Complete Solutions to Exercises 7.5

1. (a) The eigenvalues and eigenvectors of matrix $\mathbf{A}^T\mathbf{A}$ are

$$\lambda_1 = 1,\quad \mathbf{v}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \text{ and } \lambda_2 = 4,\quad \mathbf{v}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

The singular values of matrix **A** are $\sigma_1 = \sqrt{1} = 1$ and $\sigma_2 = \sqrt{4} = 2$. Using $\sigma_1\mathbf{u}_1 = \mathbf{A}\mathbf{v}_1$ and $\sigma_2\mathbf{u}_2 = \mathbf{A}\mathbf{v}_2$ we have

$$\mathbf{u}_1 = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \text{ and } 2\mathbf{u}_2 = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 2 \end{pmatrix}$$

Hence from this last result $2\mathbf{u}_2 = \begin{pmatrix} 0 \\ 2 \end{pmatrix}$ we have $\mathbf{u}_2 = \frac{1}{2}\begin{pmatrix} 0 \\ 2 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

We have $\mathbf{U} = \begin{pmatrix} \mathbf{u}_1 & \mathbf{u}_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $\mathbf{D} = \begin{pmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$ and $\mathbf{V} = \begin{pmatrix} \mathbf{v}_1 & \mathbf{v}_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

Note that $\mathbf{U} = \mathbf{V} = \mathbf{I}$. The triple factorization of the given matrix **A** is

$$\mathbf{A} = \mathbf{U}\mathbf{D}\mathbf{V}^T = \mathbf{I}\mathbf{D}\mathbf{I} = \mathbf{D}$$

(b) The eigenvalues and normalized eigenvectors of matrix $\mathbf{A}^T\mathbf{A}$ are

$$\lambda_1 = 81,\quad \mathbf{v}_1 = \frac{1}{\sqrt{5}}\begin{pmatrix} 1 \\ 2 \end{pmatrix} \text{ and } \lambda_2 = 1,\quad \mathbf{v}_2 = \frac{1}{\sqrt{5}}\begin{pmatrix} -2 \\ 1 \end{pmatrix}$$

The singular values of matrix **A** are $\sigma_1 = \sqrt{81} = 9$ and $\sigma_2 = 1$. Using $\sigma_1\mathbf{u}_1 = \mathbf{A}\mathbf{v}_1$ and $\sigma_2\mathbf{u}_2 = \mathbf{A}\mathbf{v}_2$ we have

$$9\mathbf{u}_1 = \frac{1}{\sqrt{5}}\begin{pmatrix} 1 & 4 \\ 4 & 7 \end{pmatrix}\begin{pmatrix} 1 \\ 2 \end{pmatrix} = \frac{1}{\sqrt{5}}\begin{pmatrix} 9 \\ 18 \end{pmatrix} \text{ and } \mathbf{u}_2 = \frac{1}{\sqrt{5}}\begin{pmatrix} 1 & 4 \\ 4 & 7 \end{pmatrix}\begin{pmatrix} -2 \\ 1 \end{pmatrix} = \frac{1}{\sqrt{5}}\begin{pmatrix} 2 \\ -1 \end{pmatrix}$$

We have $\mathbf{u}_1 = \frac{1}{9\sqrt{5}}\begin{pmatrix} 9 \\ 18 \end{pmatrix} = \frac{1}{\sqrt{5}}\begin{pmatrix} 1 \\ 2 \end{pmatrix}$. Substituting this and $\mathbf{u}_2$ gives

$$\mathbf{U} = \begin{pmatrix} \mathbf{u}_1 & \mathbf{u}_2 \end{pmatrix} = \frac{1}{\sqrt{5}}\begin{pmatrix} 1 & 2 \\ 2 & -1 \end{pmatrix},\ \mathbf{D} = \begin{pmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \end{pmatrix} = \begin{pmatrix} 9 & 0 \\ 0 & 1 \end{pmatrix} \text{ and}$$

$$\mathbf{V}^T = \begin{pmatrix} \mathbf{v}_1 & \mathbf{v}_2 \end{pmatrix}^T = \frac{1}{\sqrt{5}}\begin{pmatrix} 1 & -2 \\ 2 & 1 \end{pmatrix}^T = \frac{1}{\sqrt{5}}\begin{pmatrix} 1 & 2 \\ -2 & 1 \end{pmatrix}.$$

You may like to check the factorization $\mathbf{A} = \mathbf{U}\mathbf{D}\mathbf{V}^T$.

