Complete Solutions to **Miscellaneous Exercises 4**

1. The given statement is false:
'Every linearly independent set in $\mathbb{R}^n$ is an orthogonal set'.

Consider two linearly independent vectors such as $\mathbf{u}=\begin{pmatrix}1\\2\\0\\\vdots\\0\end{pmatrix}$ and $\mathbf{v}=\begin{pmatrix}2\\1\\0\\\vdots\\0\end{pmatrix}$ in $\mathbb{R}^n$.

These vectors are linearly independent but the inner (dot) product is

$$\mathbf{u}\cdot\mathbf{v}=2+2=4$$

Hence **u** and **v** are not orthogonal so the given statement is false.

2. *How do find an orthogonal basis for U from*

$$\mathbf{X}_1=[1,\ \ 1,\ \ 1,\ \ 1],\ \mathbf{X}_2=[1,\ \ 0,\ \ 0,\ \ 1],\ \mathbf{X}_3=[0,\ \ 2,\ \ 1,\ \ -1]?$$

Use the Gram Schmidtt Process (4-10) which is given by

Let $\mathbf{w}_1=\mathbf{X}_1$, $\mathbf{w}_2=\mathbf{X}_2-\dfrac{\langle\mathbf{X}_2,\ \mathbf{w}_1\rangle}{\|\mathbf{w}_1\|^2}\mathbf{w}_1$ and $\mathbf{w}_3=\mathbf{X}_3-\dfrac{\langle\mathbf{X}_3,\ \mathbf{w}_1\rangle}{\|\mathbf{w}_1\|^2}\mathbf{w}_1-\dfrac{\langle\mathbf{X}_3,\ \mathbf{w}_2\rangle}{\|\mathbf{w}_2\|^2}\mathbf{w}_2$:

Applying these we have $\mathbf{w}_1=\mathbf{X}_1=\begin{bmatrix}1\\1\\1\\1\end{bmatrix}$.

Evaluating each component of $\mathbf{w}_2=\mathbf{X}_2-\dfrac{\langle\mathbf{X}_2,\ \mathbf{w}_1\rangle}{\|\mathbf{w}_1\|^2}\mathbf{w}_1$ gives:

$$\langle\mathbf{X}_2,\ \mathbf{w}_1\rangle=\begin{bmatrix}1\\0\\0\\1\end{bmatrix}\cdot\begin{bmatrix}1\\1\\1\\1\end{bmatrix}=1+0+0+1=2 \text{ and } \|\mathbf{w}_1\|^2=\begin{bmatrix}1\\1\\1\\1\end{bmatrix}\cdot\begin{bmatrix}1\\1\\1\\1\end{bmatrix}=1^2+1^2+1^2+1^2=4$$

Substituting these $\mathbf{X}_2=\begin{bmatrix}1\\0\\0\\1\end{bmatrix}$, $\mathbf{w}_1=\begin{bmatrix}1\\1\\1\\1\end{bmatrix}$, $\langle\mathbf{X}_2,\ \mathbf{w}_1\rangle=2$ and $\|\mathbf{w}_1\|^2=4$ into

$\mathbf{w}_2=\mathbf{X}_2-\dfrac{\langle\mathbf{X}_2,\ \mathbf{w}_1\rangle}{\|\mathbf{w}_1\|^2}\mathbf{w}_1$ gives

$$\mathbf{w}_2=\begin{bmatrix}1\\0\\0\\1\end{bmatrix}-\frac{2}{4}\begin{bmatrix}1\\1\\1\\1\end{bmatrix}=\begin{bmatrix}1-1/2\\0-1/2\\0-1/2\\1-1/2\end{bmatrix}=\begin{bmatrix}1/2\\-1/2\\-1/2\\1/2\end{bmatrix}=\frac{1}{2}\begin{bmatrix}1\\-1\\-1\\1\end{bmatrix}$$

Remember we can ignore the fraction ½ and find $\mathbf{w}_3$ that is orthogonal to

$$\mathbf{w}_1=\begin{bmatrix}1\\1\\1\\1\end{bmatrix} \text{ and } \mathbf{w}'_2=\begin{bmatrix}1\\-1\\-1\\1\end{bmatrix} \qquad [\mathbf{w}'_2 \text{ is } 2\mathbf{w_2} \text{ that is we ignore } \tfrac{1}{2}]$$

Determining each component of $\mathbf{w}_3=\mathbf{X}_3-\dfrac{\langle\mathbf{X}_3,\ \mathbf{w}_1\rangle}{\|\mathbf{w}_1\|^2}\mathbf{w}_1-\dfrac{\langle\mathbf{X}_3,\ \mathbf{w}'_2\rangle}{\|\mathbf{w}'_2\|^2}\mathbf{w}'_2$ gives

$$\langle\mathbf{X}_3,\ \mathbf{w}_1\rangle=\mathbf{X}_3\cdot\mathbf{w}_1=\begin{bmatrix}0\\2\\1\\-1\end{bmatrix}\cdot\begin{bmatrix}1\\1\\1\\1\end{bmatrix}=0+2+1-1=2$$

$$\langle\mathbf{X}_3,\ \mathbf{w}'_2\rangle=\mathbf{X}_3\cdot\mathbf{w}'_2=\begin{bmatrix}0\\2\\1\\-1\end{bmatrix}\cdot\begin{bmatrix}1\\-1\\-1\\1\end{bmatrix}=0+\left[2\times(-1)\right]+\left[1\times(-1)\right]+\left[-1\times1\right]=-4$$

$$\|\mathbf{w}'_2\|^2=\begin{bmatrix}1\\-1\\-1\\1\end{bmatrix}\cdot\begin{bmatrix}1\\-1\\-1\\1\end{bmatrix}=1^2+(-1)^2+(-1)^2+1^2=4$$

Substituting these, $\mathbf{X}_3=\begin{bmatrix}0\\2\\1\\-1\end{bmatrix}$, $\mathbf{w}_1=\begin{bmatrix}1\\1\\1\\1\end{bmatrix}$, $\mathbf{w}'_2=\begin{bmatrix}1\\-1\\-1\\1\end{bmatrix}$, $\langle\mathbf{X}_3,\ \mathbf{w}_1\rangle=2$, $\langle\mathbf{X}_3,\ \mathbf{w}'_2\rangle=-4$,

$\|\mathbf{w}_1\|^2=4$ and $\|\mathbf{w}'_2\|^2=4$, into $\mathbf{w}_3=\mathbf{X}_3-\dfrac{\langle\mathbf{X}_3,\ \mathbf{w}_1\rangle}{\|\mathbf{w}_1\|^2}\mathbf{w}_1-\dfrac{\langle\mathbf{X}_3,\ \mathbf{w}'_2\rangle}{\|\mathbf{w}'_2\|^2}\mathbf{w}'_2$ yields

$$\mathbf{w}_3=\begin{bmatrix}0\\2\\1\\-1\end{bmatrix}-\frac{2}{4}\begin{bmatrix}1\\1\\1\\1\end{bmatrix}-\frac{-4}{4}\begin{bmatrix}1\\-1\\-1\\1\end{bmatrix}$$

$$=\begin{bmatrix}0\\2\\1\\-1\end{bmatrix}-\frac{1}{2}\begin{bmatrix}1\\1\\1\\1\end{bmatrix}+\begin{bmatrix}1\\-1\\-1\\1\end{bmatrix}=\begin{bmatrix}0-1/2+1\\2-1/2-1\\1-1/2-1\\-1-1/2+1\end{bmatrix}=\begin{bmatrix}1/2\\1/2\\-1/2\\-1/2\end{bmatrix}=\frac{1}{2}\begin{bmatrix}1\\1\\-1\\-1\end{bmatrix}$$

Again we can remove the fraction and let $\mathbf{w}'_3 = \begin{bmatrix} 1 \\ 1 \\ -1 \\ -1 \end{bmatrix}$. Thus our orthogonal vectors are

$\mathbf{w}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$, $\mathbf{w}'_2 = \begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \end{bmatrix}$ and $\mathbf{w}'_3 = \begin{bmatrix} 1 \\ 1 \\ -1 \\ -1 \end{bmatrix}$ which means that $\{\mathbf{w}_1,\ \mathbf{w}'_2,\ \mathbf{w}'_3\}$ forms an orthogonal set of vectors which is a orthogonal basis for *U*.

3. (a) We are given that $S = Span\left\{\begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 2 \\ 2 \end{bmatrix}\right\}$ and if these vectors are linearly independent then they form a basis for *S*.
Let $k_1,\ k_2$ and k_3 be scalars such that

$$k_1\begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} + k_2\begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix} + k_3\begin{bmatrix} 1 \\ 1 \\ 2 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

For the given vectors to be linearly independent we need to prove that $k_1 = k_2 = k_3 = 0$.
Writing the above as equations we have

$$\begin{array}{ccccccc} k_1 & & & + & k_3 & = & 0 \quad (1) \\ & & k_2 & + & k_3 & = & 0 \quad (2) \\ k_1 & + & k_2 & + & 2k_3 & = & 0 \quad (3) \\ & & k_2 & + & 2k_3 & = & 0 \quad (4) \end{array}$$

From equations (2) and (3) we have $k_2 = k_3 = 0$. Substituting this into equation (3) gives $k_1 = 0$. Hence the given vectors are linearly independent because $k_1 = k_2 = k_3 = 0$ therefore they form a basis for *S*. [Remember they already span the space *S*].
(b) We need to apply the Gram Schmidtt Process (4-10) which is:

$$\mathbf{w}_1 = \mathbf{v}_1,\ \mathbf{w}_2 = \mathbf{v}_2 - \frac{\langle \mathbf{v}_2,\ \mathbf{w}_1 \rangle}{\|\mathbf{w}_1\|^2}\mathbf{w}_1 \text{ and } \mathbf{w}_3 = \mathbf{v}_3 - \frac{\langle \mathbf{v}_3,\ \mathbf{w}_1 \rangle}{\|\mathbf{w}_1\|^2}\mathbf{w}_1 - \frac{\langle \mathbf{v}_3,\ \mathbf{w}_2 \rangle}{\|\mathbf{w}_2\|^2}\mathbf{w}_2$$

Let $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix}$ and $\mathbf{v}_3 = \begin{bmatrix} 1 \\ 1 \\ 2 \\ 2 \end{bmatrix}$ which are the given vectors.

By using (4-10) we have $\mathbf{w}_1 = \mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}$. For $\mathbf{w}_2 = \mathbf{v}_2 - \frac{\langle \mathbf{v}_2,\ \mathbf{w}_1 \rangle}{\|\mathbf{w}_1\|^2}\mathbf{w}_1$ so we need to find $\langle \mathbf{v}_2,\ \mathbf{w}_1 \rangle$ and $\|\mathbf{w}_1\|^2$:

$$\langle \mathbf{v}_2,\ \mathbf{w}_1 \rangle = \mathbf{v}_2 \cdot \mathbf{w}_1 = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} = 0+0+1+0 = 1$$

$$\|\mathbf{w}_1\|^2 = \mathbf{w}_1 \cdot \mathbf{w}_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} = 1+0+1+0 = 2$$

Putting $\mathbf{v}_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix}$, $\mathbf{w}_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}$, $\langle \mathbf{v}_2,\ \mathbf{w}_1 \rangle = 1$ and $\|\mathbf{w}_1\|^2 = 2$ into $\mathbf{w}_2 = \mathbf{v}_2 - \frac{\langle \mathbf{v}_2,\ \mathbf{w}_1 \rangle}{\|\mathbf{w}_1\|^2}\mathbf{w}_1$:

$$\mathbf{w}_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix} - \frac{1}{2}\begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0-1/2 \\ 1-0 \\ 1-1/2 \\ 1-0 \end{bmatrix} = \begin{bmatrix} -1/2 \\ 1 \\ 1/2 \\ 1 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} -1 \\ 2 \\ 1 \\ 2 \end{bmatrix}$$

Remove the fraction so that the arithmetic for evaluating $\mathbf{w}_3$ is easier, that is let

$$\mathbf{w}'_2 = \begin{bmatrix} -1 \\ 2 \\ 1 \\ 2 \end{bmatrix}$$

How do we determine $\mathbf{w}_3$*?*

By using the above (4-10), that is $\mathbf{w}_3 = \mathbf{v}_3 - \frac{\langle \mathbf{v}_3,\ \mathbf{w}_1 \rangle}{\|\mathbf{w}_1\|^2}\mathbf{w}_1 - \frac{\langle \mathbf{v}_3,\ \mathbf{w}'_2 \rangle}{\|\mathbf{w}'_2\|^2}\mathbf{w}'_2$:

$$\langle \mathbf{v}_3,\ \mathbf{w}_1 \rangle = \begin{bmatrix} 1 \\ 1 \\ 2 \\ 2 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} = 1+0+2+0 = 3$$

$$\langle \mathbf{v}_3,\ \mathbf{w}'_2 \rangle = \begin{bmatrix} 1 \\ 1 \\ 2 \\ 2 \end{bmatrix} \cdot \begin{bmatrix} -1 \\ 2 \\ 1 \\ 2 \end{bmatrix} = \left[1\times(-1)\right] + \left[1\times 2\right] + \left[2\times 1\right] + \left[2\times 2\right] = 7$$

$$\|\mathbf{w}'_2\|^2 = \begin{bmatrix} -1 \\ 2 \\ 1 \\ 2 \end{bmatrix} \cdot \begin{bmatrix} -1 \\ 2 \\ 1 \\ 2 \end{bmatrix} = (-1)^2 + 2^2 + 1^2 + 2^2 = 10$$

Substituting $\mathbf{v}_3 = \begin{bmatrix} 1 \\ 1 \\ 2 \\ 2 \end{bmatrix}$, $\mathbf{w}_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}$, $\mathbf{w}'_2 = \begin{bmatrix} -1 \\ 2 \\ 1 \\ 2 \end{bmatrix}$, $\langle \mathbf{v}_3,\ \mathbf{w}_1 \rangle = 3$, $\langle \mathbf{v}_3,\ \mathbf{w}'_2 \rangle = 7$, $\|\mathbf{w}_1\|^2 = 2$

and $\|\mathbf{w}'_2\|^2 = 10$ into $\mathbf{w}_3 = \mathbf{v}_3 - \dfrac{\langle \mathbf{v}_3,\ \mathbf{w}_1 \rangle}{\|\mathbf{w}_1\|^2}\mathbf{w}_1 - \dfrac{\langle \mathbf{v}_3,\ \mathbf{w}'_2 \rangle}{\|\mathbf{w}'_2\|^2}\mathbf{w}'_2$ gives

$$\mathbf{w}_3 = \begin{bmatrix} 1 \\ 1 \\ 2 \\ 2 \end{bmatrix} - \frac{3}{2}\begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} - \frac{7}{10}\begin{bmatrix} -1 \\ 2 \\ 1 \\ 2 \end{bmatrix}$$

$$= \begin{bmatrix} 1-3/2+7/10 \\ 1-0-14/10 \\ 2-3/2-7/10 \\ 2-0-14/10 \end{bmatrix} = \begin{bmatrix} 1/5 \\ -2/5 \\ -1/5 \\ 3/5 \end{bmatrix} = \frac{1}{5}\begin{bmatrix} 1 \\ -2 \\ -1 \\ 3 \end{bmatrix}$$

Again we can ignore the fraction to simplify our calculations, that is let

$$\mathbf{w}'_3 = \begin{bmatrix} 1 \\ -2 \\ -1 \\ 3 \end{bmatrix}$$

Our orthogonal basis is $\{\mathbf{w}_1,\ \mathbf{w}'_2,\ \mathbf{w}'_3\}$. *How do we convert these into orthonormal basis?*

By normalising each of these vectors:

$$\mathbf{w}_1 = \frac{1}{\|\mathbf{w}_1\|}\mathbf{w}_1 = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix},\ \mathbf{w}'_2 = \frac{1}{\|\mathbf{w}'_2\|}\mathbf{w}'_2 = \frac{1}{\sqrt{10}}\begin{bmatrix} -1 \\ 2 \\ 1 \\ 2 \end{bmatrix} \text{ and } \mathbf{w}'_3 = \frac{1}{\|\mathbf{w}'_3\|}\mathbf{w}'_3 = \frac{1}{\sqrt{15}}\begin{bmatrix} 1 \\ -2 \\ -1 \\ 3 \end{bmatrix}$$

Hence our orthonormal basis for *S* is $\left\{\mathbf{w}_1,\ \mathbf{w}'_2,\ \mathbf{w}'_3\right\}$.

4. (a) Let $\{\mathbf{v}_1,\ \mathbf{v}_2,\ \cdots,\ \mathbf{v}_n\}$ be a basis of a Euclidean space *V*. This basis is orthogonal if each of the vectors in the basis are orthogonal to each other, that is

$$\langle \mathbf{v}_i,\ \mathbf{v}_j \rangle = 0 \text{ where } i \neq j$$

This basis is orthonormal if we also have

$$\|\mathbf{v}_j\| = 1 \text{ for all } j = 1,\ 2,\ \cdots,\ n$$

(b) To show that the given vectors $\mathbf{f}_1 = \begin{pmatrix} -1 \\ 0 \\ 2 \end{pmatrix}$, $\mathbf{f}_2 = \begin{pmatrix} 0 \\ -2 \\ 3 \end{pmatrix}$ and $\mathbf{f}_3 = \begin{pmatrix} 1 \\ 1 \\ 4 \end{pmatrix}$ form a basis of $\mathbb{R}^3$ it is enough to show that they are linearly independent.

