(T)+\text{rank}(T)=0+3=3=n$$

(c) For $T:\mathbb{R}^3\to\mathbb{R}^3$ given by $T(\mathbf{v})=\mathbf{Av}$ where $\mathbf{A}=\begin{pmatrix}1&3&5\\2&6&10\\4&12&20\end{pmatrix}$ we need to verify the dimension theorem. *What is the dimension, n, of the domain* $\mathbb{R}^3$ *?*

$n=3$. *What is the* $\textit{nullity}(T)$ *and* $\textit{rank}(T)$ *equal to in this case?*

By solution to question 1(c) parts (ii) and (iv) we have

$$\text{nullity}(T)=2 \text{ and } \text{rank}(T)=1$$

Thus

$$\text{nullity}(T)+\text{rank}(T)=2+1=3=n$$

(d) We are given $T:P_3\to P_3$ defined by $T(\mathbf{p})=x\mathbf{p}'$. *What is the dimension, n, of the domain* P_3 *?*

Since we need 4 vectors to form a basis for P_3 therefore $n=4$. *What is the* $\textit{nullity}(T)$ *and* $\textit{rank}(T)$ *equal to?*

$\text{nullity}(T)=1$ and $\text{rank}(T)=3$. Thus

$$\text{nullity}(T)+\text{rank}(T)=1+3=4=n$$

(e) We have $T:P_3\to P_2$ given by $T(\mathbf{p})=\mathbf{p}'$. *What is the dimension, n, of the domain* P_3 *?*

$n=4$. *What is the* $\textit{nullity}(T)$ *and* $\textit{rank}(T)$ *equal to?*

$\text{nullity}(T)=1$ and $\text{rank}(T)=3$. Thus

$$\text{nullity}(T)+\text{rank}(T)=1+3=4=n$$

(f) We have $T:P_3\to\mathbb{R}$ given by $T(\mathbf{p})=\int_0^1 p(x)\,dx$. *What is the dimension, n, of the domain* P_3 *?*

$n=4$. *What is the* $nullity(T)$ *and* $rank(T)$ *equal to?*

$\text{nullity}(T)=3$ and $\text{rank}(T)=1$. Thus

$$\text{nullity}(T)+\text{rank}(T)=3+1=4=n$$

(g) The given linear transformation $T: M_{22} \to P_1$ is

$$T\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right)=(a+c)x+(b+d)$$

What is the dimension, n, of the domain M_{22}*?*

$n=4$. *What is the* $nullity(T)$ *and* $rank(T)$ *equal to?*

$\text{nullity}(T)=2$ and $\text{rank}(T)=2$. Thus

$$\text{nullity}(T)+\text{rank}(T)=2+2=4=n$$

3. (i) The kernel is a subspace of $\mathbb{R}^5$ because $\mathbb{R}^5$ is the domain of *T*. *How do we find a basis for* $\ker(T)$ *?*

We need to place the above matrix **A** into (reduced) row echelon form **R** and then solve $\mathbf{Rx}=\mathbf{O}$. The general solution **x** gives a basis for the kernel.

$$\begin{array}{c} R_1 \\ R_2 \\ R_3 \end{array}\begin{pmatrix} 1 & 3 & 4 & 2 & 1 \\ 2 & 6 & -7 & -2 & 5 \\ 4 & 12 & 6 & 4 & 6 \end{pmatrix} \Rightarrow \mathbf{R}=\begin{pmatrix} 1 & 3 & 9 & 0.4 & 1.8 \\ 0 & 0 & 1 & 0.4 & -0.2 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

From the first two rows of $\mathbf{Rx}=\mathbf{O}$ we have simultaneous equations:

$$x_1+3x_2+9x_3+0.4x_4+1.8x_5=0 \qquad (1)$$

$$x_3+0.4x_4-0.2x_5=0 \qquad (2)$$

From the bottom equation (2) we have $x_3=-0.4x_4+0.2x_5$. Let $x_4=s$ and $x_5=t$ where *s* and *t* are any real numbers. We have $x_3=-0.4s+0.2t$. Substituting $x_4=s$, $x_5=t$ and $x_3=-0.4s+0.2t$ into the top equation (1) gives

$$\begin{aligned} x_1+3x_2+9(-0.4s+0.2t)+0.4s+1.8t&=0 \\ x_1&=-3x_2+3.6s-1.8t-0.4s-1.8t \\ &=-3x_2+3.2s-3.6t \end{aligned}$$