(c) The eigenvalues and normalized eigenvectors of matrix $\mathbf{A}^T\mathbf{A}$ are

$$\lambda_1 = 6,\quad \mathbf{v}_1 = \frac{1}{\sqrt{5}}\begin{pmatrix} 1 \\ 2 \end{pmatrix} \text{ and } \lambda_2 = 1,\quad \mathbf{v}_2 = \frac{1}{\sqrt{5}}\begin{pmatrix} -2 \\ 1 \end{pmatrix}$$

The singular values of matrix **A** are $\sigma_1 = \sqrt{6}$ and $\sigma_2 = 1$.

Using $\mathbf{u}_1 = \frac{1}{\sigma_1}\mathbf{A}\mathbf{v}_1$ and $\mathbf{u}_2 = \frac{1}{\sigma_2}\mathbf{A}\mathbf{v}_2$ we have

$$\mathbf{u}_1 = \frac{1}{\sqrt{6}}\mathbf{A}\mathbf{v}_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 2 \end{pmatrix} \frac{1}{\sqrt{5}}\begin{pmatrix} 1 \\ 2 \end{pmatrix} = \frac{1}{\sqrt{30}}\begin{pmatrix} 1 \\ 2 \\ 5 \end{pmatrix} \text{ and } \mathbf{u}_2 = \mathbf{A}\mathbf{v}_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 2 \end{pmatrix} \frac{1}{\sqrt{5}}\begin{pmatrix} -2 \\ 1 \end{pmatrix} = \frac{1}{\sqrt{5}}\begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix}$$

Since **A** is a 3 by 2 matrix so **U** is a 3 by 3 matrix which means we need to find the vector $\mathbf{u}_3$ which is orthogonal to both $\mathbf{u}_1$ and $\mathbf{u}_2$:

$$\begin{pmatrix} 1 & 2 & 5 \\ -2 & 1 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \Rightarrow \quad x = 1,\ y = 2,\ z = -1 \Rightarrow \mathbf{u}_3 = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$$

Normalizing the vector $\mathbf{u}_3$ gives

$$\mathbf{u}_3 = \frac{1}{\sqrt{6}}\begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$$

We have $\mathbf{u}_1 = \frac{1}{\sqrt{30}}\begin{pmatrix} 1 \\ 2 \\ 5 \end{pmatrix}$, $\mathbf{u}_2 = \frac{1}{\sqrt{5}}\begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix}$ and $\mathbf{u}_3 = \frac{1}{\sqrt{6}}\begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$.

Substituting these and the above into **U**, **D** and **V** gives:

$$\mathbf{U} = \begin{pmatrix} \mathbf{u}_1 & \mathbf{u}_2 & \mathbf{u}_3 \end{pmatrix} = \begin{pmatrix} 1/\sqrt{30} & -2/\sqrt{5} & 1/\sqrt{6} \\ 2/\sqrt{30} & 1/\sqrt{5} & 2/\sqrt{6} \\ 5/\sqrt{30} & 0 & -1/\sqrt{6} \end{pmatrix}, \quad \mathbf{D} = \begin{pmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} \sqrt{6} & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} \text{ and}$$

$$\mathbf{V}^T = \begin{pmatrix} \mathbf{v}_1 & \mathbf{v}_2 \end{pmatrix}^T = \frac{1}{\sqrt{5}}\begin{pmatrix} 1 & -2 \\ 2 & 1 \end{pmatrix}^T = \frac{1}{\sqrt{5}}\begin{pmatrix} 1 & 2 \\ -2 & 1 \end{pmatrix}$$

You may like to check the factorization $\mathbf{A} = \mathbf{U}\mathbf{D}\mathbf{V}^T$.