Let k_1, k_2 and k_3 be scalars such that

$$k_1\begin{pmatrix} -1 \\ 0 \\ 2 \end{pmatrix} + k_2\begin{pmatrix} 0 \\ -2 \\ 3 \end{pmatrix} + k_3\begin{pmatrix} 1 \\ 1 \\ 4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Writing these out as equations we have

$$\begin{array}{rcrcrcll} -k_1 & & & + & k_3 & = & 0 & \quad (1) \\ & - & 2k_2 & + & k_3 & = & 0 & \quad (2) \\ 2k_1 & + & 3k_2 & + & 4k_3 & = & 0 & \quad (3) \end{array}$$

From equation (1) we have $k_3 = k_1$ and equation (2) we have $k_3 = 2k_2$ implies $k_1 = 2k_2$.

Substituting $k_1 = 2k_2$ and $k_3 = 2k_2$ into the bottom equation:

$$2(2k_2) + 3k_2 + 4(2k_2) = 15k_2 = 0 \text{ gives } k_2 = 0$$

Substituting $k_2 = 0$ into $k_1 = 2k_2$ and $k_3 = 2k_2$ gives $k_1 = 0$ and $k_3 = 0$.

Thus the given vectors $\mathbf{f}_1 = \begin{pmatrix} -1 \\ 0 \\ 2 \end{pmatrix}$, $\mathbf{f}_2 = \begin{pmatrix} 0 \\ -2 \\ 3 \end{pmatrix}$ and $\mathbf{f}_3 = \begin{pmatrix} 1 \\ 1 \\ 4 \end{pmatrix}$ are linearly independent so they form a basis for $\mathbb{R}^3$ because by question 12 of Exercises 3.5 which says:

n linearly independent vectors in $\mathbb{R}^n$ form a basis for $\mathbb{R}^n$.

We only need three linearly independent vectors.

(c) We need to apply the Gram Schmidt Process (4-10) which is:

$$\mathbf{w}_1 = \mathbf{v}_1,\ \mathbf{w}_2 = \mathbf{v}_2 - \frac{\langle \mathbf{v}_2, \mathbf{w}_1 \rangle}{\|\mathbf{w}_1\|^2}\mathbf{w}_1 \text{ and } \mathbf{w}_3 = \mathbf{v}_3 - \frac{\langle \mathbf{v}_3, \mathbf{w}_1 \rangle}{\|\mathbf{w}_1\|^2}\mathbf{w}_1 - \frac{\langle \mathbf{v}_3, \mathbf{w}_2 \rangle}{\|\mathbf{w}_2\|^2}\mathbf{w}_2$$

By replacing **v** with **f** in the above we have $\mathbf{w}_1 = \mathbf{f}_1 = \begin{pmatrix} -1 \\ 0 \\ 2 \end{pmatrix}$ and $\mathbf{w}_2 = \mathbf{f}_2 - \dfrac{\langle \mathbf{f}_2, \mathbf{w}_1 \rangle}{\|\mathbf{w}_1\|^2}\mathbf{w}_1$:

$$\langle \mathbf{f}_2, \mathbf{w}_1 \rangle = \mathbf{f}_2 \cdot \mathbf{w}_1 = \begin{pmatrix} 0 \\ -2 \\ 3 \end{pmatrix} \cdot \begin{pmatrix} -1 \\ 0 \\ 2 \end{pmatrix} = 0 + 0 + 6 = 6$$

$$\|\mathbf{w}_1\|^2 = \begin{pmatrix} -1 \\ 0 \\ 2 \end{pmatrix} \cdot \begin{pmatrix} -1 \\ 0 \\ 2 \end{pmatrix} = (-1)^2 + 0^2 + 2^2 = 5$$

Putting $\mathbf{w}_1 = \begin{pmatrix} -1 \\ 0 \\ 2 \end{pmatrix}$, $\mathbf{f}_2 = \begin{pmatrix} 0 \\ -2 \\ 3 \end{pmatrix}$, $\langle \mathbf{f}_2, \mathbf{w}_1 \rangle = 6$ and $\|\mathbf{w}_1\|^2 = 5$ into $\mathbf{w}_2 = \mathbf{f}_2 - \dfrac{\langle \mathbf{f}_2, \mathbf{w}_1 \rangle}{\|\mathbf{w}_1\|^2}\mathbf{w}_1$:

$$\mathbf{w}_2 = \begin{pmatrix} 0 \\ -2 \\ 3 \end{pmatrix} - \frac{6}{5}\begin{pmatrix} -1 \\ 0 \\ 2 \end{pmatrix}$$

$$= \begin{pmatrix} 0+6/5 \\ -2-0 \\ 3-12/5 \end{pmatrix} = \begin{pmatrix} 6/5 \\ -2 \\ 3/5 \end{pmatrix} = \frac{1}{5}\begin{pmatrix} 6 \\ -10 \\ 3 \end{pmatrix}$$

We can ignore the fraction and write this as

$$\mathbf{w}'_2 = \begin{pmatrix} 6 \\ -10 \\ 3 \end{pmatrix}$$

What else do we need to find?

$\mathbf{w}_3$ which is given by $\mathbf{w}_3 = \mathbf{f}_3 - \frac{\langle \mathbf{f}_3,\ \mathbf{w}_1\rangle}{\|\mathbf{w}_1\|^2}\mathbf{w}_1 - \frac{\langle \mathbf{f}_3,\ \mathbf{w}'_2\rangle}{\|\mathbf{w}'_2\|^2}\mathbf{w}'_2$. Finding each component:

$$\langle \mathbf{f}_3,\ \mathbf{w}_1\rangle = \mathbf{f}_3 \cdot \mathbf{w}_1 = \begin{pmatrix} 1 \\ 1 \\ 4 \end{pmatrix} \cdot \begin{pmatrix} -1 \\ 0 \\ 2 \end{pmatrix} = -1+0+8 = 7$$

$$\langle \mathbf{f}_3,\ \mathbf{w}'_2\rangle = \mathbf{f}_3 \cdot \mathbf{w}'_2 = \begin{pmatrix} 1 \\ 1 \\ 4 \end{pmatrix} \cdot \begin{pmatrix} 6 \\ -10 \\ 3 \end{pmatrix} = 6-10+12 = 8$$

$$\|\mathbf{w}'_2\|^2 = \mathbf{w}'_2 \cdot \mathbf{w}'_2 = \begin{pmatrix} 6 \\ -10 \\ 3 \end{pmatrix} \cdot \begin{pmatrix} 6 \\ -10 \\ 3 \end{pmatrix} = 6^2 + (-10)^2 + 3^2 = 145$$

Substituting $\mathbf{f}_3 = \begin{pmatrix} 1 \\ 1 \\ 4 \end{pmatrix}$, $\mathbf{w}_1 = \begin{pmatrix} -1 \\ 0 \\ 2 \end{pmatrix}$, $\mathbf{w}'_2 = \begin{pmatrix} 6 \\ -10 \\ 3 \end{pmatrix}$, $\langle \mathbf{f}_3,\ \mathbf{w}_1\rangle = 7$, $\langle \mathbf{f}_3,\ \mathbf{w}'_2\rangle = 8$,

$\|\mathbf{w}'_2\|^2 = 145$ and $\|\mathbf{w}_1\|^2 = 5$ into $\mathbf{w}_3 = \mathbf{f}_3 - \frac{\langle \mathbf{f}_3,\ \mathbf{w}_1\rangle}{\|\mathbf{w}_1\|^2}\mathbf{w}_1 - \frac{\langle \mathbf{f}_3,\ \mathbf{w}'_2\rangle}{\|\mathbf{w}'_2\|^2}\mathbf{w}'_2$ gives

$$\mathbf{w}_3 = \begin{pmatrix} 1 \\ 1 \\ 4 \end{pmatrix} - \frac{7}{5}\begin{pmatrix} -1 \\ 0 \\ 2 \end{pmatrix} - \frac{8}{145}\begin{pmatrix} 6 \\ -10 \\ 3 \end{pmatrix}$$

$$= \begin{pmatrix} 1+7/5-48/145 \\ 1-0+80/145 \\ 4-14/5-24/145 \end{pmatrix} = \begin{pmatrix} 60/29 \\ 45/29 \\ 30/29 \end{pmatrix} = \frac{15}{29}\begin{pmatrix} 4 \\ 3 \\ 2 \end{pmatrix}$$

Again ignore the fraction and let $\mathbf{w}'_3 = \begin{pmatrix} 4 \\ 3 \\ 2 \end{pmatrix}$.

Hence our orthogonal basis is $\left\{ \mathbf{w}_1 = \begin{pmatrix} -1 \\ 0 \\ 2 \end{pmatrix},\ \mathbf{w}'_2 = \begin{pmatrix} 6 \\ -10 \\ 3 \end{pmatrix}, \mathbf{w}'_3 = \begin{pmatrix} 4 \\ 3 \\ 2 \end{pmatrix} \right\}$.

5. (a) We need to prove that an orthogonal set of non-zero vectors in $\mathbb{R}^n$ are linearly independent.

Proof.

Let $\{\mathbf{v}_1,\ \mathbf{v}_2,\ \cdots,\ \mathbf{v}_m\}$ be an orthogonal set of non-zero vectors in $\mathbb{R}^n$ and $k_1,\ k_2,\ \cdots,\ k_m$ be scalars such that

$$k_1\mathbf{v}_1 + k_2\mathbf{v}_2 + \cdots + k_m\mathbf{v}_m = \mathbf{O} \qquad (*)$$

What do we need to prove?

Need to show that $\{\mathbf{v}_1,\ \mathbf{v}_2,\ \cdots,\ \mathbf{v}_m\}$ is linearly independent which means prove that $k_1 = k_2 = \cdots = k_m = 0$.

Consider the inner product $\langle \mathbf{v}_1,\ \mathbf{O} \rangle = 0$. Substituting (*) for the zero vector into this gives

$$\begin{aligned} \langle \mathbf{v}_1,\ \mathbf{O} \rangle &= \langle \mathbf{v}_1,\ k_1\mathbf{v}_1 + k_2\mathbf{v}_2 + \cdots + k_m\mathbf{v}_m \rangle \\ &= k_1\langle \mathbf{v}_1,\ \mathbf{v}_1 \rangle + k_2\langle \mathbf{v}_1,\ \mathbf{v}_2 \rangle + \cdots + k_m\langle \mathbf{v}_1,\ \mathbf{v}_m \rangle \\ &= k_1\|\mathbf{v}_1\|^2 + k_2(0) + \cdots + k_m(0) \qquad [\text{Since } \mathbf{v}_1,\ \mathbf{v}_2,\ \cdots,\ \mathbf{v}_m \text{ are orthogonal}] \\ &= k_1\|\mathbf{v}_1\|^2 \end{aligned}$$

We have $k_1\|\mathbf{v}_1\|^2 = 0$ but $\mathbf{v}_1 \neq \mathbf{O}$ [Not Zero] therefore $k_1 = 0$.

By expanding $\langle \mathbf{v}_2,\ \mathbf{O} \rangle = \langle \mathbf{v}_3,\ \mathbf{O} \rangle = \cdots = \langle \mathbf{v}_n,\ \mathbf{O} \rangle = 0$ and repeating the above argument we have $k_2 = k_3 = \cdots = k_m = 0$. Thus we have $k_1 = k_2 = \cdots = k_m = 0$ which means that $\{\mathbf{v}_1,\ \mathbf{v}_2,\ \cdots,\ \mathbf{v}_m\}$ is linearly independent.

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(b) We need to prove that

$$\mathbf{w} = (\mathbf{v}_1 \cdot \mathbf{w})\mathbf{v}_1 + (\mathbf{v}_2 \cdot \mathbf{w})\mathbf{v}_2 + \cdots + (\mathbf{v}_n \cdot \mathbf{w})\mathbf{v}_n$$

Proof.

We are given that $B = \{\mathbf{v}_1,\ \cdots,\ \mathbf{v}_n\}$ is an orthonormal basis of $\mathbb{R}^n$. There exists scalars $k_1,\ k_2,\ \cdots,\ k_n$ such that

$$\mathbf{w} = k_1\mathbf{v}_1 + k_2\mathbf{v}_2 + \cdots + k_n\mathbf{v}_n$$

Required to prove that $k_1 = (\mathbf{v}_1 \cdot \mathbf{w}),\ k_2 = (\mathbf{v}_2 \cdot \mathbf{w}),\ \cdots,\ k_n = (\mathbf{v}_n \cdot \mathbf{w})$. Consider the inner product of the vector $\mathbf{v}_j$ (where *j* is any integer from 1 to n) and **w**:

$$\begin{aligned} \mathbf{v}_j \cdot \mathbf{w} &= \mathbf{v}_j \cdot (k_1\mathbf{v}_1 + k_2\mathbf{v}_2 + \cdots + k_n\mathbf{v}_n) \qquad [\text{Because } \mathbf{w} = k_1\mathbf{v}_1 + k_2\mathbf{v}_2 + \cdots + k_n\mathbf{v}_n] \\ &= k_1(\mathbf{v}_j \cdot \mathbf{v}_1) + k_2(\mathbf{v}_j \cdot \mathbf{v}_2) + \cdots + k_j(\mathbf{v}_j \cdot \mathbf{v}_j) + \cdots + k_n(\mathbf{v}_j \cdot \mathbf{v}_n) \\ &= k_1(0) + k_2(0) + \cdots + k_j\|\mathbf{v}_j\|^2 + \cdots + k_n(0) \qquad \begin{bmatrix} \text{Because } \{\mathbf{v}_1,\ \cdots,\ \mathbf{v}_n\} \\ \text{is orthonormal} \end{bmatrix} \\ &= k_j\|\mathbf{v}_j\|^2 = k_j \qquad [\text{Because } \mathbf{v}_j \text{ is orthonormal, normalised vector}] \end{aligned}$$

Hence we have $k_j = \mathbf{v}_j \cdot \mathbf{w}$. Since j was an arbitrary integer between 1 and n therefore the result is true for any j, that is $k_1 = (\mathbf{v}_1 \cdot \mathbf{w}),\ k_2 = (\mathbf{v}_2 \cdot \mathbf{w}), \cdots,\ k_n = (\mathbf{v}_n \cdot \mathbf{w})$.
This completes our proof.

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6. (a) The vectors $\mathbf{w}_1,\ \mathbf{w}_2,\ \mathbf{w}_3$ in $\mathbb{R}^3$ are orthonormal means that **both** the following conditions are satisfied:

$$\langle \mathbf{w}_1,\ \mathbf{w}_2 \rangle = \langle \mathbf{w}_1,\ \mathbf{w}_3 \rangle = \langle \mathbf{w}_2,\ \mathbf{w}_3 \rangle = 0 \text{ and}$$

$$\|\mathbf{w}_1\| = \|\mathbf{w}_2\| = \|\mathbf{w}_3\| = 1$$

(b) We need to convert $\mathbf{v}_1 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix},\ \mathbf{v}_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix},\ \mathbf{v}_3 = \begin{bmatrix} 5 \\ 4 \\ 6 \end{bmatrix}$ into an orthonormal set by applying the Gram Schmidt Process (4-10) which is

$$\mathbf{w}_1 = \mathbf{v}_1,\ \mathbf{w}_2 = \mathbf{v}_2 - \frac{\langle \mathbf{v}_2,\ \mathbf{w}_1 \rangle}{\|\mathbf{w}_1\|^2}\mathbf{w}_1 \text{ and } \mathbf{w}_3 = \mathbf{v}_3 - \frac{\langle \mathbf{v}_3,\ \mathbf{w}_1 \rangle}{\|\mathbf{w}_1\|^2}\mathbf{w}_1 - \frac{\langle \mathbf{v}_3,\ \mathbf{w}_2 \rangle}{\|\mathbf{w}_2\|^2}\mathbf{w}_2$$

We have $\mathbf{w}_1 = \mathbf{v}_1 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$. We need to evaluate $\mathbf{w}_2 = \mathbf{v}_2 - \frac{\langle \mathbf{v}_2,\ \mathbf{w}_1 \rangle}{\|\mathbf{w}_1\|^2}\mathbf{w}_1$:

$$\langle \mathbf{v}_2,\ \mathbf{w}_1 \rangle = \mathbf{v}_2 \cdot \mathbf{w}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} = 0 + 1 + 0 = 1$$

$$\|\mathbf{w}_1\|^2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} = 0^2 + 1^2 + 1^2 = 2$$

Substituting $\mathbf{v}_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix},\ \mathbf{w}_1 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix},\ \langle \mathbf{v}_2,\ \mathbf{w}_1 \rangle = 1,\ \|\mathbf{w}_1\|^2 = 2$ into $\mathbf{w}_2 = \mathbf{v}_2 - \frac{\langle \mathbf{v}_2,\ \mathbf{w}_1 \rangle}{\|\mathbf{w}_1\|^2}\mathbf{w}_1$ gives

$$\begin{aligned} \mathbf{w}_2 &= \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} - \frac{1}{2}\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \\ &= \begin{bmatrix} 1-0 \\ 1-1/2 \\ 0-1/2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1/2 \\ -1/2 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix} \end{aligned}$$

Ignore the fraction so that evaluating $\mathbf{w}_3$ is made easier, that is let

$$\mathbf{w}'_2 = \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}$$

What else do we need to find?