Let $x_2=r$ where *r* is any real number. We have

$$x_1=-3r+3.2s-3.6t$$

Thus

$$\mathbf{x}=\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix}=\begin{pmatrix} -3r+3.2s-3.6t \\ r \\ -0.4s+0.2t \\ s \\ t \end{pmatrix}=r\begin{pmatrix} -3 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}+s\begin{pmatrix} 3.2 \\ 0 \\ -0.4 \\ 1 \\ 0 \end{pmatrix}+t\begin{pmatrix} -3.6 \\ 0 \\ 0.2 \\ 0 \\ 1 \end{pmatrix}$$

Thus a basis *B* for the kernel of *T* is given by

$$B=\left\{\begin{pmatrix} -3 & 1 & 0 & 0 & 0 \end{pmatrix}^T,\ \begin{pmatrix} 3.2 & 0 & -0.4 & 1 & 0 \end{pmatrix}^T,\ \begin{pmatrix} -3.6 & 0 & 0.2 & 0 & 1 \end{pmatrix}^T\right\}$$

(ii) A basis *B'* for the range of *T* can be found by determining the reduced row echelon form of $\mathbf{A}^T$ (matrix **A** transposed). The non-zero rows of this reduced row echelon form gives a basis for the range of *T*. We have

$$\mathbf{A}^t = \begin{pmatrix} 1 & 2 & 4 \\ 3 & 6 & 12 \\ 4 & -7 & 6 \\ 2 & -2 & 4 \\ 1 & 5 & 6 \end{pmatrix} \Longrightarrow \begin{pmatrix} 1 & 0 & 8/3 \\ 0 & 1 & 2/3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \mathbf{R}'$$

The first two rows of the reduced row echelon form matrix $\mathbf{R}'$ is a basis for the range of *T*:

$$B' = \left\{ \begin{pmatrix} 1 & 0 & 8/3 \end{pmatrix}^T, \ \begin{pmatrix} 0 & 1 & 2/3 \end{pmatrix}^T \right\}$$

4. We need to prove for a linear transform $T(\mathbf{x}) = \mathbf{A}\mathbf{x}$ that

$$range(T) = \text{ column space of matrix } \mathbf{A}$$

Proof.

The range of *T* is the actual arrival space in the codomain. Let **A** be a *m* by *n* matrix and $\mathbf{x} = \begin{pmatrix} x_1 & x_2 & \cdots & x_n \end{pmatrix}$ then

$$T(\mathbf{x}) = \mathbf{A}\mathbf{x} = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{pmatrix} \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$$

$$= x_1 \begin{pmatrix} a_{11} \\ \vdots \\ a_{m1} \end{pmatrix} + x_2 \begin{pmatrix} a_{12} \\ \vdots \\ a_{m2} \end{pmatrix} + x_3 \begin{pmatrix} a_{13} \\ \vdots \\ a_{m3} \end{pmatrix} + \cdots + x_n \begin{pmatrix} a_{1n} \\ \vdots \\ a_{mn} \end{pmatrix} \qquad (*)$$

Note that $\begin{pmatrix} a_{11} \\ \vdots \\ a_{m1} \end{pmatrix}, \begin{pmatrix} a_{12} \\ \vdots \\ a_{m2} \end{pmatrix}, \begin{pmatrix} a_{13} \\ \vdots \\ a_{m3} \end{pmatrix}, \cdots, \begin{pmatrix} a_{1n} \\ \vdots \\ a_{mn} \end{pmatrix}$ are the column vectors of matrix **A** and (*) is a linear combination of these which means it is the column space of matrix **A**. Hence we have our result.

■