(d) The product $\mathbf{A}^T\mathbf{A} = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 1 & 2 & 5 \end{pmatrix}$. The eigenvalues and normalized eigenvectors of matrix $\mathbf{A}^T\mathbf{A}$ are

$$\lambda_1 = 6,\quad \mathbf{v}_1 = \frac{1}{\sqrt{30}}\begin{pmatrix} 1 \\ 2 \\ 5 \end{pmatrix},\quad \lambda_2 = 1,\quad \mathbf{v}_2 = \frac{1}{\sqrt{5}}\begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix} \text{ and } \lambda_3 = 0,\quad \mathbf{v}_3 = \frac{1}{\sqrt{6}}\begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$$

The singular values of matrix **A** are $\sigma_1 = \sqrt{6}$ and $\sigma_2 = 1$.

Using $\mathbf{u}_1 = \frac{1}{\sigma_1}\mathbf{A}\mathbf{v}_1$ and $\mathbf{u}_2 = \frac{1}{\sigma_2}\mathbf{A}\mathbf{v}_2$ we have

$$\mathbf{u}_1 = \frac{1}{\sqrt{6}}\mathbf{A}\mathbf{v}_1 = \frac{1}{\sqrt{6}}\begin{pmatrix}1 & 0 & 1\\ 0 & 1 & 2\end{pmatrix}\frac{1}{\sqrt{30}}\begin{pmatrix}1\\ 2\\ 5\end{pmatrix} = \frac{1}{\sqrt{180}}\begin{pmatrix}6\\ 12\end{pmatrix} \text{ and}$$

$$\mathbf{u}_2 = \frac{1}{1}\mathbf{A}\mathbf{v}_2 = \begin{pmatrix}1 & 0 & 1\\ 0 & 1 & 2\end{pmatrix}\frac{1}{\sqrt{5}}\begin{pmatrix}-2\\ 1\\ 0\end{pmatrix} = \frac{1}{\sqrt{5}}\begin{pmatrix}-2\\ 1\end{pmatrix}$$

Since **A** is a 2 by 3 matrix so **U** is a 2 by 2 matrix, **D** is a 2 by 3 matrix and **V** is a 3 by 3 matrix:
Substituting these and the above into **U**, **D** and **V** gives:

$$\mathbf{U} = \begin{pmatrix}\mathbf{u}_1 & \mathbf{u}_2\end{pmatrix} = \begin{pmatrix}6/\sqrt{180} & -2/\sqrt{5}\\ 12/\sqrt{180} & 1/\sqrt{5}\end{pmatrix},\quad \mathbf{D} = \begin{pmatrix}\sigma_1 & 0 & 0\\ 0 & \sigma_2 & 0\end{pmatrix} = \begin{pmatrix}\sqrt{6} & 0 & 0\\ 0 & 1 & 0\end{pmatrix} \text{ and}$$

$$\mathbf{V} = \begin{pmatrix}\mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3\end{pmatrix} = \begin{pmatrix}1/\sqrt{30} & -2/\sqrt{5} & 1/\sqrt{6}\\ 2/\sqrt{30} & 1/\sqrt{5} & 2/\sqrt{6}\\ 5/\sqrt{30} & 0 & -1/\sqrt{6}\end{pmatrix}$$

You may like to check the factorization $\mathbf{A} = \mathbf{U}\mathbf{D}\mathbf{V}^T$ by first transposing matrix **V**.

(e) The product $\mathbf{A}^T\mathbf{A} = \begin{pmatrix}1 & 3\\ 1 & 3\end{pmatrix}\begin{pmatrix}1 & 1\\ 3 & 3\end{pmatrix} = \begin{pmatrix}10 & 10\\ 10 & 10\end{pmatrix}$. The eigenvalues and normalized eigenvectors of matrix $\mathbf{A}^T\mathbf{A}$ are

$$\lambda_1 = 20,\quad \mathbf{v}_1 = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\ 1\end{pmatrix} \text{ and } \lambda_2 = 0,\quad \mathbf{v}_2 = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\ -1\end{pmatrix}$$

The positive singular value of matrix **A** is $\sigma_1 = \sqrt{20}$.