The vector $\mathbf{w}_3$ which is given by $\mathbf{w}_3 = \mathbf{v}_3 - \dfrac{\langle \mathbf{v}_3, \mathbf{w}_1 \rangle}{\|\mathbf{w}_1\|^2}\mathbf{w}_1 - \dfrac{\langle \mathbf{v}_3, \mathbf{w}'_2 \rangle}{\|\mathbf{w}'_2\|^2}\mathbf{w}'_2$:

$$\langle \mathbf{v}_3, \mathbf{w}_1 \rangle = \mathbf{v}_3 \cdot \mathbf{w}_1 = \begin{bmatrix} 5 \\ 4 \\ 6 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} = (5\times 0) + (4\times 1) + (6\times 1) = 10$$

$$\langle \mathbf{v}_3, \mathbf{w}'_2 \rangle = \mathbf{v}_3 \cdot \mathbf{w}'_2 = \begin{bmatrix} 5 \\ 4 \\ 6 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix} = (5\times 2) + (4\times 1) + (6\times(-1)) = 8$$

$$\|\mathbf{w}'_2\|^2 = \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix} = 2^2 + 1^2 + (-1)^2 = 6$$

Substituting $\mathbf{v}_3 = \begin{bmatrix} 5 \\ 4 \\ 6 \end{bmatrix}$, $\mathbf{w}_1 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$, $\mathbf{w}'_2 = \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}$, $\langle \mathbf{v}_3, \mathbf{w}_1 \rangle = 10$, $\|\mathbf{w}_1\|^2 = 2$, $\langle \mathbf{v}_3, \mathbf{w}'_2 \rangle = 8$ and $\|\mathbf{w}'_2\|^2 = 6$ into $\mathbf{w}_3 = \mathbf{v}_3 - \dfrac{\langle \mathbf{v}_3, \mathbf{w}_1 \rangle}{\|\mathbf{w}_1\|^2}\mathbf{w}_1 - \dfrac{\langle \mathbf{v}_3, \mathbf{w}'_2 \rangle}{\|\mathbf{w}'_2\|^2}\mathbf{w}'_2$ gives

$$\begin{aligned}
\mathbf{w}_3 &= \begin{bmatrix} 5 \\ 4 \\ 6 \end{bmatrix} - \frac{10}{2}\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} - \frac{8}{6}\begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix} \\
&= \begin{bmatrix} 5 \\ 4 \\ 6 \end{bmatrix} - 5\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} - \frac{4}{3}\begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 5-0-8/3 \\ 4-5-4/3 \\ 6-5+4/3 \end{bmatrix} = \begin{bmatrix} 7/3 \\ -7/3 \\ 7/3 \end{bmatrix} = \frac{7}{3}\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}
\end{aligned}$$

Again ignore the fraction and let $\mathbf{w}'_3 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$.

Hence our orthogonal vectors in $\mathbb{R}^3$ are $\mathbf{w}_1 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$, $\mathbf{w}'_2 = \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}$ and $\mathbf{w}'_3 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$.

We need to normalise these vectors so that we have an orthonormal set:

$$\mathbf{w}_1 = \frac{1}{\|\mathbf{w}_1\|}\mathbf{w}_1 = \frac{1}{\sqrt{2}}\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix},\quad \mathbf{w}'_2 = \frac{1}{\|\mathbf{w}'_2\|}\mathbf{w}'_2 = \frac{1}{\sqrt{6}}\begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix} \text{ and } \mathbf{w}'_3 = \frac{1}{\|\mathbf{w}'_3\|}\mathbf{w}'_3 = \frac{1}{\sqrt{3}}\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

Hence our orthonormal set in $\mathbb{R}^3$ is $\left\{ \dfrac{1}{\sqrt{2}}\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \dfrac{1}{\sqrt{6}}\begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}, \dfrac{1}{\sqrt{3}}\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \right\}$.

7. (a) The norm $\|\mathbf{x}\|$ of $\mathbf{x}=(x_1,\ \ x_2,\ \ x_3,\ \ x_4)$ is defined as

$$\begin{aligned}\|\mathbf{x}\| &= \sqrt{\langle \mathbf{x},\ \mathbf{x}\rangle}\\ &= \sqrt{\mathbf{x}\cdot\mathbf{x}} = \sqrt{x_1^{\,2}+x_2^{\,2}+x_3^{\,2}+x_4^{\,2}}\end{aligned}$$

We have

$$\|(2,\ \ -3,\ \ 5,\ \ 1)\| = \sqrt{2^2+(-3)^2+5^2+1^2} = \sqrt{39}$$

(b) We say two vectors $\mathbf{x},\ \mathbf{y}\in\mathbb{R}^4$ are orthogonal if

$$\langle \mathbf{x},\ \mathbf{y}\rangle = \mathbf{x}\cdot\mathbf{y} = 0$$

Checking the given vectors:

$$\langle (0,\ \ 1,\ \ -1,\ \ 1),\ (6,\ \ 3,\ \ 3,\ \ 27)\rangle = (0\times 6)+(1\times 3)+(-1\times 3)+(1\times 27) = 27$$

Since $\langle (0,\ \ 1,\ \ -1,\ \ 1),\ (6,\ \ 3,\ \ 3,\ \ 27)\rangle = 27 \neq 0$ therefore the given vectors are **not** orthogonal.

(c) An orthonormal set of vectors in $\mathbb{R}^4$ is a set of vectors where each vector is orthogonal to each other and every vector has a norm of 1.

Checking Orthogonality:

$$\left\langle (0,\ \ 0,\ \ 1,\ \ 0),\ \left(\frac{1}{\sqrt{6}},\ \ \frac{-2}{\sqrt{6}},\ \ 0,\ \ \frac{1}{\sqrt{6}}\right)\right\rangle = 0+0+0+0 = 0$$

$$\left\langle (0,\ \ 0,\ \ 1,\ \ 0),\ \left(\frac{2}{\sqrt{5}},\ \ \frac{1}{\sqrt{5}},\ \ 0,\ \ 0\right)\right\rangle = 0$$

$$\left\langle \left(\frac{1}{\sqrt{6}},\ \ \frac{-2}{\sqrt{6}},\ \ 0,\ \ \frac{1}{\sqrt{6}}\right),\ \left(\frac{2}{\sqrt{5}},\ \ \frac{1}{\sqrt{5}},\ \ 0,\ \ 0\right)\right\rangle = \frac{2}{\sqrt{30}}-\frac{2}{\sqrt{30}}+0+0 = 0$$

Hence the given vectors are orthogonal.

Checking Normality:

$$\|(0,\ \ 0,\ \ 1,\ \ 0)\| = 1$$

$$\left\|\left(\frac{1}{\sqrt{6}},\ \ \frac{-2}{\sqrt{6}},\ \ 0,\ \ \frac{1}{\sqrt{6}}\right)\right\| = \frac{1}{6}+\frac{4}{6}+0+\frac{1}{6} = 1$$

$$\left\|\left(\frac{2}{\sqrt{5}},\ \ \frac{1}{\sqrt{5}},\ \ 0,\ \ 0\right)\right\| = \frac{4}{5}+\frac{1}{5}+0+0 = 1$$

Each vector has a norm of 1.

Since all three of the given vectors are orthogonal and each has a norm of 1 therefore $\left\{(0,\ \ 0,\ \ 1,\ \ 0),\ \left(\frac{1}{\sqrt{6}},\ \ \frac{-2}{\sqrt{6}},\ \ 0,\ \ \frac{1}{\sqrt{6}}\right),\ \left(\frac{2}{\sqrt{5}},\ \ \frac{1}{\sqrt{5}},\ \ 0,\ \ 0\right)\right\}$ is an orthonormal set.

(d) The Gram-Schmidt process (4-10) is given by

Let $\mathbf{w}_1=\mathbf{v}_1$, $\mathbf{w}_2=\mathbf{v}_2-\dfrac{\langle \mathbf{v}_2,\ \mathbf{w}_1\rangle}{\|\mathbf{w}_1\|^2}\mathbf{w}_1$

$$\mathbf{w}_3 = \mathbf{v}_3 - \frac{\langle \mathbf{v}_3,\ \mathbf{w}_1 \rangle}{\|\mathbf{w}_1\|^2}\mathbf{w}_1 - \frac{\langle \mathbf{v}_3,\ \mathbf{w}_2 \rangle}{\|\mathbf{w}_2\|^2}\mathbf{w}_2$$

$$\mathbf{w}_4 = \mathbf{v}_4 - \frac{\langle \mathbf{v}_4,\ \mathbf{w}_1 \rangle}{\|\mathbf{w}_1\|^2}\mathbf{w}_1 - \frac{\langle \mathbf{v}_4,\ \mathbf{w}_2 \rangle}{\|\mathbf{w}_2\|^2}\mathbf{w}_2 - \frac{\langle \mathbf{v}_4,\ \mathbf{w}_3 \rangle}{\|\mathbf{w}_3\|^2}\mathbf{w}_3$$

$$\vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots$$

Applying this to the given vectors:

$\mathbf{v}_1 = (1,\ 0,\ 0,\ 0)$, $\mathbf{v}_2 = (1,\ 1,\ 0,\ 1)$, $\mathbf{v}_3 = (0,\ 1,\ 1,\ 1)$, $\mathbf{v}_4 = (0,\ 1,\ -1,\ 0)$

we have $\mathbf{w}_1 = \mathbf{v}_1 = (1,\ 0,\ 0,\ 0)$. We need to find $\mathbf{w}_2 = \mathbf{v}_2 - \dfrac{\langle \mathbf{v}_2,\ \mathbf{w}_1 \rangle}{\|\mathbf{w}_1\|^2}\mathbf{w}_1$.

Evaluating each component of this we have

$$\langle \mathbf{v}_2,\ \mathbf{w}_1 \rangle = \mathbf{v}_2 \cdot \mathbf{w}_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} = 1 + 0 + 0 + 0 = 1$$

$$\|\mathbf{w}_1\|^2 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} = 1$$

Putting $\mathbf{v}_2 = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 1 \end{pmatrix}$, $\mathbf{w}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$, $\langle \mathbf{v}_2,\ \mathbf{w}_1 \rangle = 1$ and $\|\mathbf{w}_1\|^2 = 1$ into $\mathbf{w}_2 = \mathbf{v}_2 - \dfrac{\langle \mathbf{v}_2,\ \mathbf{w}_1 \rangle}{\|\mathbf{w}_1\|^2}\mathbf{w}_1$:

$$\mathbf{w}_2 = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 1 \end{pmatrix} - \frac{1}{1}\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1-1 \\ 1-0 \\ 0-0 \\ 1-0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix}$$

Next we need to find $\mathbf{w}_3 = \mathbf{v}_3 - \dfrac{\langle \mathbf{v}_3,\ \mathbf{w}_1 \rangle}{\|\mathbf{w}_1\|^2}\mathbf{w}_1 - \dfrac{\langle \mathbf{v}_3,\ \mathbf{w}_2 \rangle}{\|\mathbf{w}_2\|^2}\mathbf{w}_2$:

$$\langle \mathbf{v}_3,\ \mathbf{w}_1 \rangle = \mathbf{v}_3 \cdot \mathbf{w}_1 = \begin{pmatrix} 0 \\ 1 \\ 1 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} = 0$$

$$\langle \mathbf{v}_3,\ \mathbf{w}_2 \rangle = \mathbf{v}_3 \cdot \mathbf{w}_2 = \begin{pmatrix} 0 \\ 1 \\ 1 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix} = 0 + 1 + 0 + 1 = 2$$

$$\|\mathbf{w}_2\|^2 = \mathbf{w}_2 \cdot \mathbf{w}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix} = 0^2 + 1^2 + 0^2 + 1^2 = 2$$

Substituting $\mathbf{v}_3 = \begin{pmatrix} 0 \\ 1 \\ 1 \\ 1 \end{pmatrix}$, $\mathbf{w}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix}$, $\langle \mathbf{v}_3,\ \mathbf{w}_1 \rangle = 0$, $\langle \mathbf{v}_3,\ \mathbf{w}_2 \rangle = 2$ and $\|\mathbf{w}_2\|^2 = 2$ into $\mathbf{w}_3 = \mathbf{v}_3 - \dfrac{\langle \mathbf{v}_3,\ \mathbf{w}_1 \rangle}{\|\mathbf{w}_1\|^2}\mathbf{w}_1 - \dfrac{\langle \mathbf{v}_3,\ \mathbf{w}_2 \rangle}{\|\mathbf{w}_2\|^2}\mathbf{w}_2$ gives:

$$\mathbf{w}_3 = \begin{pmatrix} 0 \\ 1 \\ 1 \\ 1 \end{pmatrix} - 0\mathbf{w}_1 - \frac{2}{2}\begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0-0 \\ 1-1 \\ 1-0 \\ 1-1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}$$

We need to find $\mathbf{w}_4 = \mathbf{v}_4 - \dfrac{\langle \mathbf{v}_4,\ \mathbf{w}_1 \rangle}{\|\mathbf{w}_1\|^2}\mathbf{w}_1 - \dfrac{\langle \mathbf{v}_4,\ \mathbf{w}_2 \rangle}{\|\mathbf{w}_2\|^2}\mathbf{w}_2 - \dfrac{\langle \mathbf{v}_4,\ \mathbf{w}_3 \rangle}{\|\mathbf{w}_3\|^2}\mathbf{w}_3$:

$$\langle \mathbf{v}_4,\ \mathbf{w}_1 \rangle = \begin{pmatrix} 0 \\ 1 \\ -1 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} = (0\times 1) + (1\times 0) + (-1\times 0) + (0\times 0) = 0$$

$$\langle \mathbf{v}_4,\ \mathbf{w}_2 \rangle = \begin{pmatrix} 0 \\ 1 \\ -1 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix} = (0\times 0) + (1\times 1) + (-1\times 0) + (0\times 1) = 1$$

$$\langle \mathbf{v}_4,\ \mathbf{w}_3 \rangle = \begin{pmatrix} 0 \\ 1 \\ -1 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} = (0\times 0) + (1\times 0) + (-1\times 1) + (0\times 0) = -1$$

$$\|\mathbf{w}_3\|^2 = \langle \mathbf{w}_3,\ \mathbf{w}_3 \rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} = 0^2 + 0^2 + 1^2 + 0^2 = 1$$

Substituting $\mathbf{v}_4 = \begin{pmatrix} 0 \\ 1 \\ -1 \\ 0 \end{pmatrix}$, $\mathbf{w}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$, $\mathbf{w}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix}$, $\mathbf{w}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}$, $\langle \mathbf{v}_4,\ \mathbf{w}_1 \rangle = 0$, $\|\mathbf{w}_1\|^2 = 1$,

$\langle \mathbf{v}_4, \mathbf{w}_2 \rangle = 1$, $\|\mathbf{w}_2\|^2 = 2$, $\langle \mathbf{v}_4, \mathbf{w}_3 \rangle = -1$ and $\|\mathbf{w}_3\|^2 = 1$ into

$\mathbf{w}_4 = \mathbf{v}_4 - \dfrac{\langle \mathbf{v}_4, \mathbf{w}_1 \rangle}{\|\mathbf{w}_1\|^2}\mathbf{w}_1 - \dfrac{\langle \mathbf{v}_4, \mathbf{w}_2 \rangle}{\|\mathbf{w}_2\|^2}\mathbf{w}_2 - \dfrac{\langle \mathbf{v}_4, \mathbf{w}_3 \rangle}{\|\mathbf{w}_3\|^2}\mathbf{w}_3$ gives

$$\mathbf{w}_4 = \begin{pmatrix} 0 \\ 1 \\ -1 \\ 0 \end{pmatrix} - 0\mathbf{w}_1 - \frac{1}{2}\begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix} - \frac{-1}{1}\begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0-0+0 \\ 1-1/2+0 \\ -1-0+1 \\ 0-1/2+0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1/2 \\ 0 \\ -1/2 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 0 \\ 1 \\ 0 \\ -1 \end{pmatrix}$$

Ignore the fraction and let $\mathbf{w}'_4 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ -1 \end{pmatrix}$. We have our orthogonal set of vectors

$$\mathbf{w}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \ \mathbf{w}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix}, \ \mathbf{w}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \text{ and } \mathbf{w}'_4 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ -1 \end{pmatrix}$$

We need to normalise these vectors. Clearly $\mathbf{w}_1$ and $\mathbf{w}_3$ have a norm of 1 therefore we do **not** need to normalise these.

$$\mathbf{w}_2 = \frac{1}{\|\mathbf{w}_2\|}\mathbf{w}_2 = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix}$$

$$\mathbf{w}'_4 = \frac{1}{\|\mathbf{w}'_4\|}\mathbf{w}'_4 = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ 1 \\ 0 \\ -1 \end{pmatrix}$$

Hence our orthonormal basis is $\{\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3, \mathbf{w}'_4\}$.