Using $\mathbf{u}_1 = \dfrac{1}{\sigma_1}\mathbf{A}\mathbf{v}_1$ we have

$$\mathbf{u}_1 = \frac{1}{\sqrt{20}}\mathbf{A}\mathbf{v}_1 = \frac{1}{\sqrt{20}}\begin{pmatrix}1 & 1\\ 3 & 3\end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix}1\\ 1\end{pmatrix}$$

$$= \frac{1}{\sqrt{40}}\begin{pmatrix}2\\ 6\end{pmatrix} = \frac{1}{\sqrt{10}}\begin{pmatrix}1\\ 3\end{pmatrix} \qquad \left[\text{Because } \sqrt{40} = \sqrt{4\times 10} = 2\sqrt{10}\right]$$

Since **A** is a 2 by 2 matrix so **U** is a 2 by 2 matrix. *What is* $\mathbf{u}_2$ *equal to?*
$\mathbf{u}_2$ needs to be orthogonal to $\mathbf{u}_1$ which means $\mathbf{u}_2 \cdot \mathbf{u}_1 = 0$ therefore by inspection and normalizing we have

$$\mathbf{u}_2 = \frac{1}{\sqrt{10}}\begin{pmatrix}3\\ -1\end{pmatrix}$$

Substituting these and the above into **U**, **D** and **V** gives:

$$\mathbf{U} = \begin{pmatrix}\mathbf{u}_1 & \mathbf{u}_2\end{pmatrix} = \frac{1}{\sqrt{10}}\begin{pmatrix}1 & 3\\ 3 & -1\end{pmatrix},\quad \mathbf{D} = \begin{pmatrix}\sigma_1 & 0\\ 0 & 0\end{pmatrix} = \begin{pmatrix}\sqrt{20} & 0\\ 0 & 0\end{pmatrix} \text{ and}$$

$$\mathbf{V}^T = \begin{pmatrix}\mathbf{v}_1 & \mathbf{v}_2\end{pmatrix}^T = \frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix}^T = \frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix}$$

You may like to check the factorization $\mathbf{A} = \mathbf{UDV}^T$.

(f) The product $\mathbf{A}^T\mathbf{A} = \begin{pmatrix} 1 & 3 \\ 1 & 3 \\ 1 & 3 \end{pmatrix}\begin{pmatrix} 1 & 1 & 1 \\ 3 & 3 & 3 \end{pmatrix} = \begin{pmatrix} 10 & 10 & 10 \\ 10 & 10 & 10 \\ 10 & 10 & 10 \end{pmatrix}$. The eigenvalues and normalized eigenvectors of matrix $\mathbf{A}^T\mathbf{A}$ are

$$\lambda_1 = 30, \quad \mathbf{v}_1 = \frac{1}{\sqrt{3}}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \; \lambda_2 = 0, \quad \mathbf{v}_2 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} \text{ and } \lambda_3 = 0, \quad \mathbf{v}_3 = \frac{1}{6}\begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix}$$

The positive singular value of matrix **A** is $\sigma_1 = \sqrt{30}$.

Using $\mathbf{u}_1 = \dfrac{1}{\sigma_1}\mathbf{A}\mathbf{v}_1$ we have

$$\mathbf{u}_1 = \frac{1}{\sqrt{20}}\mathbf{A}\mathbf{v}_1 = \frac{1}{\sqrt{30}}\begin{pmatrix} 1 & 1 & 1 \\ 3 & 3 & 3 \end{pmatrix}\frac{1}{\sqrt{3}}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

$$= \frac{1}{\sqrt{90}}\begin{pmatrix} 3 \\ 9 \end{pmatrix} = \frac{1}{\sqrt{10}}\begin{pmatrix} 1 \\ 3 \end{pmatrix} \qquad \left[\text{Because } \sqrt{90} = \sqrt{9\times 10} = 3\sqrt{10}\right]$$