8. (a) Let $\mathbf{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. The norm $\|\mathbf{A}\|$ of $\mathbf{A} \in M(2, 2)$ is defined as

$$\|\mathbf{A}\| = \sqrt{\langle \mathbf{A}, \mathbf{A} \rangle} = \sqrt{tr(\mathbf{A}^T\mathbf{A})} \qquad (*)$$

We have

$$tr(\mathbf{A}^T\mathbf{A}) = tr\left(\begin{pmatrix} a & c \\ b & d \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) \qquad \left[\text{Because } \mathbf{A}^{\mathrm{T}} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}^T = \begin{pmatrix} a & c \\ b & d \end{pmatrix}\right]$$

$$= tr\begin{pmatrix} a^2+c^2 & ab+cd \\ ab+cd & b^2+d^2 \end{pmatrix}$$

$$= a^2+b^2+c^2+d^2 \qquad \left[\begin{array}{l}\text{Because the trace is the addition of} \\ \text{leading diagonal entries}\end{array}\right]$$

Substituting this $tr\left(\mathbf{A}^T\mathbf{A}\right)=a^2+b^2+c^2+d^2$ into (*) gives

$$\|\mathbf{A}\|=\sqrt{a^2+b^2+c^2+d^2} \qquad (☼)$$

For the given matrix we have

$$\left\|\begin{pmatrix} 2 & -3 \\ 5 & 1 \end{pmatrix}\right\|=\sqrt{2^2+(-3)^2+5^2+1^2}=\sqrt{39}$$

(b) Matrices **A** and **B** are orthogonal if $\langle \mathbf{A},\ \mathbf{B}\rangle = tr\left(\mathbf{B}^T\mathbf{A}\right)=0$.

For the given matrices we have

$$\left\langle \begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix}, \begin{pmatrix} 6 & 3 \\ 3 & 27 \end{pmatrix} \right\rangle = tr\left(\begin{pmatrix} 6 & 3 \\ 3 & 27 \end{pmatrix}^T \begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix}\right)$$

$$= tr\left(\begin{pmatrix} 6 & 3 \\ 3 & 27 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix}\right) = tr\begin{pmatrix} -3 & 9 \\ -27 & 30 \end{pmatrix} = -3+30=27$$

Thus the given matrices are **not** orthogonal.

(c) Orthonormal set of matrices are a set of matrices which are orthogonal and have a norm of 1.

<u>Checking Orthogonality</u>: We need to find $\langle \mathbf{A}_2,\ \mathbf{A}_1\rangle$, $\langle \mathbf{A}_3,\ \mathbf{A}_1\rangle$, $\langle \mathbf{A}_3,\ \mathbf{A}_2\rangle$:

$$\langle \mathbf{A}_2,\ \mathbf{A}_1\rangle = tr\left(\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}^T \begin{pmatrix} 0 & 1/\sqrt{2} \\ 0 & 1/\sqrt{2} \end{pmatrix}\right)$$

$$= tr\left(\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1/\sqrt{2} \\ 0 & 1/\sqrt{2} \end{pmatrix}\right) = tr\begin{pmatrix} 0 & 1/\sqrt{2} \\ 0 & 0 \end{pmatrix} = 0+0=0$$

$$\langle \mathbf{A}_3,\ \mathbf{A}_1\rangle = tr\left(\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}^T \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\right)$$

$$= tr\left(\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\right) = tr\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = 0+0=0$$

$$\langle \mathbf{A}_3,\ \mathbf{A}_2\rangle = tr\left(\begin{pmatrix} 0 & 1/\sqrt{2} \\ 0 & 1/\sqrt{2} \end{pmatrix}^T \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\right)$$

$$= tr\left(\begin{pmatrix} 0 & 0 \\ 1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\right) = tr\begin{pmatrix} 0 & 0 \\ 1/\sqrt{2} & 0 \end{pmatrix} = 0+0=0$$

Since $\langle \mathbf{A}_2,\ \mathbf{A}_1\rangle = \langle \mathbf{A}_3,\ \mathbf{A}_1\rangle = \langle \mathbf{A}_3,\ \mathbf{A}_2\rangle = 0$ therefore the given set

$$\left\{\mathbf{A}_1=\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix},\ \mathbf{A}_2=\begin{pmatrix} 0 & 1/\sqrt{2} \\ 0 & 1/\sqrt{2} \end{pmatrix},\ \mathbf{A}_3=\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\right\}$$

is an orthogonal set of matrices.

<u>Checking Normality</u>:

We also need to check that each of these have a norm of 1 to ensure they are orthonormal. *How?*

We can use (☼) above which is

$$\|\mathbf{A}\| = \sqrt{a^2+b^2+c^2+d^2} \quad \text{where } \mathbf{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

$$\|\mathbf{A}_1\| = \left\|\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\right\| = \sqrt{1^2+0^2+0^2+0^2} = 1$$

$$\|\mathbf{A}_2\|^2 = \left\|\begin{pmatrix} 0 & 1/\sqrt{2} \\ 0 & 1/\sqrt{2} \end{pmatrix}\right\| = \sqrt{0^2+\left(\frac{1}{\sqrt{2}}\right)^2+0^2+\left(\frac{1}{\sqrt{2}}\right)^2} = 1$$

$$\|\mathbf{A}_3\|^2 = \left\|\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\right\| = \sqrt{0^2+0^2+1^2+0^2} = 1$$

Taking the square root of each we have $\|\mathbf{A}_1\| = \|\mathbf{A}_2\| = \|\mathbf{A}_3\| = 1$. Hence the given set of matrices are orthonormal.

(d) The first three matrices of the set $\{\mathbf{A}_1,\ \mathbf{A}_2,\ \mathbf{A}_3,\ \mathbf{A}_4\}$ are orthonormal as shown in part (c) above. We need to find a matrix $\mathbf{A}_4$ which is orthogonal to $\mathbf{A}_1,\ \mathbf{A}_2,\ \mathbf{A}_3$ and is of norm 1. *How do we find this* $\mathbf{A}_4$ *?*

Use Gram-Schmidt Process (4-10) which is

$$\mathbf{w}_4 = \mathbf{v}_4 - \frac{\langle \mathbf{v}_4,\ \mathbf{w}_1\rangle}{\|\mathbf{w}_1\|^2}\mathbf{w}_1 - \frac{\langle \mathbf{v}_4,\ \mathbf{w}_2\rangle}{\|\mathbf{w}_2\|^2}\mathbf{w}_2 - \frac{\langle \mathbf{v}_4,\ \mathbf{w}_3\rangle}{\|\mathbf{w}_3\|^2}\mathbf{w}_3 \qquad (*)$$

Let $\mathbf{w}_1 = \mathbf{A}_1,\ \mathbf{w}_2 = \mathbf{A}_2,\ \mathbf{w}_3 = \mathbf{A}_3$ because these are already orthonormal which also means that $\|\mathbf{w}_1\|^2 = \|\mathbf{A}_1\|^2 = 1,\ \|\mathbf{w}_2\|^2 = \|\mathbf{A}_2\|^2 = 1$ and $\|\mathbf{w}_3\|^2 = \|\mathbf{A}_3\|^2 = 1$.

Let $\mathbf{v}_4 = \mathbf{B}$ and $\mathbf{w}_4 = \mathbf{A}_4$ so substituting $\mathbf{w}_1 = \mathbf{A}_1,\ \mathbf{w}_2 = \mathbf{A}_2,\ \mathbf{w}_3 = \mathbf{A}_3$ and $\|\mathbf{w}_1\|^2 = \|\mathbf{w}_2\|^2 = \|\mathbf{w}_3\|^2 = 1$ into (*) gives

$$\mathbf{A}_4 = \mathbf{B} - \langle \mathbf{B},\ \mathbf{A}_1\rangle \mathbf{A}_1 - \langle \mathbf{B},\ \mathbf{A}_2\rangle \mathbf{A}_2 - \langle \mathbf{B},\ \mathbf{A}_3\rangle \mathbf{A}_3 \qquad (**)$$

Evaluating each component of (**):

$$\langle \mathbf{B},\ \mathbf{A}_1\rangle = \langle \mathbf{A}_1,\ \mathbf{B}\rangle = tr\left[\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}^T \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\right] = tr\left[\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\right] = tr\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = 0$$

$$\langle \mathbf{B},\ \mathbf{A}_2\rangle = \langle \mathbf{A}_2,\ \mathbf{B}\rangle = tr\left[\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1/\sqrt{2} \\ 0 & 1/\sqrt{2} \end{pmatrix}\right] = tr\begin{pmatrix} 0 & -1/\sqrt{2} \\ 0 & 1/\sqrt{2} \end{pmatrix} = \frac{1}{\sqrt{2}}$$

$$\langle \mathbf{B},\ \mathbf{A}_3\rangle = \langle \mathbf{A}_3,\ \mathbf{B}\rangle = tr\left[\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\right] = tr\begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix} = -1$$

Substituting $\langle \mathbf{B},\ \mathbf{A}_1\rangle = 0,\ \langle \mathbf{B},\ \mathbf{A}_2\rangle = \frac{1}{\sqrt{2}},\ \langle \mathbf{B},\ \mathbf{A}_3\rangle = -1,\ \mathbf{B} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},\ \mathbf{A}_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix},$

$\mathbf{A}_2 = \begin{pmatrix} 0 & 1/\sqrt{2} \\ 0 & 1/\sqrt{2} \end{pmatrix}$ and $\mathbf{A}_3 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ into

$$\mathbf{A}_4 = \mathbf{B} - \langle \mathbf{B},\ \mathbf{A}_1\rangle \mathbf{A}_1 - \langle \mathbf{B},\ \mathbf{A}_2\rangle \mathbf{A}_2 - \langle \mathbf{B},\ \mathbf{A}_3\rangle \mathbf{A}_3 \qquad (**)$$

gives

$$\mathbf{A}_4 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} - 0\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} - \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 1/\sqrt{2} \\ 0 & 1/\sqrt{2} \end{pmatrix} - (-1)\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$
$$= \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} - \frac{1}{2}\begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$
$$= \begin{pmatrix} 0-0+0 & 1-1/2+0 \\ -1-0+1 & 0-1/2+0 \end{pmatrix}$$
$$= \begin{pmatrix} 0 & 1/2 \\ 0 & -1/2 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ 0 & -1 \end{pmatrix}$$

Remember we can ignore the fraction and let $\mathbf{A}'_4 = \begin{pmatrix} 0 & 1 \\ 0 & -1 \end{pmatrix}$. This matrix is orthogonal to $\{\mathbf{A}_1, \ \mathbf{A}_2, \ \mathbf{A}_3\}$. We only need to normalise $\mathbf{A}'_4$:

$$\|\mathbf{A}'_4\| = \left\|\begin{pmatrix} 0 & 1 \\ 0 & -1 \end{pmatrix}\right\| = \sqrt{0^2 + 1^2 + 0^2 + (-1)^2} = \sqrt{2}$$

Hence the normalised matrix is

$$\mathbf{A}'_4 = \frac{1}{\|\mathbf{A}'_4\|}\mathbf{A}'_4 = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 1 \\ 0 & -1 \end{pmatrix}$$

(e) *How do we find the given scalars* $\lambda_1, \ \lambda_2, \ \lambda_3, \ \lambda_4$*?*

By using:

Proposition (4-11). Let $\{\mathbf{v}_1, \ \mathbf{v}_2, \ \mathbf{v}_3, \ \cdots, \ \mathbf{v}_n\}$ be an **orthonormal** set of vectors and $\mathbf{u} \in V$ then

$$\mathbf{u} = \langle \mathbf{u}, \ \mathbf{v}_1 \rangle \mathbf{v}_1 + \langle \mathbf{u}, \ \mathbf{v}_2 \rangle \mathbf{v}_2 + \langle \mathbf{u}, \ \mathbf{v}_3 \rangle \mathbf{v}_3 + \cdots + \langle \mathbf{u}, \ \mathbf{v}_4 \rangle \mathbf{v}_4$$

Let $\mathbf{C} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. Note that $\mathbf{C}$ is the identity matrix, that is $\mathbf{C} = \mathbf{I}$.

$$\mathbf{C} = \lambda_1\mathbf{A}_1 + \lambda_2\mathbf{A}_2 + \lambda_3\mathbf{A}_3 + \lambda_4\mathbf{A}'_4$$

where $\lambda_1 = \langle \mathbf{C}, \ \mathbf{A}_1 \rangle$, $\lambda_2 = \langle \mathbf{C}, \ \mathbf{A}_2 \rangle$, $\lambda_3 = \langle \mathbf{C}, \ \mathbf{A}_3 \rangle$ and $\lambda_4 = \langle \mathbf{C}, \ \mathbf{A}'_4 \rangle$.

Since $\mathbf{C} = \mathbf{I}$ we have for $k = 1, \ 2, \ 3$ and 4:

$$\lambda_k = \langle \mathbf{C}, \ \mathbf{A}_k \rangle = \langle \mathbf{I}, \ \mathbf{A}_k \rangle$$
$$= \langle \mathbf{A}_k, \ \mathbf{I} \rangle = tr\left(\mathbf{I}^T\mathbf{A}_k\right) = tr\left(\mathbf{I}\mathbf{A}_k\right) = tr\left(\mathbf{A}_k\right)$$

We use this $\lambda_k = tr\left(\mathbf{A}_k\right)$ to find $\lambda_1, \ \lambda_2, \ \lambda_3, \ \lambda_4$:

$$\lambda_1 = tr\left[\mathbf{A}_1\right] = tr\left[\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\right] = 1$$

$$\lambda_2 = tr\left[\mathbf{A}_2\right] = tr\left[\begin{pmatrix} 0 & 1/\sqrt{2} \\ 0 & 1/\sqrt{2} \end{pmatrix}\right] = \frac{1}{\sqrt{2}}$$

$$\lambda_3 = tr\left[\mathbf{A}_3\right] = tr\left[\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\right] = 0$$

$$\lambda_4 = tr\left[\mathbf{A}'_4\right] = tr\left[\frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 1 \\ 0 & -1 \end{pmatrix}\right] = -\frac{1}{\sqrt{2}}$$

Hence $\lambda_1 = 1,\ \lambda_2 = \frac{1}{\sqrt{2}},\ \lambda_3 = 0$ and $\lambda_4 = -\frac{1}{\sqrt{2}}$.

9. (a) The length of the vector $\mathbf{v} = 2\mathbf{q}_1 - 3\mathbf{q}_2 + 2\mathbf{q}_3$ is given by evaluating the norm of this vector. We have

$$\begin{aligned} \|\mathbf{v}\|^2 &= \langle 2\mathbf{q}_1 - 3\mathbf{q}_2 + 2\mathbf{q}_3,\ 2\mathbf{q}_1 - 3\mathbf{q}_2 + 2\mathbf{q}_3 \rangle \\ &= 4\langle \mathbf{q}_1,\ \mathbf{q}_1 \rangle + 9\langle \mathbf{q}_2,\ \mathbf{q}_2 \rangle + 4\langle \mathbf{q}_3,\ \mathbf{q}_3 \rangle && \left[\text{Because } \mathbf{q}_1,\ \mathbf{q}_2,\ \mathbf{q}_3 \text{ are orthonormal}\right] \\ &= 4\|\mathbf{q}_1\|^2 + 9\|\mathbf{q}_2\|^2 + 4\|\mathbf{q}_3\|^2 \\ &= 4(1) + 9(1) + 4(1) = 17 && \left[\text{Because } \mathbf{q}_1,\ \mathbf{q}_2,\ \mathbf{q}_3 \text{ are orthonormal}\right] \end{aligned}$$

Taking the square root gives $\|\mathbf{v}\| = \sqrt{17}$.

(b) We are given that $\mathbf{q}_1,\ \mathbf{q}_2,\ \mathbf{q}_3$ are orthonormal vectors in $\mathbb{R}^4$. We need to convert **u** so that it is an orthogonal vector to $\mathbf{q}_1,\ \mathbf{q}_2,\ \mathbf{q}_3$ by applying Gram-Schmidt (4-10):

$$\mathbf{w} = \mathbf{u} - \frac{\langle \mathbf{u},\ \mathbf{q}_1 \rangle}{\|\mathbf{q}_1\|^2}\mathbf{q}_1 - \frac{\langle \mathbf{u},\ \mathbf{q}_2 \rangle}{\|\mathbf{q}_2\|^2}\mathbf{q}_2 - \frac{\langle \mathbf{u},\ \mathbf{q}_3 \rangle}{\|\mathbf{q}_3\|^2}\mathbf{q}_3$$

Since $\mathbf{q}_1$, $\mathbf{q}_2$ and $\mathbf{q}_3$ are orthonormal vectors therefore $\|\mathbf{q}_1\|^2 = \|\mathbf{q}_2\|^2 = \|\mathbf{q}_3\|^2 = 1$. Thus the above can be written as

$$\mathbf{w} = \mathbf{u} - \langle \mathbf{u},\ \mathbf{q}_1 \rangle \mathbf{q}_1 - \langle \mathbf{u},\ \mathbf{q}_2 \rangle \mathbf{q}_2 - \langle \mathbf{u},\ \mathbf{q}_3 \rangle \mathbf{q}_3 \qquad \left\lfloor \text{Because } \|\mathbf{q}_1\|^2 = \|\mathbf{q}_2\|^2 = \|\mathbf{q}_3\|^2 = 1 \right\rfloor$$

However this vector **w** is **not** normalized. *How do we normalize this?*
The normalized vector is given by

$$\mathbf{w} = \frac{\mathbf{w}}{\|\mathbf{w}\|} \qquad \text{where } \mathbf{w} = \mathbf{u} - \langle \mathbf{u},\ \mathbf{q}_1 \rangle \mathbf{q}_1 - \langle \mathbf{u},\ \mathbf{q}_2 \rangle \mathbf{q}_2 - \langle \mathbf{u},\ \mathbf{q}_3 \rangle \mathbf{q}_3$$

(c) We have $\mathbf{u} = 2\mathbf{q}_1 - 3\mathbf{q}_2 + 2\mathbf{q}_3$ which means that the vector **u** is linearly dependent on $\mathbf{q}_1,\ \mathbf{q}_2,\ \mathbf{q}_3$. Gram Schmidt **cannot** be applied to linearly **dependent** vectors.

10. Finding $\mathbf{w}_1,\ \mathbf{w}_2,\ \mathbf{w}_3$ in $\mathbb{R}^5$ such that $\mathbf{w}_i \cdot \mathbf{w}_j = 0$ means that we need to find orthogonal vectors. *How?*
By using the Gram-Schmidt Process (4-10):

$$\mathbf{w}_1 = \mathbf{v}_1,\ \mathbf{w}_2 = \mathbf{v}_2 - \frac{\langle \mathbf{v}_2,\ \mathbf{w}_1 \rangle}{\|\mathbf{w}_1\|^2}\mathbf{w}_1 \text{ and } \mathbf{w}_3 = \mathbf{v}_3 - \frac{\langle \mathbf{v}_3,\ \mathbf{w}_1 \rangle}{\|\mathbf{w}_1\|^2}\mathbf{w}_1 - \frac{\langle \mathbf{v}_3,\ \mathbf{w}_2 \rangle}{\|\mathbf{w}_2\|^2}\mathbf{w}_2$$

We have $\mathbf{w}_1 = \mathbf{v}_1 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}$. Substituting $\mathbf{v}_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}$ and $\mathbf{w}_1 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}$ into $\langle \mathbf{v}_2, \ \mathbf{w}_1 \rangle$ gives

$$\langle \mathbf{v}_2, \ \mathbf{w}_1 \rangle = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} = 0+1+0+0+0 = 1$$

$$\|\mathbf{w}_1\|^2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} = 0^2 + 1^2 + 0^2 + 1^2 + 0^2 = 2$$

Hence we have

$$\mathbf{w}_2 = \mathbf{v}_2 - \frac{\langle \mathbf{v}_2, \ \mathbf{w}_1 \rangle}{\|\mathbf{w}_1\|^2}\mathbf{w}_1 = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} - \frac{1}{2}\begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0-0 \\ 1-1/2 \\ 1-0 \\ 0-1/2 \\ 0-0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1/2 \\ 1 \\ -1/2 \\ 0 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 0 \\ 1 \\ 2 \\ -1 \\ 0 \end{bmatrix}$$

Ignore the fraction and let $\mathbf{w}'_2 = \begin{bmatrix} 0 \\ 1 \\ 2 \\ -1 \\ 0 \end{bmatrix}$. *What else do we need to find?*

The vector $\mathbf{w}_3$ which is given by $\mathbf{w}_3 = \mathbf{v}_3 - \frac{\langle \mathbf{v}_3, \ \mathbf{w}_1 \rangle}{\|\mathbf{w}_1\|^2}\mathbf{w}_1 - \frac{\langle \mathbf{v}_3, \ \mathbf{w}'_2 \rangle}{\|\mathbf{w}'_2\|^2}\mathbf{w}'_2$:

$$\langle \mathbf{v}_3, \ \mathbf{w}_1 \rangle = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} = 0+1+0+1+0 = 2$$

$$\langle \mathbf{v}_3, \ \mathbf{w}'_2 \rangle = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 1 \\ 2 \\ -1 \\ 0 \end{bmatrix} = 0+1+0-1+0 = 0$$

We do **not** need to evaluate the remaining parts because $\langle \mathbf{v}_3, \ \mathbf{w}'_2 \rangle = 0$ which means $\mathbf{w}_3 = \mathbf{v}_3 - \frac{\langle \mathbf{v}_3, \ \mathbf{w}_1 \rangle}{\|\mathbf{w}_1\|^2}\mathbf{w}_1 - \frac{0}{\|\mathbf{w}'_2\|^2}\mathbf{w}'_2 = \mathbf{v}_3 - \frac{\langle \mathbf{v}_3, \ \mathbf{w}_1 \rangle}{\|\mathbf{w}_1\|^2}\mathbf{w}_1$:

$$\mathbf{w}_3 = \mathbf{v}_3 - \frac{\langle \mathbf{v}_3, \ \mathbf{w}_1 \rangle}{\|\mathbf{w}_1\|^2}\mathbf{w}_1 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \\ 1 \end{bmatrix} - \frac{2}{2}\begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0-0 \\ 1-1 \\ 0-0 \\ 1-1 \\ 1-0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

Thus our orthogonal vectors are $\mathbf{w}_1 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}$, $\mathbf{w}'_2 = \begin{bmatrix} 0 \\ 1 \\ 2 \\ -1 \\ 0 \end{bmatrix}$ and $\mathbf{w}_3 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$.

11. (a) and (b). We need to find the vectors $\mathbf{v} = \begin{bmatrix} v_1, & v_2 \end{bmatrix}$ and $\mathbf{w} = \begin{bmatrix} w_1, & w_2 \end{bmatrix}$ such that

$$\langle \mathbf{v}, \ \mathbf{w} \rangle = 3v_1w_1 + 2v_2w_2 = 0$$

Rearranging this we have

$$3v_1w_1 = -2v_2w_2$$
$$v_1w_1 = -\frac{2}{3}v_2w_2$$

Let $v_2w_2 = -3$. Let $v_2 = -1$ and $w_2 = 3$. For these values we have $v_1w_1 = 2$. Let $v_1 = 2$ and $w_1 = 1$. Our vectors are

$$\mathbf{v} = \begin{bmatrix} v_1, & v_2 \end{bmatrix} = \begin{bmatrix} 2, & -1 \end{bmatrix} \text{ and } \mathbf{w} = \begin{bmatrix} w_1, & w_2 \end{bmatrix} = \begin{bmatrix} 1, & 3 \end{bmatrix}$$

Since we are given that dot product of $\mathbf{v} = \begin{bmatrix} v_1, & v_2 \end{bmatrix}$ and $\mathbf{w} = \begin{bmatrix} w_1, & w_2 \end{bmatrix}$ is **not** zero which means that:

$$\langle \mathbf{v}, \ \mathbf{w} \rangle = v_1w_1 + v_2w_2 \neq 0$$

We need to check this holds for $\mathbf{v} = \begin{bmatrix} 2, & -1 \end{bmatrix}$ and $\mathbf{w} = \begin{bmatrix} 1, & 3 \end{bmatrix}$:

$$\begin{aligned} \langle \mathbf{v}, \ \mathbf{w} \rangle &= v_1w_1 + v_2w_2 \\ &= (2 \times 1) + (-1 \times 3) = -1 \neq 0 \end{aligned}$$

12. To find an orthonormal basis we apply Gram-Schmidt Process (4-10) which is

$$\mathbf{w}_1 = \mathbf{v}_1, \ \mathbf{w}_2 = \mathbf{v}_2 - \frac{\langle \mathbf{v}_2, \ \mathbf{w}_1 \rangle}{\|\mathbf{w}_1\|^2}\mathbf{w}_1 \text{ and } \mathbf{w}_3 = \mathbf{v}_3 - \frac{\langle \mathbf{v}_3, \ \mathbf{w}_1 \rangle}{\|\mathbf{w}_1\|^2}\mathbf{w}_1 - \frac{\langle \mathbf{v}_3, \ \mathbf{w}_2 \rangle}{\|\mathbf{w}_2\|^2}\mathbf{w}_2$$

The given inner product is $\langle \mathbf{f}, \ \mathbf{g} \rangle = \int_{-1}^{1} f(x)g(x)\,dx$.

We need to find an orthonormal basis of space V spanned by $\{\mathbf{1}, \ \mathbf{x}\}$.

Let $\mathbf{w}_1 = \mathbf{v}_1 = \mathbf{1}$ and $\mathbf{v}_2 = \mathbf{x}$. We need to determine $\mathbf{w}_2 = \mathbf{v}_2 - \frac{\langle \mathbf{v}_2, \ \mathbf{w}_1 \rangle}{\|\mathbf{w}_1\|^2}\mathbf{w}_1$:

$$\begin{aligned}\langle \mathbf{v}_2,\ \mathbf{w}_1\rangle &= \langle \mathbf{x},\ \mathbf{1}\rangle\\ &= \int_{-1}^{1} x(1)\,dx\\ &= \left[\frac{x^2}{2}\right]_{-1}^{1} = \frac{1}{2}[1-1] = 0\end{aligned}$$

This means that our given vectors were already orthogonal. Since $\mathbf{v}_2 = \mathbf{x}$ and $\mathbf{w}_1 = \mathbf{1}$ are orthogonal therefore $\mathbf{w}_2 = \mathbf{v}_2 = \mathbf{x}$.

Our orthogonal basis is $\{\mathbf{1},\ \mathbf{x}\}$. We need to normalise these vectors to obtain a set of orthonormal basis.

Evaluating $\|\mathbf{w}_1\|^2$:

$$\begin{aligned}\|\mathbf{w}_1\|^2 &= \langle \mathbf{1},\ \mathbf{1}\rangle\\ &= \int_{-1}^{1} (1\times 1)\,dx = [x]_{-1}^{1} = 1-(-1) = 2\end{aligned}$$

$$\mathbf{w}_1 = \frac{\mathbf{w}_1}{\|\mathbf{w}_1\|} = \frac{1}{\sqrt{2}} \qquad \left[\text{Because } \|\mathbf{w}_1\|^2 = 2 \text{ from above}\right]$$

Similarly we have $\mathbf{w}_2 = \frac{\mathbf{w}_2}{\|\mathbf{w}_2\|}$. From above $\mathbf{w}_2 = \mathbf{x}$ but we need to find $\|\mathbf{w}_2\|$:

$$\begin{aligned}\|\mathbf{w}_2\|^2 &= \langle \mathbf{x},\ \mathbf{x}\rangle\\ &= \int_{-1}^{1} xx\ \ dx\\ &= \int_{-1}^{1} x^2\ dx = \left[\frac{x^3}{3}\right]_{-1}^{1} = \frac{1}{3}\left[1-(-1)^3\right] = \frac{2}{3}\end{aligned}$$

Taking the square root gives $\|\mathbf{w}_2\| = \sqrt{\frac{2}{3}}$. Substituting $\mathbf{w}_2 = \mathbf{x}$ and $\|\mathbf{w}_2\| = \sqrt{\frac{2}{3}}$ into $\mathbf{w}_2 = \frac{\mathbf{w}_2}{\|\mathbf{w}_2\|}$ gives

$$\mathbf{w}_2 = \frac{\mathbf{x}}{\sqrt{2/3}} = \sqrt{\frac{3}{2}}\mathbf{x}$$

Hence our orthonormal basis is $\left\{\frac{1}{\sqrt{2}},\ \sqrt{\frac{3}{2}}\mathbf{x}\right\}$.

13. (a) We first need to show that $\langle \mathbf{A},\ \mathbf{B}\rangle = tr\left(\mathbf{A}^T\mathbf{B}\right)$ is an inner product (tr( ) is the trace of the matrix).

By definition (4-1) an inner product satisfies:

(a) $\langle \mathbf{u},\ \mathbf{v}\rangle = \langle \mathbf{v},\ \mathbf{u}\rangle$ [Commutative Law]

(b) $\langle \mathbf{u}+\mathbf{v},\ \mathbf{w}\rangle = \langle \mathbf{u},\ \mathbf{w}\rangle + \langle \mathbf{v},\ \mathbf{w}\rangle$ [Distributive Law]

(c) $\langle k\mathbf{u},\mathbf{v}\rangle = k\langle \mathbf{u},\ \mathbf{v}\rangle$

(d) $\langle \mathbf{u},\ \mathbf{u}\rangle \ge 0$ and we have $\langle \mathbf{u},\ \mathbf{u}\rangle = 0$ if and only if $\mathbf{u}=\mathbf{O}$

<u>Checking (a)</u>:

$\mathbf{A}=\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ Let and $\mathbf{B}=\begin{bmatrix} e & f \\ g & h \end{bmatrix}$ then

$$\begin{aligned}\langle \mathbf{A},\ \mathbf{B}\rangle &= tr\left(\mathbf{A}^T\mathbf{B}\right)\\ &= tr\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}^T \begin{bmatrix} e & f \\ g & h \end{bmatrix}\right)\\ &= tr\left(\begin{bmatrix} a & c \\ b & d \end{bmatrix}\begin{bmatrix} e & f \\ g & h \end{bmatrix}\right)\\ &= tr\begin{pmatrix} ae+cg & * \\ ** & bf+dh \end{pmatrix} = ae+cg+bf+dh\end{aligned}$$

The * and ** signify that we do **not** need to evaluate these entries because the trace of a matrix is the addition of the leading diagonal elements.

Evaluating $\langle \mathbf{B},\ \mathbf{A}\rangle = tr\left(\mathbf{B}^T\mathbf{A}\right)$:

$$\begin{aligned}\langle \mathbf{B},\ \mathbf{A}\rangle &= tr\left(\mathbf{B}^T\mathbf{A}\right)\\ &= tr\left(\begin{bmatrix} e & f \\ g & h \end{bmatrix}^T \begin{bmatrix} a & b \\ c & d \end{bmatrix}\right)\\ &= tr\left(\begin{bmatrix} e & g \\ f & h \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right)\\ &= tr\begin{pmatrix} ae+gc & \dagger \\ \dagger\dagger & fb+hd \end{pmatrix} = ae+gc+fb+hd = \langle \mathbf{A},\ \mathbf{B}\rangle \qquad \left[\text{From Above}\right]\end{aligned}$$

Hence we have $\langle \mathbf{A},\ \mathbf{B}\rangle = \langle \mathbf{B},\ \mathbf{A}\rangle$ which means that (a) holds.

<u>Checking (b)</u>:

We are assuming $tr(\mathbf{A}+\mathbf{B}) = tr(\mathbf{A})+tr(\mathbf{B})$. Let $\mathbf{C}\in M_{2,\ 2}(\mathbb{R})$:

$$\begin{aligned}\langle \mathbf{A}+\mathbf{B},\ \mathbf{C}\rangle &= tr\left[(\mathbf{A}+\mathbf{B})^T\,\mathbf{C}\right]\\ &= tr\left[\left(\mathbf{A}^T+\mathbf{B}^T\right)\mathbf{C}\right] \qquad \left[\text{Because } (\mathbf{A}+\mathbf{B})^T = \mathbf{A}^T+\mathbf{B}^T\right]\\ &= tr\left[\mathbf{A}^T\mathbf{C}+\mathbf{B}^T\mathbf{C}\right]\\ &= tr\left[\mathbf{A}^T\mathbf{C}\right]+tr\left[\mathbf{B}^T\mathbf{C}\right] = \langle \mathbf{A},\ \mathbf{C}\rangle + \langle \mathbf{B},\ \mathbf{C}\rangle\end{aligned}$$

We have $\langle \mathbf{A}+\mathbf{B},\ \mathbf{C}\rangle = \langle \mathbf{A},\ \mathbf{C}\rangle + \langle \mathbf{B},\ \mathbf{C}\rangle$ which means that part (b) holds.

<u>Checking (c)</u>:

We assume $tr\left((k\mathbf{A})\mathbf{B}\right) = k\left[tr(\mathbf{AB})\right]$:

$$\begin{aligned}\langle k\mathbf{A},\ \mathbf{B}\rangle &= tr\left((k\mathbf{A})^T,\ \mathbf{B}\right)\\ &= tr\left(k\mathbf{A}^T,\ \mathbf{B}\right) = k\left[tr\left(\mathbf{A}^T,\ \mathbf{B}\right)\right] = k\langle \mathbf{A},\ \mathbf{B}\rangle\end{aligned}$$

Since we have $\langle k\mathbf{A},\ \mathbf{B}\rangle = k\langle \mathbf{A},\ \mathbf{B}\rangle$ therefore part (c) holds.

Checking (d):
We have

$$\begin{aligned}\langle \mathbf{A},\ \mathbf{A}\rangle &= tr\left(\mathbf{A}^T\mathbf{A}\right)\\ &= tr\left(\begin{bmatrix} a & b \\ c & d\end{bmatrix}^T \begin{bmatrix} a & b \\ c & d\end{bmatrix}\right)\\ &= tr\left(\begin{bmatrix} a & c \\ b & d\end{bmatrix}\begin{bmatrix} a & b \\ c & d\end{bmatrix}\right)\\ &= tr\left(\begin{bmatrix} a^2+c^2 & ? \\ ?? & b^2+d^2\end{bmatrix}\right) = a^2+c^2+b^2+d^2 \geq 0\end{aligned}$$

This means $\langle \mathbf{A},\ \mathbf{A}\rangle = a^2+c^2+b^2+d^2 \geq 0$. Also

$$\langle \mathbf{A},\ \mathbf{A}\rangle = a^2+c^2+b^2+d^2 = 0 \quad \Leftrightarrow \quad a=b=c=d=0$$

With $a=b=c=d=0$ which means we have $\mathbf{A} = \begin{bmatrix} 0 & 0 \\ 0 & 0\end{bmatrix} = \mathbf{O}$.