Since **A** is a 2 by 3 matrix so **U** is a 2 by 2 matrix, **D** is a 2 by 3 matrix and **V** is a 3 by 3 matrix. *What is* $\mathbf{u}_2$ *equal to?*

$\mathbf{u}_2$ needs to be orthogonal to $\mathbf{u}_1$. We need

$$\mathbf{u}_2 \cdot \mathbf{u}_1 = 0$$

As in part (e) we have $\mathbf{u}_2 = \dfrac{1}{\sqrt{10}}\begin{pmatrix} 3 \\ -1 \end{pmatrix}$. We have

$$\mathbf{U} = \begin{pmatrix} \mathbf{u}_1 & \mathbf{u}_2 \end{pmatrix} = \frac{1}{\sqrt{10}}\begin{pmatrix} 1 & 3 \\ 3 & -1 \end{pmatrix}, \quad \mathbf{D} = \begin{pmatrix} \sqrt{30} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{and}$$

$$\mathbf{V}^T = \begin{pmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3 \end{pmatrix}^T = \begin{pmatrix} 1/\sqrt{3} & 1/\sqrt{2} & 1/\sqrt{6} \\ 1/\sqrt{3} & -1/\sqrt{2} & 1/\sqrt{6} \\ 1/\sqrt{3} & 0 & -2/\sqrt{6} \end{pmatrix}^T = \begin{pmatrix} 1/\sqrt{3} & 1/\sqrt{3} & 1/\sqrt{3} \\ 1/\sqrt{2} & -1/\sqrt{2} & 0 \\ 1/\sqrt{6} & 1/\sqrt{6} & -2/\sqrt{6} \end{pmatrix}$$

You may like to check the factorization $\mathbf{A} = \mathbf{UDV}^T$.

2. We need to prove that:

Let **A** be any matrix. Then the eigenvalues of $\mathbf{A}^T\mathbf{A}$ are positive or zero.

Proof.

Let $\lambda_1, \lambda_2, \cdots, \lambda_n$ be the eigenvalues of $\mathbf{A}^T\mathbf{A}$ with eigenvectors $\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_n$ respectively. For an arbitrary eigenvector $\mathbf{v}_j$ we have

$$\mathbf{A}^T\mathbf{A}\mathbf{v}_j = \lambda_j \mathbf{v}_j \qquad (*)$$

By Proposition (7-23):

(7-23). Let **A** be any matrix. Then $\mathbf{A}^T\mathbf{A}$ is a symmetric matrix.

This means that $\mathbf{A}^T\mathbf{A}$ can be orthogonally diagonalized so the eigenvectors are orthonormal.

Consider the norm square $\|\mathbf{Av}_j\|^2$:

$$\begin{aligned}\|\mathbf{Av}_j\|^2 &= \mathbf{Av}_j \cdot \mathbf{Av}_j = \left(\mathbf{Av}_j\right)^T \mathbf{Av}_j \\ &= \mathbf{v}_j^T \mathbf{A}^T \mathbf{Av}_j \\ &\underset{\text{by }(*)}{=} \mathbf{v}_j^T \lambda_j \mathbf{v}_j = \lambda_j\left(\mathbf{v}_j \cdot \mathbf{v}_j\right) = \lambda_j \qquad \left[\text{Because } \left(\mathbf{v}_j \cdot \mathbf{v}_j\right) = \|\mathbf{v}_j\| = 1\right]\end{aligned}$$

We have $\lambda_j = \|\mathbf{Av}_j\|^2 \geq 0$. Hence all the eigenvalues of $\mathbf{A}^T\mathbf{A}$ are positive or zero.

■

3. Required to prove:

Let matrix **A** have *k* positive singular values. Then the rank of matrix **A** is *k*.

Proof.