Hence part (d) holds.
Since all four parts (a), (b), (c) and (d) of definition (4-1) is satisfied therefore $\langle \mathbf{A},\ \mathbf{B}\rangle = tr\left(\mathbf{A}^T\mathbf{B}\right)$ is an inner product on $M_{2,\ 2}\left(\mathbb{R}\right)$.

(b) The distance between $\mathbf{A} = \begin{bmatrix} 1 & 2 \\ 1 & 0\end{bmatrix}$ and $\mathbf{B} = \begin{bmatrix} 3 & 3 \\ 1 & 2\end{bmatrix}$ is given by

$$\|\mathbf{A}-\mathbf{B}\|^2 = \langle \mathbf{A}-\mathbf{B},\ \mathbf{A}-\mathbf{B}\rangle \qquad (*)$$

Carrying out the subtraction $\mathbf{A}-\mathbf{B} = \begin{bmatrix} 1 & 2 \\ 1 & 0\end{bmatrix} - \begin{bmatrix} 3 & 3 \\ 1 & 2\end{bmatrix} = \begin{bmatrix} -2 & -1 \\ 0 & -2\end{bmatrix}$. Substituting this into (*) gives

$$\begin{aligned}\|\mathbf{A}-\mathbf{B}\|^2 &= \left\langle \begin{bmatrix} -2 & -1 \\ 0 & -2\end{bmatrix},\ \begin{bmatrix} -2 & -1 \\ 0 & -2\end{bmatrix}\right\rangle\\ &= tr\left(\begin{bmatrix} -2 & -1 \\ 0 & -2\end{bmatrix}^T \begin{bmatrix} -2 & -1 \\ 0 & -2\end{bmatrix}\right)\\ &= tr\left(\begin{bmatrix} -2 & 0 \\ -1 & -2\end{bmatrix}\begin{bmatrix} -2 & -1 \\ 0 & -2\end{bmatrix}\right)\\ &= tr\left(\begin{bmatrix} 4 & 2 \\ 2 & 5\end{bmatrix}\right) = 4+5 = 9\end{aligned}$$

Taking the square root gives $\|\mathbf{A}-\mathbf{B}\| = \sqrt{9} = 3$.

(c) The angle θ between $\mathbf{A} = \begin{bmatrix} 1 & 2 \\ 1 & 0\end{bmatrix}$ and $\mathbf{B} = \begin{bmatrix} 3 & 3 \\ 1 & 2\end{bmatrix}$ is given by

$$\cos(\theta) = \frac{\langle \mathbf{A},\ \mathbf{B}\rangle}{\|\mathbf{A}\|\|\mathbf{B}\|} \qquad (£)$$

We can evaluate each component:

$$\|\mathbf{A}\|^2 = \langle \mathbf{A},\ \mathbf{A} \rangle = tr\left(\mathbf{A}^T\mathbf{A}\right)$$
$$= tr\left(\begin{bmatrix} 1 & 2 \\ 1 & 0 \end{bmatrix}^T \begin{bmatrix} 1 & 2 \\ 1 & 0 \end{bmatrix}\right)$$
$$= tr\left(\begin{bmatrix} 1 & 1 \\ 2 & 0 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 1 & 0 \end{bmatrix}\right) = tr\begin{pmatrix} 2 & 2 \\ 2 & 4 \end{pmatrix} = 2+4=6$$

Taking the square root gives $\|\mathbf{A}\| = \sqrt{6}$. Similarly we have

$$\|\mathbf{B}\|^2 = \langle \mathbf{B},\ \mathbf{B} \rangle = tr\left(\mathbf{B}^T\mathbf{B}\right)$$
$$= tr\left(\begin{bmatrix} 3 & 3 \\ 1 & 2 \end{bmatrix}^T \begin{bmatrix} 3 & 3 \\ 1 & 2 \end{bmatrix}\right)$$
$$= tr\left(\begin{bmatrix} 3 & 1 \\ 3 & 2 \end{bmatrix}\begin{bmatrix} 3 & 3 \\ 1 & 2 \end{bmatrix}\right) = tr\begin{pmatrix} 10 & 11 \\ 11 & 13 \end{pmatrix} = 10+13=23$$

Taking the square root gives $\|\mathbf{B}\| = \sqrt{23}$.

Evaluating out the numerator in the above (£):

$$\langle \mathbf{A},\ \mathbf{B} \rangle = tr\left(\mathbf{A}^T\mathbf{B}\right)$$
$$= tr\left(\begin{bmatrix} 1 & 2 \\ 1 & 0 \end{bmatrix}^T \begin{bmatrix} 3 & 3 \\ 1 & 2 \end{bmatrix}\right)$$
$$= tr\left(\begin{bmatrix} 1 & 1 \\ 2 & 0 \end{bmatrix}\begin{bmatrix} 3 & 3 \\ 1 & 2 \end{bmatrix}\right) = tr\left(\begin{bmatrix} 4 & 5 \\ 6 & 6 \end{bmatrix}\right) = 4+6=10$$

Substituting $\|\mathbf{A}\| = \sqrt{6}$, $\|\mathbf{B}\| = \sqrt{23}$ and $\langle \mathbf{A},\ \mathbf{B} \rangle = 10$ into (£) gives:

$$\cos(\theta) = \frac{\langle \mathbf{A},\ \mathbf{B} \rangle}{\|\mathbf{A}\|\|\mathbf{B}\|} = \frac{10}{\sqrt{6}\sqrt{23}}$$

Taking inverse cos yields:

$$\theta = \cos^{-1}\left(\frac{10}{\sqrt{6}\sqrt{23}}\right) = 31.65°$$

14. We are given that $\mathbf{f}(t) = c_0 + c_1 t + c_2 t^2$ and $\mathbf{f}\left(\frac{1}{2}\right) = 0$. Substituting $t = \frac{1}{2}$ into $\mathbf{f}(t) = c_0 + c_1 t + c_2 t^2$ gives

$$\mathbf{f}\left(\frac{1}{2}\right) = c_0 + c_1\left(\frac{1}{2}\right) + c_2\left(\frac{1}{2}\right)^2 = c_0 + \frac{c_1}{2} + \frac{c_2}{4} = 0$$

Multiplying both sides by 4 gives

$$4c_0 + 2c_1 + c_2 = 0 \quad \Rightarrow \quad c_2 = -4c_0 - 2c_1 \qquad (*)$$

We are also given that $\mathbf{f}(t) = c_0 + c_1 t + c_2 t^2$ is orthogonal to $1+t$:

$$\left\langle 1+t,\ c_0+c_1t+c_2t^2\right\rangle = \int_0^1 (1+t)\left(c_0+c_1t+c_2t^2\right) dt$$
$$= \int_0^1 \left[\left(c_0+c_1t+c_2t^2\right)+\left(c_0t+c_1t^2+c_2t^3\right)\right] dt$$
$$= \int_0^1 \left[c_0+(c_1+c_0)t+(c_2+c_1)t^2+c_2t^3\right] dt$$
$$= \left[c_0t+\frac{(c_1+c_0)t^2}{2}+\frac{(c_2+c_1)t^3}{3}+\frac{c_2t^4}{4}\right]_0^1$$
$$= c_0+\frac{(c_1+c_0)}{2}+\frac{(c_2+c_1)}{3}+\frac{c_2}{4}=0 \qquad \left[\text{Because of orthogonality}\right]$$

Multiplying the last equation by 12 gives

$$12c_0+6(c_1+c_0)+4(c_2+c_1)+3c_2=0$$
$$18c_0+10c_1+7c_2=0$$

Substituting $c_2=-4c_0-2c_1$ which is (*) above into this last equation $18c_0+10c_1+7c_2=0$:

$$18c_0+10c_1+7(-4c_0-2c_1)=0$$
$$-10c_0-4c_1=0$$
$$c_1=-\frac{10}{4}c_0$$

Let $c_0=4$ then $c_1=-\frac{10}{4}(4)=-10$. Substituting $c_0=4$ and $c_1=-10$ into $c_2=-4c_0-2c_1$ gives

$$c_2=-4(4)-2(-10)=4$$

Thus we have $c_0=4$, $c_1=-10$ and $c_2=4$ which yields that

$$\mathbf{f}(t)=c_0+c_1t+c_2t^2=4-10t+4t^2$$

A basis for the given subspace S is $\left\{\mathbf{f}(t)=4-10t+4t^2\right\}$.

15. We can start with standard basis for P_2 which is the set of all quadratic polynomials and then apply Gram-Schmidt process to convert into an orthogonal basis. *What is the standard basis for P_2?*

$\left\{\mathbf{1},\ \mathbf{x},\ \mathbf{x}^2\right\}$. Let $\mathbf{v}_1=\mathbf{1}$, $\mathbf{v}_2=\mathbf{x}$ and $\mathbf{v}_3=\mathbf{x}^2$. Gram-Schmidt Process (4-10) is

$$\mathbf{w}_1=\mathbf{v}_1,\ \mathbf{w}_2=\mathbf{v}_2-\frac{\left\langle \mathbf{v}_2,\ \mathbf{w}_1\right\rangle}{\|\mathbf{w}_1\|^2}\mathbf{w}_1 \text{ and } \mathbf{w}_3=\mathbf{v}_3-\frac{\left\langle \mathbf{v}_3,\ \mathbf{w}_1\right\rangle}{\|\mathbf{w}_1\|^2}\mathbf{w}_1-\frac{\left\langle \mathbf{v}_3,\ \mathbf{w}_2\right\rangle}{\|\mathbf{w}_2\|^2}\mathbf{w}_2$$

The inner product is given by $\left\langle \mathbf{f},\ \mathbf{g}\right\rangle=\int_0^1 \mathbf{f}(x)\,\mathbf{g}(1-x)\,dx$.

We have $\mathbf{w}_1=\mathbf{v}_1=\mathbf{1}$. We need to determine $\mathbf{w}_2$ by using the above formula. We have $\mathbf{v}_2=\mathbf{x}$ and $\mathbf{w}_1=1$ which means that we can evaluate

$$\langle \mathbf{v}_2,\ \mathbf{w}_1\rangle = \langle \mathbf{x},\ \mathbf{1}\rangle = \int_0^1 x(1)\,dx$$

$$= \int_0^1 x\,dx = \left[\frac{x^2}{2}\right]_0^1 = \frac{1}{2}$$

$$\|\mathbf{w}_1\|^2 = \langle \mathbf{1},\ \mathbf{1}\rangle = \int_0^1 1\,dx = [x]_0^1 = 1$$

Substituting $\mathbf{v}_2 = \mathbf{x}$, $\mathbf{w}_1 = \mathbf{1}$, $\langle \mathbf{v}_2,\ \mathbf{w}_1\rangle = \frac{1}{2}$ and $\|\mathbf{w}_1\|^2 = 1$ into $\mathbf{w}_2 = \mathbf{v}_2 - \frac{\langle \mathbf{v}_2,\ \mathbf{w}_1\rangle}{\|\mathbf{w}_1\|^2}\mathbf{w}_1$:

$$\mathbf{w}_2 = \mathbf{x} - \frac{1}{2}(1) = \mathbf{x} - \frac{1}{2}$$

What else do we need to find?

$$\mathbf{w}_3 = \mathbf{v}_3 - \frac{\langle \mathbf{v}_3,\ \mathbf{w}_1\rangle}{\|\mathbf{w}_1\|^2}\mathbf{w}_1 - \frac{\langle \mathbf{v}_3,\ \mathbf{w}_2\rangle}{\|\mathbf{w}_2\|^2}\mathbf{w}_2$$

Evaluating each of the components gives:

$$\langle \mathbf{v}_3,\ \mathbf{w}_1\rangle = \langle \mathbf{x}^2,\ \mathbf{1}\rangle = \int_0^1 x^2(1)\,dx$$

$$= \left[\frac{x^3}{3}\right]_0^1 = \frac{1}{3}$$

In the next two evaluations we need to find $\mathbf{g}(1-x)$ given that $\mathbf{g}(x) = x - \frac{1}{2}$:

$$\mathbf{g}(1-x) = 1 - x - \frac{1}{2} = \frac{1}{2} - x \qquad (\dagger)$$

We have

$$\langle \mathbf{v}_3,\ \mathbf{w}_2\rangle = \left\langle \mathbf{x}^2,\ \mathbf{x} - \frac{1}{2}\right\rangle \qquad \left[\text{Because } \mathbf{w}_2 = \mathbf{x} - \frac{1}{2} \text{ from above}\right]$$

$$= \int_0^1 x^2\left(\frac{1}{2} - x\right)dx \qquad [\text{By } (\dagger)]$$

$$= \int_0^1 \left(\frac{x^2}{2} - x^3\right)dx$$

$$= \left[\frac{x^3}{6} - \frac{x^4}{4}\right]_0^1 \underset{\text{Substituting Limits}}{=} \frac{1}{6} - \frac{1}{4} = -\frac{1}{12}$$

Also

$$\begin{aligned}\|\mathbf{w}_2\|^2 &= \langle \mathbf{w}_2,\ \mathbf{w}_2\rangle \\ &= \left\langle \mathbf{x}-\frac{1}{2},\ \mathbf{x}-\frac{1}{2}\right\rangle \\ &= \int_0^1 \left(x-\frac{1}{2}\right)\left(\frac{1}{2}-x\right)dx \\ &= \int_0^1 \left(\frac{1}{2}x - x^2 - \frac{1}{4}+\frac{1}{2}x\right)dx \\ &= \int_0^1 \left(x-x^2-\frac{1}{4}\right)dx = \left[\frac{x^2}{2}-\frac{x^3}{3}-\frac{x}{4}\right]_0^1 = \frac{1}{2}-\frac{1}{3}-\frac{1}{4} = -\frac{1}{12}\end{aligned}$$

Substituting $\mathbf{v}_3 = \mathbf{x}^2$, $\langle \mathbf{v}_3,\ \mathbf{w}_1\rangle = \frac{1}{3}$, $\|\mathbf{w}_1\|^2 = 1$, $\mathbf{w}_1 = \mathbf{1}$, $\langle \mathbf{v}_3,\ \mathbf{w}_2\rangle = -\frac{1}{12}$, $\|\mathbf{w}_2\|^2 = -\frac{1}{12}$ and $\mathbf{w}_2 = \mathbf{x}-\frac{1}{2}$ into $\mathbf{w}_3 = \mathbf{v}_3 - \frac{\langle \mathbf{v}_3,\ \mathbf{w}_1\rangle}{\|\mathbf{w}_1\|^2}\mathbf{w}_1 - \frac{\langle \mathbf{v}_3,\ \mathbf{w}_2\rangle}{\|\mathbf{w}_2\|^2}\mathbf{w}_2$ gives

$$\begin{aligned}\mathbf{w}_3 &= \mathbf{x}^2 - \frac{1}{3}(\mathbf{1}) - \frac{-1/12}{-1/12}\left(\mathbf{x}-\frac{1}{2}\right) \\ &= \mathbf{x}^2 - \frac{1}{3} - \left(\mathbf{x}-\frac{1}{2}\right) = \mathbf{x}^2 - \mathbf{x} + \frac{1}{6}\end{aligned}$$

Our orthogonal basis is $\left\{\mathbf{1},\ \mathbf{x}-\frac{1}{2},\ \mathbf{x}^2-\mathbf{x}+\frac{1}{6}\right\}$.

16. We need to show why $\langle \mathbf{A},\ \mathbf{B}\rangle = tr(\mathbf{AB})$ is **not** an inner product.

Let $\mathbf{A} = \begin{pmatrix} a & b \\ c & d\end{pmatrix}$. We can check that part (d) of definition (4-1) does **not** hold, that is

(d) $\langle \mathbf{u},\ \mathbf{u}\rangle \ge 0$ and we have $\langle \mathbf{u},\ \mathbf{u}\rangle = 0$ if and only if $\mathbf{u} = \mathbf{O}$

It is enough to show that for particular entries *a*, *b*, *c* and *d* of matrix **A** we have $\langle \mathbf{A},\ \mathbf{A}\rangle < 0$. Evaluating $\langle \mathbf{A},\ \mathbf{A}\rangle$ by using $\langle \mathbf{A},\ \mathbf{B}\rangle = tr(\mathbf{AB})$:

$$\begin{aligned}\langle \mathbf{A},\ \mathbf{A}\rangle &= tr(\mathbf{AA}) \\ &= tr\left(\begin{pmatrix} a & b \\ c & d\end{pmatrix}\begin{pmatrix} a & b \\ c & d\end{pmatrix}\right) \\ &= tr\begin{pmatrix} a^2+bc & * \\ ** & cb+d^2\end{pmatrix} = a^2+bc+cb+d^2 = a^2+2bc+d^2\end{aligned}$$

Now chose any values of *a*, *b*, *c* and *d* so that $2bc < a^2+d^2$. For example let $a = d = 1$ and $b = 1,\ c = -2$ then from above we have

$$\langle \mathbf{A},\ \mathbf{A}\rangle = a^2+2bc+d^2 = 1^2+2(1)(-2)+1^2 = -2 < 0$$

Hence we have $\langle \mathbf{A},\ \mathbf{A}\rangle < 0$ which means that part (d) of definition (4-1) fails so $\langle \mathbf{A},\ \mathbf{B}\rangle = tr(\mathbf{AB})$ **cannot** be an inner product.

17. Remember Proposition (4-11) is:

Let $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \cdots, \mathbf{v}_n\}$ be an **orthonormal** set of vectors in an inner product space *V* of dimension *n*. Let $\mathbf{u} \in V$ then

$$\mathbf{u} = \langle \mathbf{u}, \mathbf{v}_1 \rangle \mathbf{v}_1 + \langle \mathbf{u}, \mathbf{v}_2 \rangle \mathbf{v}_2 + \langle \mathbf{u}, \mathbf{v}_3 \rangle \mathbf{v}_3 + \cdots + \langle \mathbf{u}, \mathbf{v}_n \rangle \mathbf{v}_n$$

Proof.

Since $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \cdots, \mathbf{v}_n\}$ is an **orthonormal** set of vectors so they are linearly independent which means that they form a basis for the inner product space *V* because *V* has dimension *n*. Let $\mathbf{u} \in V$ then there exists scalars $k_1, k_2, k_3, \cdots, k_n$ such that

$$\mathbf{u} = k_1\mathbf{v}_1 + k_2\mathbf{v}_2 + k_3\mathbf{v}_3 + \cdots + k_n\mathbf{v}_n \qquad (*)$$

Required to prove that $k_j = \langle \mathbf{u}, \mathbf{v}_j \rangle$ for $j = 1, 2, 3, \cdots, n$. Consider the inner product $\langle \mathbf{u}, \mathbf{v}_j \rangle$ for any $j = 1, 2, 3, \cdots, n$:

$$\begin{aligned}
\langle \mathbf{u}, \mathbf{v}_j \rangle &= \langle k_1\mathbf{v}_1 + \cdots + k_j\mathbf{v}_j + \cdots + k_n\mathbf{v}_n, \mathbf{v}_j \rangle && \left[\text{By } (*) \quad \mathbf{u} = k_1\mathbf{v}_1 + \cdots + k_n\mathbf{v}_n\right] \\
&= k_1\langle \mathbf{v}_1, \mathbf{v}_j \rangle + \cdots + k_j\langle \mathbf{v}_j, \mathbf{v}_j \rangle + \cdots + k_n\langle \mathbf{v}_n, \mathbf{v}_j \rangle \\
&= k_1(0) + \cdots + k_j\|\mathbf{v}_j\|^2 + \cdots + k_n(0) && \left[\{\mathbf{v}_1, \cdots, \mathbf{v}_n\} \text{ are orthogonal}\right] \\
&= k_j\|\mathbf{v}_j\|^2 = k_j(1) = k_j && \left[\text{Because } \mathbf{v}_j \text{ is orthonormal}\right]
\end{aligned}$$

Hence we have shown that $k_j = \langle \mathbf{u}, \mathbf{v}_j \rangle$ for any $j = 1, 2, 3, \cdots, n$. Substituting this $k_j = \langle \mathbf{u}, \mathbf{v}_j \rangle$ into (*) gives us our required result:

$$\mathbf{u} = \langle \mathbf{u}, \mathbf{v}_1 \rangle \mathbf{v}_1 + \langle \mathbf{u}, \mathbf{v}_2 \rangle \mathbf{v}_2 + \langle \mathbf{u}, \mathbf{v}_3 \rangle \mathbf{v}_3 + \cdots + \langle \mathbf{u}, \mathbf{v}_n \rangle \mathbf{v}_n$$

■

18. We need to use Gram-Schmidt Process (4-10) which is:

$$\mathbf{w}_1 = \mathbf{v}_1, \ \mathbf{w}_2 = \mathbf{v}_2 - \frac{\langle \mathbf{v}_2, \mathbf{w}_1 \rangle}{\|\mathbf{w}_1\|^2}\mathbf{w}_1 \text{ and } \mathbf{w}_3 = \mathbf{v}_3 - \frac{\langle \mathbf{v}_3, \mathbf{w}_1 \rangle}{\|\mathbf{w}_1\|^2}\mathbf{w}_1 - \frac{\langle \mathbf{v}_3, \mathbf{w}_2 \rangle}{\|\mathbf{w}_2\|^2}\mathbf{w}_2$$

Let $\mathbf{v}_1 = \mathbf{1}, \mathbf{v}_2 = \mathbf{x}, \mathbf{v}_3 = \mathbf{x}^2, \mathbf{v}_4 = \mathbf{x}^3$. Applying Gram-Schmidt we have $\mathbf{w}_1 = \mathbf{v}_1 = \mathbf{1}$.

The inner product is given by $\langle \mathbf{f}, \mathbf{g} \rangle = \int_{-1}^{1} f(x)\, g(x)\, dx$. We need to find $\mathbf{w}_2 = \mathbf{v}_2 - \frac{\langle \mathbf{v}_2, \mathbf{w}_1 \rangle}{\|\mathbf{w}_1\|^2}\mathbf{w}_1$ by evaluating each component:

$$\begin{aligned}
\langle \mathbf{v}_2, \mathbf{w}_1 \rangle &= \langle \mathbf{x}, \mathbf{1} \rangle \\
&= \int_{-1}^{1} x(1)\, dx = \left[\frac{x^2}{2}\right]_{-1}^{1} \underset{\text{Substituting Limits}}{=} \frac{1}{2}\left[1^2 - (-1)^2\right] = 0
\end{aligned}$$

What does the zero result signify?

Means that **x** and **1** are already orthogonal therefore $\mathbf{w}_2 = \mathbf{x}$.

Next we determine each of the components of $\mathbf{w}_3 = \mathbf{v}_3 - \frac{\langle \mathbf{v}_3, \mathbf{w}_1 \rangle}{\|\mathbf{w}_1\|^2}\mathbf{w}_1 - \frac{\langle \mathbf{v}_3, \mathbf{w}_2 \rangle}{\|\mathbf{w}_2\|^2}\mathbf{w}_2$:

$$\langle \mathbf{v}_3,\ \mathbf{w}_1\rangle = \langle \mathbf{x}^2,\ \mathbf{1}\rangle$$

$$= \int_{-1}^{1} x^2\,(1)\,dx = \left[\frac{x^3}{3}\right]_{-1}^{1} \underset{\text{Substituting Limits}}{=} \frac{1}{3}\left[1^3 - (-1)^3\right] = \frac{2}{3}$$

$$\|\mathbf{w}_1\|^2 = \langle \mathbf{w}_1,\ \mathbf{w}_1\rangle$$

$$= \langle \mathbf{1},\ \mathbf{1}\rangle = \int_{-1}^{1} 1(1)\,dx = [x]_{-1}^{1} \underset{\text{Substituting Limits}}{=} [1-(-1)] = 2$$

$$\langle \mathbf{v}_3,\ \mathbf{w}_2\rangle = \langle \mathbf{x}^2,\ \mathbf{x}\rangle$$

$$= \int_{-1}^{1} x^2\,(x)\,dx = \int_{-1}^{1} x^3 dx = \left[\frac{x^4}{4}\right]_{-1}^{1} \underset{\text{Substituting Limits}}{=} \frac{1}{4}\left[1^4 - (-1)^4\right] = 0$$

As above this $\langle \mathbf{v}_3,\ \mathbf{w}_2\rangle = \langle \mathbf{x}^2,\ \mathbf{x}\rangle = 0$ means $\mathbf{x}^2$ and $\mathbf{x}$ are orthogonal.

Substituting $\mathbf{v}_3 = \mathbf{x}^2$, $\langle \mathbf{v}_3,\ \mathbf{w}_1\rangle = \frac{2}{3}$, $\|\mathbf{w}_1\|^2 = 2$, $\mathbf{w}_1 = \mathbf{1}$ and $\langle \mathbf{v}_3,\ \mathbf{w}_2\rangle = 0$ into

$\mathbf{w}_3 = \mathbf{v}_3 - \dfrac{\langle \mathbf{v}_3,\ \mathbf{w}_1\rangle}{\|\mathbf{w}_1\|^2}\mathbf{w}_1 - \dfrac{\langle \mathbf{v}_3,\ \mathbf{w}_2\rangle}{\|\mathbf{w}_2\|^2}\mathbf{w}_2$ gives

$$\mathbf{w}_3 = \mathbf{x}^2 - \frac{2/3}{2}(\mathbf{1}) - 0\mathbf{w}_2$$

$$= \mathbf{x}^2 - \frac{1}{3}$$

What else do we need to find?

$\mathbf{w}_4$ which is given by $\mathbf{w}_4 = \mathbf{v}_4 - \dfrac{\langle \mathbf{v}_4,\ \mathbf{w}_1\rangle}{\|\mathbf{w}_1\|^2}\mathbf{w}_1 - \dfrac{\langle \mathbf{v}_4,\ \mathbf{w}_2\rangle}{\|\mathbf{w}_2\|^2}\mathbf{w}_2 - \dfrac{\langle \mathbf{v}_4,\ \mathbf{w}_3\rangle}{\|\mathbf{w}_3\|^2}\mathbf{w}_3$. Evaluating each of these we have

$$\langle \mathbf{v}_4,\ \mathbf{w}_1\rangle = \langle \mathbf{x}^3,\ \mathbf{1}\rangle$$

$$= \int_{-1}^{1} x^3\,(1)\,dx = \left[\frac{x^4}{4}\right]_{-1}^{1} = \frac{1}{4}\left[1^4 - (-1)^4\right] = 0$$

$$\langle \mathbf{v}_4,\ \mathbf{w}_2\rangle = \langle \mathbf{x}^3,\ \mathbf{x}\rangle$$

$$= \int_{-1}^{1} x^3\,(x)\,dx = \left[\frac{x^5}{5}\right]_{-1}^{1} = \frac{1}{5}\left[1^5 - (-1)^5\right] = \frac{2}{5}$$

$$\langle \mathbf{v}_4,\ \ \mathbf{w}_3\rangle = \left\langle \mathbf{x}^3,\ \ \mathbf{x}^2 - \frac{1}{3}\right\rangle$$

$$= \int_{-1}^{1} x^3\left(x^2 - \frac{1}{3}\right)dx$$

$$= \int_{-1}^{1}\left(x^5 - \frac{x^3}{3}\right)dx$$

$$= \left[\frac{x^6}{6} - \frac{x^4}{12}\right]_{-1}^{1} = \frac{1}{12}\left[2(1)^6 - (1)^4 - \left(2(-1)^6 - (-1)^4\right)\right]_{-1}^{1} = 0$$

$$\begin{aligned}\|\mathbf{w}_2\|^2 &= \langle \mathbf{w}_2, \mathbf{w}_2 \rangle \\ &= \langle \mathbf{x},\ \mathbf{x} \rangle \\ &= \int_{-1}^{1} x(x)\,dx \\ &= \int_{-1}^{1} x^2\,dx = \left[\frac{x^3}{3}\right]_{-1}^{1} = \frac{1}{3}\left[1^3 - (-1)^3\right] = \frac{2}{3}\end{aligned}$$

Substituting $\mathbf{v}_4 = \mathbf{x}^3$, $\langle \mathbf{v}_4,\ \mathbf{w}_1 \rangle = 0$, $\langle \mathbf{v}_4,\ \mathbf{w}_2 \rangle = \frac{2}{5}$, $\|\mathbf{w}_2\|^2 = \frac{2}{3}$, $\mathbf{w}_2 = \mathbf{x}$ and $\langle \mathbf{v}_4,\ \ \mathbf{w}_3 \rangle = 0$ into $\mathbf{w}_4 = \mathbf{v}_4 - \frac{\langle \mathbf{v}_4,\ \mathbf{w}_1 \rangle}{\|\mathbf{w}_1\|^2}\mathbf{w}_1 - \frac{\langle \mathbf{v}_4,\ \mathbf{w}_2 \rangle}{\|\mathbf{w}_2\|^2}\mathbf{w}_2 - \frac{\langle \mathbf{v}_4,\ \mathbf{w}_3 \rangle}{\|\mathbf{w}_3\|^2}\mathbf{w}_3$ gives

$$\begin{aligned}\mathbf{w}_4 &= \mathbf{x}^3 - 0\mathbf{w}_1 - \frac{2/5}{2/3}\mathbf{x} - 0\mathbf{w}_3 \\ &= \mathbf{x}^3 - \frac{3}{5}\mathbf{x}\end{aligned}$$

Thus our orthogonal vectors are $\mathbf{w}_1 = \mathbf{1}$, $\mathbf{w}_2 = \mathbf{x}$, $\mathbf{w}_3 = \mathbf{x}^2 - \frac{1}{3}$ and $\mathbf{w}_4 = \mathbf{x}^3 - \frac{3}{5}\mathbf{x}$.

How do we find an orthonormal basis?

We need to normalise these vectors:

$$\mathbf{w}_1 = \frac{1}{\|\mathbf{w}_1\|}\mathbf{w}_1 = \frac{1}{\sqrt{2}}(\mathbf{1})$$

$$\mathbf{w}_2 = \frac{1}{\|\mathbf{w}_2\|}\mathbf{w}_2 = \frac{1}{\sqrt{2/3}}(\mathbf{x}) = \sqrt{\frac{3}{2}}\mathbf{x} \qquad \left[\text{Because } \|\mathbf{w}_2\|^2 = \frac{2}{3}\right]$$

For the remaining two vectors we need to find $\|\mathbf{w}_3\|$ and $\|\mathbf{w}_4\|$:

$$\begin{aligned}\|\mathbf{w}_3\|^2 &= \langle \mathbf{w}_3,\ \mathbf{w}_3 \rangle \\ &= \left\langle \mathbf{x}^2 - \frac{1}{3},\ \mathbf{x}^2 - \frac{1}{3} \right\rangle \\ &= \int_{-1}^{1}\left(x^2 - \frac{1}{3}\right)\left(x^2 - \frac{1}{3}\right)dx \\ &= \int_{-1}^{1}\left(x^4 - \frac{2}{3}x^2 + \frac{1}{9}\right)dx \\ &= \left[\frac{x^5}{5} - \frac{2x^3}{9} + \frac{x}{9}\right]_{-1}^{1} \\ &= \left(\frac{1^5}{5} - \frac{2(1)^3}{9} + \frac{1}{9}\right) - \left(\frac{(-1)^5}{5} - \frac{2(-1)^3}{9} - \frac{1}{9}\right) = \frac{8}{45}\end{aligned}$$

Taking the square root gives $\|\mathbf{w}_3\| = \sqrt{\frac{8}{45}}$. Similarly we have

$$\begin{aligned}
\|\mathbf{w}_4\|^2 &= \langle \mathbf{w}_4,\ \mathbf{w}_4 \rangle \\
&= \left\langle \mathbf{x}^3 - \frac{3}{5}\mathbf{x},\ \mathbf{x}^3 - \frac{3}{5}\mathbf{x} \right\rangle \qquad \left[\text{Because } \mathbf{w}_4 = \mathbf{x}^3 - \frac{3}{5}\mathbf{x}\right] \\
&= \int_{-1}^{1} \left(x^3 - \frac{3}{5}x\right)\left(x^3 - \frac{3}{5}x\right) dx \\
&= \int_{-1}^{1} \left(x^6 - \frac{6}{5}x^4 + \frac{9}{25}x^2\right) dx \\
&= \left[\frac{x^7}{7} - \frac{6x^5}{25} + \frac{3x^3}{25}\right]_{-1}^{1} \\
&= \left(\frac{1^7}{7} - \frac{6(1)^5}{25} + \frac{3(1)^3}{25}\right) - \left(\frac{(-1)^7}{7} - \frac{6(-1)^5}{25} + \frac{3(-1)^3}{25}\right) = \frac{8}{175}
\end{aligned}$$

Taking the square root gives $\|\mathbf{w}_4\| = \sqrt{\frac{8}{175}}$. We have

$$\begin{aligned}
\mathbf{w}_3 &= \frac{1}{\|\mathbf{w}_3\|}\mathbf{w}_3 = \frac{1}{\sqrt{8/45}}\left(\mathbf{x}^2 - \frac{1}{3}\right) \qquad \left[\text{Because } \|\mathbf{w}_3\| = \sqrt{\frac{8}{45}}\right] \\
&= \sqrt{\frac{45}{8}}\left(\mathbf{x}^2 - \frac{1}{3}\right) \\
&= \sqrt{\frac{9\times 5}{8}}\left(\frac{3\mathbf{x}^2 - 1}{3}\right) = \cancel{3}\sqrt{\frac{5}{8}}\left(\frac{3\mathbf{x}^2 - 1}{\cancel{3}}\right) = \sqrt{\frac{5}{8}}\left(3\mathbf{x}^2 - 1\right)
\end{aligned}$$

$$\begin{aligned}
\mathbf{w}_4 &= \frac{1}{\|\mathbf{w}_4\|}\mathbf{w}_4 = \frac{1}{\sqrt{8/175}}\left(\mathbf{x}^3 - \frac{3}{5}\mathbf{x}\right) \qquad \left[\text{Because } \|\mathbf{w}_4\| = \sqrt{\frac{8}{175}}\right] \\
&= \sqrt{\frac{175}{8}}\left(\frac{5\mathbf{x}^3 - 3\mathbf{x}}{5}\right) \\
&= \sqrt{\frac{25\times 7}{8}}\left(\frac{5\mathbf{x}^3 - 3\mathbf{x}}{5}\right) = \cancel{5}\sqrt{\frac{7}{8}}\left(\frac{5\mathbf{x}^3 - 3\mathbf{x}}{\cancel{5}}\right) = \sqrt{\frac{7}{8}}\left(5\mathbf{x}^3 - 3\mathbf{x}\right)
\end{aligned}$$

Collecting our orthonormal basis vectors we have

$$\left\{\frac{1}{\sqrt{2}},\ \sqrt{\frac{3}{2}}\mathbf{x},\ \ \sqrt{\frac{5}{8}}\left(3\mathbf{x}^2 - 1\right),\ \sqrt{\frac{7}{8}}\left(5\mathbf{x}^3 - 3\mathbf{x}\right)\right\}$$

Note that the first two vectors are the normalised vectors found in question 12.