Let **A** be a *m* by *n* matrix. By SVD:

(7-22). We can decompose any given matrix **A** of size *m* by *n* with singular values $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_k > 0$ where $k \leq m$, into $\mathbf{UDV}^T$, that is

$$\mathbf{A} = \mathbf{UDV}^T$$

where **U** is a *m* by *m* *orthogonal* matrix, **D** is a *m* by *n* matrix and **V** is an *n* by *n* *orthogonal* matrix.

We have $\mathbf{A} = \mathbf{UDV}^T$ where **U** and **V** are orthogonal matrices. Since **U** is orthogonal so it is invertible because $\mathbf{U}^{-1} = \mathbf{U}^T$ and similarly **V** is orthogonal so $\mathbf{V}^{-1} = \mathbf{V}^T$. Hence $\left(\mathbf{V}^T\right)^{-1} = \left(\mathbf{V}^{-1}\right)^{-1} = \mathbf{V}$ which means that $\mathbf{V}^T$ is invertible. By hint we have:

$$\begin{aligned}rank(\mathbf{A}) &= rank\left(\mathbf{UDV}^T\right) \\ &= rank\left(\mathbf{DV}^T\right) = rank(\mathbf{D}) = k\end{aligned}$$

This completes our proof.

■

4. (a) We need to prove that $\{\mathbf{u}_1,\ \mathbf{u}_2,\ \cdots,\ \mathbf{u}_k\}$ form an orthonormal basis for the column space of matrix **A**.

Proof.

U is a *m* by *m* orthogonal matrix so the column vectors of matrix **U** are orthonormal:

$$\mathbf{U} = \begin{pmatrix}\mathbf{u}_1 & \mathbf{u}_2 & \cdots & \mathbf{u}_n\end{pmatrix}$$

By result of question 3 we have the rank of matrix **A** is *k*.

By Proposition (3-18) of chapter 3:

(3-18). $rank(\mathbf{A}) = \text{Row rank of } \mathbf{A} = \text{Column rank of } \mathbf{A}$

Hence the dimension of the column space is *k* which means we need *k* basis vectors for the column space of matrix **A**.

We are given that $\sigma_1,\ \sigma_2,\ \cdots,\ \sigma_k$ are positive and from the main theorem of the section (7-22) for $j = 1,\ 2,\ 3,\ \cdots,\ k$ we have

$$\mathbf{u}_j = \frac{1}{\sigma_j}\mathbf{Av}_j$$

By Proposition (3-24):

(3-24). The linear system $\mathbf{Ax} = \mathbf{b}$ has a solution $\Leftrightarrow$ $\mathbf{b}$ can be generated by the column space of matrix **A**.

Therefore $\mathbf{u}_j$ is in the column space of matrix **A**. We have *k* orthonormal vectors $\{\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_k\}$ which are in the column space of matrix **A**. Since orthogonal vectors are linearly independent so they $\{\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_k\}$ form an orthonormal basis for the column space of matrix **A**.

■

(b) We need to prove that $\{\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_k\}$ form an orthonormal basis for the row space of matrix **A**.

Proof.

Since **V** is an *n* by *n* orthogonal matrix whose columns are the eigenvectors of $\mathbf{A}^T\mathbf{A}$:

$$\mathbf{V} = \begin{pmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \cdots & \mathbf{v}_n \end{pmatrix}$$

These eigenvectors are an orthonormal set of vectors because **V** is orthogonal.

By result of question 3 we have the rank of matrix **A** is *k*. By Definition (3-12) of chapter 3:

(3-12). The **rank** of a matrix **A** is the row rank of **A**.

Hence the dimension of the row space is *k* which means we need *k* basis vectors for the row space. A subset of *k* vectors in the above, $S = \{\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_k\}$, is also orthonormal. As these vectors are orthogonal so they are linearly independent which means they form a basis. Therefore *S* forms an orthonormal basis for the row space of matrix **A**.

■

(c) Required to prove:

The set of vectors $\{\mathbf{v}_{k+1}, \mathbf{v}_{k+2}, \cdots, \mathbf{v}_n\}$ form an orthonormal basis for the null space of matrix **A**.

Proof.

By Theorem (3-22):

(3-22). If **A** is a matrix with *n* columns (number of unknowns) then

$$nullity(\mathbf{A}) + rank(\mathbf{A}) = n$$

We have the dimension of null space is $n-k$ and there are $n-k$ vectors in the set $\{\mathbf{v}_{k+1}, \mathbf{v}_{k+2}, \cdots, \mathbf{v}_n\}$. Remember this set $\{\mathbf{v}_{k+1}, \mathbf{v}_{k+2}, \cdots, \mathbf{v}_n\}$ represent the orthonormal eigenvectors of $\mathbf{A}^T\mathbf{A}$ which correspond to the zero eigenvalues $\lambda_{k+1} = \lambda_{k+2} = \cdots = \lambda_k = 0$.

This means that for $j = k+1, \ k+2, \ \cdots, \ k+n$ we have

$$\left(\mathbf{A}^T\mathbf{A}\right)\mathbf{v}_j = \lambda_j \mathbf{v}_j = 0\mathbf{v}_j = \mathbf{O}$$

These vectors $\{\mathbf{v}_{k+1}, \mathbf{v}_{k+2}, \cdots, \mathbf{v}_n\}$ form an orthonormal basis for the null space of $\mathbf{A}^T\mathbf{A}$. The null space of $\mathbf{A}^T\mathbf{A}$ and **A** are identical. Hence $\{\mathbf{v}_{k+1}, \mathbf{v}_{k+2}, \cdots, \mathbf{v}_n\}$ form an orthonormal basis for the null space of matrix **A**.

■

5. *Proof.*

The eigenvalues of $\mathbf{A}^T\mathbf{A}$ are unique. *Why?*

By Question 9 of Exercises 7.2:

Let **A** be a square matrix and λ be an eigenvalue with the corresponding eigenvector **u**. The eigenvalue λ is unique for the eigenvector **u**.

The singular values are given by the positive roots:

$$\sigma_1 = \sqrt{\lambda_1},\ \sigma_2 = \sqrt{\lambda_2},\ \cdots,\ \sigma_n = \sqrt{\lambda_n}$$

Therefore the singular values are unique.

■

6. We need to prove that the column vectors of matrix **U** in $\mathbf{A} = \mathbf{UDV}^T$ are the eigenvectors of $\mathbf{AA}^T$.

Proof.

Using the singular value decomposition $\mathbf{A} = \mathbf{UDV}^T$ we have

$$\begin{aligned}
\mathbf{AA}^T &= \left(\mathbf{UDV}^T\right)\left(\mathbf{UDV}^T\right)^T \\
&= \left(\mathbf{UDV}^T\right)\left(\mathbf{V}^T\right)^T \mathbf{D}^T\mathbf{U}^T \qquad \left[\text{By using } \left(\mathbf{XYZ}\right)^T = \mathbf{Z}^T\mathbf{Y}^T\mathbf{X}^T\right] \\
&= \mathbf{UD}\underbrace{\mathbf{V}^T\mathbf{V}}_{=\mathbf{I}}\mathbf{D}^T\mathbf{U}^T \qquad \left[\text{Because } \left(\mathbf{X}^T\right)^T = \mathbf{X}\right] \\
&= \mathbf{U}\left(\mathbf{DD}^T\right)\mathbf{U}^T = \mathbf{U}\left(\mathbf{D}'\right)\mathbf{U}^T \qquad \text{where } \mathbf{D}' = \mathbf{DD}^T \text{ is a diagonal matrix}
\end{aligned}$$

Remember **U** is an orthogonal matrix so it inverse is given by $\mathbf{U}^T$. Left-multiplying the above result $\mathbf{AA}^T = \mathbf{U}\left(\mathbf{D}'\right)\mathbf{U}^T$ by $\mathbf{U}^T$ and right-multiplying by **U** gives

$$\mathbf{U}^T\left(\mathbf{AA}^T\right)\mathbf{U} = \underbrace{\mathbf{U}^T\mathbf{U}}_{=\mathbf{I}}\left(\mathbf{D}'\right)\underbrace{\mathbf{U}^T\mathbf{U}}_{=\mathbf{I}} = \mathbf{D}'$$

Since $\mathbf{U}^T\left(\mathbf{AA}^T\right)\mathbf{U} = \mathbf{D}'$, the matrix **U** diagonalizes $\mathbf{AA}^T$ and the columns of **U** are the eigenvectors of $\mathbf{AA}^T$. This completes our proof.