19. We need to find an orthonormal basis for the column space $C(\mathbf{A})$ of the matrix

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & 2 & 1 \\ 0 & 1 & 3 & 3 \\ 2 & 5 & 1 & 0 \\ 0 & 4 & 0 & 0 \end{bmatrix}$$

What is the column space $C(\mathbf{A})$ equal to?

It is the space spanned by the vectors in the columns of the given matrix **A**:

$$C(\mathbf{A}) = Span\left\{\begin{pmatrix}1\\0\\2\\0\end{pmatrix}, \begin{pmatrix}0\\1\\5\\4\end{pmatrix}, \begin{pmatrix}2\\3\\1\\0\end{pmatrix}, \begin{pmatrix}1\\3\\0\\0\end{pmatrix}\right\}$$

You can show that these vectors are linearly independent. *Why do we need to check that these vectors are linearly independent?*

Because we need to convert these vectors into an orthogonal basis which means that we need to use Gram-Schmidt Process (4-10) and this can only be applied if the given vectors are a basis to start of with.

Let $\mathbf{v}_1 = \begin{pmatrix}1\\0\\2\\0\end{pmatrix}$, $\mathbf{v}_2 = \begin{pmatrix}0\\1\\5\\4\end{pmatrix}$, $\mathbf{v}_3 = \begin{pmatrix}2\\3\\1\\0\end{pmatrix}$, $\mathbf{v}_4 = \begin{pmatrix}1\\3\\0\\0\end{pmatrix}$. Applying the Gram-Schmidt Process (4-10) we have $\mathbf{w}_1 = \mathbf{v}_1 = \begin{pmatrix}1\\0\\2\\0\end{pmatrix}$. We need to find $\mathbf{w}_2 = \mathbf{v}_2 - \frac{\langle \mathbf{v}_2,\ \mathbf{w}_1\rangle}{\|\mathbf{w}_1\|^2}\mathbf{w}_1$:

$$\langle \mathbf{v}_2,\ \mathbf{w}_1\rangle = \begin{pmatrix}0\\1\\5\\4\end{pmatrix}\cdot\begin{pmatrix}1\\0\\2\\0\end{pmatrix} = 0+0+10+0 = 10$$

$$\|\mathbf{w}_1\|^2 = \begin{pmatrix}1\\0\\2\\0\end{pmatrix}\cdot\begin{pmatrix}1\\0\\2\\0\end{pmatrix} = 1^2+0^2+2^2+0^2 = 5$$

Substituting $\mathbf{v}_2 = \begin{pmatrix}0\\1\\5\\4\end{pmatrix}$, $\mathbf{w}_1 = \begin{pmatrix}1\\0\\2\\0\end{pmatrix}$ and $\frac{\langle \mathbf{v}_2,\ \mathbf{w}_1\rangle}{\|\mathbf{w}_1\|^2} = \frac{10}{5} = 2$ into $\mathbf{w}_2 = \mathbf{v}_2 - \frac{\langle \mathbf{v}_2,\ \mathbf{w}_1\rangle}{\|\mathbf{w}_1\|^2}\mathbf{w}_1$:

$$\mathbf{w}_2 = \begin{pmatrix}0\\1\\5\\4\end{pmatrix} - 2\begin{pmatrix}1\\0\\2\\0\end{pmatrix} = \begin{pmatrix}0-2\\1-0\\5-4\\4-0\end{pmatrix} = \begin{pmatrix}-2\\1\\1\\4\end{pmatrix}$$

Next we find $\mathbf{w}_3 = \mathbf{v}_3 - \frac{\langle \mathbf{v}_3,\ \mathbf{w}_1\rangle}{\|\mathbf{w}_1\|^2}\mathbf{w}_1 - \frac{\langle \mathbf{v}_3,\ \mathbf{w}_2\rangle}{\|\mathbf{w}_2\|^2}\mathbf{w}_2$:

$$\langle \mathbf{v}_3,\ \mathbf{w}_1\rangle = \begin{pmatrix}2\\3\\1\\0\end{pmatrix}\cdot\begin{pmatrix}1\\0\\2\\0\end{pmatrix} = 2+0+2+0 = 4$$

$$\langle \mathbf{v}_3,\ \mathbf{w}_2 \rangle = \begin{pmatrix} 2 \\ 3 \\ 1 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} -2 \\ 1 \\ 1 \\ 4 \end{pmatrix} = -4+3+1+0=0$$

This means that $\mathbf{v}_3$ and $\mathbf{w}_2$ are orthogonal. Substituting $\mathbf{v}_3 = \begin{pmatrix} 2 \\ 3 \\ 1 \\ 0 \end{pmatrix}$, $\mathbf{w}_1 = \begin{pmatrix} 1 \\ 0 \\ 2 \\ 0 \end{pmatrix}$,

$\dfrac{\langle \mathbf{v}_3,\ \mathbf{w}_1 \rangle}{\|\mathbf{w}_1\|^2} = \dfrac{4}{5}$, and $\langle \mathbf{v}_3,\ \mathbf{w}_2 \rangle = 0$ into $\mathbf{w}_3 = \mathbf{v}_3 - \dfrac{\langle \mathbf{v}_3,\ \mathbf{w}_1 \rangle}{\|\mathbf{w}_1\|^2}\mathbf{w}_1 - \dfrac{\langle \mathbf{v}_3,\ \mathbf{w}_2 \rangle}{\|\mathbf{w}_2\|^2}\mathbf{w}_2$ gives

$$\mathbf{w}_3 = \begin{pmatrix} 2 \\ 3 \\ 1 \\ 0 \end{pmatrix} - \frac{4}{5}\begin{pmatrix} 1 \\ 0 \\ 2 \\ 0 \end{pmatrix} - 0\mathbf{w}_2 = \begin{pmatrix} 2-4/5 \\ 3-0 \\ 1-8/5 \\ 0-0 \end{pmatrix} = \begin{pmatrix} 6/5 \\ 3 \\ -3/5 \\ 0 \end{pmatrix} = \frac{3}{5}\begin{pmatrix} 2 \\ 5 \\ -1 \\ 0 \end{pmatrix}$$

Ignore the fraction and let $\mathbf{w}'_3 = \begin{pmatrix} 2 \\ 5 \\ -1 \\ 0 \end{pmatrix}$. *What else do we need to find?*

$\mathbf{w}_4$ where $\mathbf{w}_4 = \mathbf{v}_4 - \dfrac{\langle \mathbf{v}_4,\ \mathbf{w}_1 \rangle}{\|\mathbf{w}_1\|^2}\mathbf{w}_1 - \dfrac{\langle \mathbf{v}_4,\ \mathbf{w}_2 \rangle}{\|\mathbf{w}_2\|^2}\mathbf{w}_2 - \dfrac{\langle \mathbf{v}_4,\ \mathbf{w}'_3 \rangle}{\|\mathbf{w}'_3\|^2}\mathbf{w}'_3$. We have

$$\langle \mathbf{v}_4,\ \mathbf{w}_1 \rangle = \begin{pmatrix} 1 \\ 3 \\ 0 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 0 \\ 2 \\ 0 \end{pmatrix} = 1+0+0+0=1$$

$$\langle \mathbf{v}_4,\ \mathbf{w}_2 \rangle = \begin{pmatrix} 1 \\ 3 \\ 0 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} -2 \\ 1 \\ 1 \\ 4 \end{pmatrix} = -2+3+0+0=1$$

$$\langle \mathbf{v}_4,\ \mathbf{w}'_3 \rangle = \begin{pmatrix} 1 \\ 3 \\ 0 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ 5 \\ -1 \\ 0 \end{pmatrix} = 2+15+0+0=17$$

$$\|\mathbf{w}_2\|^2 = \begin{pmatrix} -2 \\ 1 \\ 1 \\ 4 \end{pmatrix} \cdot \begin{pmatrix} -2 \\ 1 \\ 1 \\ 4 \end{pmatrix} = 4+1+1+16=22$$

$$\left\|\mathbf{w}'_3\right\|^2=\begin{pmatrix}2\\5\\-1\\0\end{pmatrix}\cdot\begin{pmatrix}2\\5\\-1\\0\end{pmatrix}=4+25+1+0=30$$

Substituting $\mathbf{v}_4=\begin{pmatrix}1\\3\\0\\0\end{pmatrix}$, $\mathbf{w}_1=\begin{pmatrix}1\\0\\2\\0\end{pmatrix}$, $\mathbf{w}_2=\begin{pmatrix}-2\\1\\1\\4\end{pmatrix}$, $\mathbf{w}'_3=\begin{pmatrix}2\\5\\-1\\0\end{pmatrix}$, $\dfrac{\langle\mathbf{v}_4,\ \mathbf{w}_1\rangle}{\left\|\mathbf{w}_1\right\|^2}=\dfrac{1}{5}$, $\dfrac{\langle\mathbf{v}_4,\ \mathbf{w}_2\rangle}{\left\|\mathbf{w}_2\right\|^2}=\dfrac{1}{22}$ and $\dfrac{\langle\mathbf{v}_4,\ \mathbf{w}'_3\rangle}{\left\|\mathbf{w}'_3\right\|^2}=\dfrac{17}{30}$ into

$$\mathbf{w}_4=\mathbf{v}_4-\frac{\langle\mathbf{v}_4,\ \mathbf{w}_1\rangle}{\left\|\mathbf{w}_1\right\|^2}\mathbf{w}_1-\frac{\langle\mathbf{v}_4,\ \mathbf{w}_2\rangle}{\left\|\mathbf{w}_2\right\|^2}\mathbf{w}_2-\frac{\langle\mathbf{v}_4,\ \mathbf{w}'_3\rangle}{\left\|\mathbf{w}'_3\right\|^2}\mathbf{w}'_3$$

we have

$$\mathbf{w}_4=\begin{pmatrix}1\\3\\0\\0\end{pmatrix}-\frac{1}{5}\begin{pmatrix}1\\0\\2\\0\end{pmatrix}-\frac{1}{22}\begin{pmatrix}-2\\1\\1\\4\end{pmatrix}-\frac{17}{30}\begin{pmatrix}2\\5\\-1\\0\end{pmatrix}$$

$$=\begin{pmatrix}1-1/5+2/22-34/30\\3-0-1/22-85/30\\0-2/5-1/22+17/30\\0-0-4/22-0\end{pmatrix}$$

$$=\begin{pmatrix}-8/33\\4/33\\4/33\\-4/22\end{pmatrix}=\begin{pmatrix}-16/66\\8/66\\8/66\\-12/66\end{pmatrix}=\frac{4}{66}\begin{pmatrix}-4\\2\\2\\-3\end{pmatrix}$$

Again ignoring the fraction and let $\mathbf{w}'_4=\begin{pmatrix}-4\\2\\2\\-3\end{pmatrix}$. Our orthogonal basis is

$$\left\{\begin{pmatrix}1\\0\\2\\0\end{pmatrix},\begin{pmatrix}-2\\1\\1\\4\end{pmatrix},\begin{pmatrix}2\\5\\-1\\0\end{pmatrix},\begin{pmatrix}-4\\2\\2\\-3\end{pmatrix}\right\}$$

20. Need to prove that if **u** is orthogonal for every $\mathbf{v}\in V$ then $\mathbf{u}=\mathbf{O}$.

Proof.
Since we are given that the vector **u** is orthogonal for every $\mathbf{v} \in V$ therefore

$$\langle \mathbf{u},\ \mathbf{v} \rangle = 0 \text{ for every } \mathbf{v} \in V$$

Let $\mathbf{u} = \mathbf{v}$ then by the above we have $\langle \mathbf{u},\ \mathbf{v} \rangle = \langle \mathbf{u},\ \mathbf{u} \rangle = 0$. By Definition (4-1) part (d) we have $\langle \mathbf{u},\ \mathbf{u} \rangle = 0 \iff \mathbf{u} = \mathbf{O}$. This completes our proof.

■

21. We need to prove that if **u** is orthogonal to **v** then every scalar multiple of **u** is also orthogonal to **v**.
Proof.
We are given that $\langle \mathbf{u},\ \mathbf{v} \rangle = 0$ because **u** and **v** are orthogonal. Consider $\langle k\mathbf{u},\ \mathbf{v} \rangle = 0$ where *k* is any scalar. We have

$$\langle k\mathbf{u},\ \mathbf{v} \rangle = k\langle \mathbf{u},\ \mathbf{v} \rangle = k(0) = 0$$

Thus $k\mathbf{u}$ and **v** are orthogonal which means every scalar multiple of **u** is orthogonal to **v**.

■

22. We need to prove

$$\langle x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + \cdots + x_n\mathbf{v}_n,\ y_1\mathbf{v}_1 + y_2\mathbf{v}_2 + \cdots + y_n\mathbf{v}_n \rangle = x_1y_1 + x_2y_2 + \cdots + x_ny_n$$

Proof.
We are given that $\{\mathbf{v}_1,\ \mathbf{v}_2,\ \mathbf{v}_3,\ \cdots,\ \mathbf{v}_n\}$ is a set of orthonormal vectors. This means that $\langle \mathbf{v}_i,\ \mathbf{v}_j \rangle = 0 \quad (i \neq j)$ and $\|\mathbf{v}_i\| = 1$ for $i = 1,\ 2, \cdots,\ n$. Applying this we have

$$\begin{aligned}\langle x_1\mathbf{v}_1 + \cdots + x_n\mathbf{v}_n,\ y_1\mathbf{v}_1 + \cdots + y_n\mathbf{v}_n \rangle &= x_1y_1\langle \mathbf{v}_1,\ \mathbf{v}_1 \rangle + \cdots + x_ny_n\langle \mathbf{v}_n,\ \mathbf{v}_n \rangle \\ &= x_1y_1\|\mathbf{v}_1\|^2 + \cdots + x_ny_n\|\mathbf{v}_n\|^2 \\ &= x_1y_1 + \cdots + x_ny_n\end{aligned}$$

Thus we have our required result.

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23. Need to prove that if $\mathbf{u} \neq \mathbf{v}$ then $d(\mathbf{u},\ \mathbf{v}) > 0$.
Proof.
By using the definition of distance function we have

$$d(\mathbf{u},\ \mathbf{v}) = \|\mathbf{u} - \mathbf{v}\|$$

By the properties of norm given in Proposition (4-2) part (a) which says:

$$\|\mathbf{u}\| \geq 0 \quad [\text{Non-negative}]$$

We have $d(\mathbf{u},\ \mathbf{v}) = \|\mathbf{u} - \mathbf{v}\| \geq 0$. Since we are given that $\mathbf{u} \neq \mathbf{v}$ which implies $\mathbf{u} - \mathbf{v} \neq \mathbf{O}$ [Not Zero] therefore by the same proposition part (ii) which says:

$$\|\mathbf{u}\| = 0 \iff \mathbf{u} = \mathbf{O}$$

We must have $d(\mathbf{u},\ \mathbf{v}) = \|\mathbf{u} - \mathbf{v}\| > 0$ which is our required result.

■

(b) We need to prove $d(\mathbf{u},\ \mathbf{v}) \leq d(\mathbf{u},\ \mathbf{w}) + d(\mathbf{w},\ \mathbf{v})$.
How do we prove this result?
By using Minkowski's Inequality (4-4) which says

$$\|\mathbf{u}+\mathbf{v}\| \le \|\mathbf{u}\|+\|\mathbf{v}\|$$

Proof.

We have $d(\mathbf{u},\ \mathbf{v}) = \|\mathbf{u}-\mathbf{v}\|$. By rewriting this

$$\|\mathbf{u}-\mathbf{v}\| = \|\mathbf{u}-\mathbf{w}+\mathbf{w}-\mathbf{v}\|$$

Applying Minkowski's Inequality (4-4) to this we have

$$\begin{aligned} \|\mathbf{u}-\mathbf{v}\| &= \|(\mathbf{u}-\mathbf{w})+(\mathbf{w}-\mathbf{v})\| \\ &\le \|\mathbf{u}-\mathbf{w}\|+\|\mathbf{w}-\mathbf{v}\| \\ &= d(\mathbf{u},\ \mathbf{w})+d(\mathbf{w},\ \mathbf{v}) \end{aligned}$$

Hence we have our result $d(\mathbf{u},\ \mathbf{v}) \le d(\mathbf{u},\ \mathbf{w})+d(\mathbf{w},\ \mathbf{v})$.

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