■

7. Required to prove that:

The singular values of **A** and $\mathbf{A}^T$ are identical.

Proof.

The singular values of a matrix **A** are given by the square roots of the eigenvalues $\lambda_1,\ \lambda_2,\ \cdots,\ \lambda_n$ of $\mathbf{AA}^T$

$$\sigma_1 = \sqrt{\lambda_1},\ \sigma_2 = \sqrt{\lambda_2},\ \cdots,\ \sigma_n = \sqrt{\lambda_n}$$

The singular values of a matrix $\mathbf{A}^T$ are given by the square roots of the eigenvalues $t_1,\ t_2,\ \cdots,\ t_n$ of $\left(\mathbf{AA}^T\right)^T$.

By Question 16 of Exercises 7.2:

The eigenvalues of the transposed matrix, $\mathbf{A}^T$, are exactly the eigenvalues of the matrix **A**.

Hence $\left(\mathbf{AA}^T\right)^T$ will have the same eigenvalues as $\mathbf{AA}^T$ which means:

$$t_1 = \lambda_1,\ t_2 = \lambda_2,\ \cdots,\ t_n = \lambda_n$$

Therefore the singular values of $\mathbf{A}^T$ are the same. Hence the singular values of both **A** and the transposed matrix $\mathbf{A}^T$ are identical.

■

8. (a) We have to prove that:

The set of vectors $\{\mathbf{u}_1,\ \mathbf{u}_2,\ \cdots,\mathbf{u}_k\}$ form an orthonormal basis for the range of T.

Proof.

Let $\{\mathbf{u}_1,\ \mathbf{u}_2,\ \cdots,\mathbf{u}_k\}$ be the first k column vectors of matrix $\mathbf{U}$. By Proposition (7-26) part(a):

(7-26) (a) The set of vectors $\{\mathbf{u}_1,\ \mathbf{u}_2,\ \cdots,\mathbf{u}_k\}$ is an orthonormal basis for the column space of matrix $\mathbf{A}$.

Also we are given that $T(\mathbf{x}) = \mathbf{A}\mathbf{x}$ so by Proposition (5-6) of chapter 5:

(5-6). Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation given by $T(\mathbf{x}) = \mathbf{A}\mathbf{x}$. Then $range(T)$ is the column space of $\mathbf{A}$.

So the range of the transformation T is the column space of matrix $\mathbf{A}$ therefore $\{\mathbf{u}_1,\ \mathbf{u}_2,\ \cdots,\mathbf{u}_k\}$ is an orthonormal basis for the range.

■

(b) Required to prove that:

The set of vectors $\{\mathbf{v}_{k+1},\ \mathbf{v}_{k+2},\ \cdots,\ \mathbf{v}_n\}$ form an orthonormal basis for the kernel of T.

Proof.

Remember the kernel of a transformation T are the vectors in the start vector space which are mapped to the zero vector. In our case we have $T(\mathbf{x}) = \mathbf{A}\mathbf{x}$ so it is the vectors $\mathbf{x}$ which satisfy $T(\mathbf{x}) = \mathbf{A}\mathbf{x} = \mathbf{O}$. Of course this is the null space of matrix $\mathbf{A}$. By Proposition (7-26):

(7-26) (c) The set of vectors $\{\mathbf{v}_{k+1},\ \mathbf{v}_{k+2},\ \cdots,\ \mathbf{v}_n\}$ form an orthonormal basis for the null space of matrix $\mathbf{A}$.

Hence $\{\mathbf{v}_{k+1},\ \mathbf{v}_{k+2},\ \cdots,\ \mathbf{v}_n\}$ is an orthonormal basis for kernel of T.

■