Complete solutions to Exercises 1.4

1. We are given the following matrices, $\mathbf{A}=\begin{pmatrix}1 & 2\\ 3 & -1\end{pmatrix}$, $\mathbf{B}=\begin{pmatrix}6 & -1\\ 5 & 3\end{pmatrix}$ and $\mathbf{C}=\begin{pmatrix}-1\\ 1\end{pmatrix}$.

(a) We have

$$\mathbf{A}+\mathbf{B}=\begin{pmatrix}1 & 2\\ 3 & -1\end{pmatrix}+\begin{pmatrix}6 & -1\\ 5 & 3\end{pmatrix}$$
$$=\begin{pmatrix}1+6 & 2-1\\ 3+5 & -1+3\end{pmatrix}=\begin{pmatrix}7 & 1\\ 8 & 2\end{pmatrix}$$

(b) **B**+**A** is similar to part (a):

$$\mathbf{B}+\mathbf{A}=\begin{pmatrix}6 & -1\\ 5 & 3\end{pmatrix}+\begin{pmatrix}1 & 2\\ 3 & -1\end{pmatrix}$$
$$=\begin{pmatrix}6+1 & -1+2\\ 5+3 & 3-1\end{pmatrix}=\begin{pmatrix}7 & 1\\ 8 & 2\end{pmatrix}$$

(c) **B**+**B**+**B** is given by

$$\mathbf{B}+\mathbf{B}+\mathbf{B}=\begin{pmatrix}6 & -1\\ 5 & 3\end{pmatrix}+\begin{pmatrix}6 & -1\\ 5 & 3\end{pmatrix}+\begin{pmatrix}6 & -1\\ 5 & 3\end{pmatrix}$$
$$=\begin{pmatrix}6+6+6 & -1-1-1\\ 5+5+5 & 3+3+3\end{pmatrix}=\begin{pmatrix}18 & -3\\ 15 & 9\end{pmatrix}$$

(d) $3\mathbf{B}=3\begin{pmatrix}6 & -1\\ 5 & 3\end{pmatrix}=\begin{pmatrix}(3\times 6) & (3\times(-1))\\ (3\times 5) & (3\times 3)\end{pmatrix}=\begin{pmatrix}18 & -3\\ 15 & 9\end{pmatrix}$

(e) We have

$$3\mathbf{A}+2\mathbf{B}=3\begin{pmatrix}1 & 2\\ 3 & -1\end{pmatrix}+2\begin{pmatrix}6 & -1\\ 5 & 3\end{pmatrix}$$
$$=\begin{pmatrix}3 & 6\\ 9 & -3\end{pmatrix}+\begin{pmatrix}12 & -2\\ 10 & 6\end{pmatrix}=\begin{pmatrix}3+12 & 6-2\\ 9+10 & -3+6\end{pmatrix}=\begin{pmatrix}15 & 4\\ 19 & 3\end{pmatrix}$$

(f) **A**+**C** is impossible because the matrix sizes are different.
(g) $\mathbf{B}+\mathbf{C}$ is impossible because the matrix sizes are different.
(h) **AC** is evaluated by row times column:

$$\mathbf{AC}=\begin{pmatrix}1 & 2\\ 3 & -1\end{pmatrix}\begin{pmatrix}-1\\ 1\end{pmatrix}=\begin{pmatrix}(1\times -1)+(2\times 1)\\ (3\times -1)+(-1\times 1)\end{pmatrix}=\begin{pmatrix}1\\ -4\end{pmatrix}$$

(i) Similarly **BC** equals

$$\mathbf{BC}=\begin{pmatrix}6 & -1\\ 5 & 3\end{pmatrix}\begin{pmatrix}-1\\ 1\end{pmatrix}=\begin{pmatrix}(6\times -1)+(-1\times 1)\\ (5\times -1)+(3\times 1)\end{pmatrix}=\begin{pmatrix}-7\\ -2\end{pmatrix}$$

(j) We have

$$5\mathbf{A}-7\mathbf{BC}=5\begin{pmatrix}1 & 2\\ 3 & -1\end{pmatrix}-7\underbrace{\begin{pmatrix}-7\\ -2\end{pmatrix}}_{\text{Evaluated above}}$$
$$=\begin{pmatrix}5 & 10\\ 15 & -5\end{pmatrix}-\begin{pmatrix}-49\\ -14\end{pmatrix}$$

Cannot subtract the above matrices because they are of different size. Therefore it is impossible to evaluate $5\mathbf{A}-7\mathbf{BC}$.

(k) By parts (h) and (i) above we have

$$3\mathbf{AC}-2\mathbf{BC}=3\underbrace{\begin{pmatrix}1\\-4\end{pmatrix}}_{\text{By Part (h)}}-2\underbrace{\begin{pmatrix}-7\\-2\end{pmatrix}}_{\text{By Part (i)}}$$

$$=\begin{pmatrix}3\\-12\end{pmatrix}-\begin{pmatrix}-14\\-4\end{pmatrix}=\begin{pmatrix}3-(-14)\\-12-(-4)\end{pmatrix}=\begin{pmatrix}17\\-8\end{pmatrix}$$

2. $\mathbf{A}=\begin{pmatrix}1&-1&7\\2&9&6\end{pmatrix}$, $\mathbf{B}=\begin{pmatrix}5&1&4\\8&2&7\\1&4&9\end{pmatrix}$, $\mathbf{C}=\begin{pmatrix}2\\4\\7\end{pmatrix}$ and $\mathbf{D}=\begin{pmatrix}7&9&4\\1&3&-5\\2&-1&-3\end{pmatrix}$

(a) We have

$$\mathbf{A}-\mathbf{A}=\begin{pmatrix}1&-1&7\\2&9&6\end{pmatrix}-\begin{pmatrix}1&-1&7\\2&9&6\end{pmatrix}$$

$$=\begin{pmatrix}1-1&-1-(-1)&7-7\\2-2&9-9&6-6\end{pmatrix}=\begin{pmatrix}0&0&0\\0&0&0\end{pmatrix}$$

(b) Similarly we have

$$3\mathbf{A}-2\mathbf{A}=3\begin{pmatrix}1&-1&7\\2&9&6\end{pmatrix}-2\begin{pmatrix}1&-1&7\\2&9&6\end{pmatrix}$$

$$=\begin{pmatrix}3&-3&21\\6&27&18\end{pmatrix}-\begin{pmatrix}2&-2&14\\4&18&12\end{pmatrix}$$

$$=\begin{pmatrix}3-2&-3-(-2)&21-14\\6-4&27-18&18-12\end{pmatrix}=\begin{pmatrix}1&-1&7\\2&9&6\end{pmatrix}=\mathbf{A}$$

(c) **BC** which is matrix times a vector:

$$\mathbf{BC}=\begin{pmatrix}5&1&4\\8&2&7\\1&4&9\end{pmatrix}\begin{pmatrix}2\\4\\7\end{pmatrix}=\left(2\begin{pmatrix}5\\8\\1\end{pmatrix}+4\begin{pmatrix}1\\2\\4\end{pmatrix}+7\begin{pmatrix}4\\7\\9\end{pmatrix}\right)=\begin{pmatrix}42\\73\\81\end{pmatrix}$$

(d) **CB** is impossible because vector **C** has only one column but matrix **B** has 3 rows. Remember for multiplication of matrices the number of columns of the first (Left Hand) matrix must equal the number of rows of the second (Right Hand) matrix.

(e) $\mathbf{B}+\mathbf{D}$ is given by

$$\mathbf{B}+\mathbf{D}=\begin{pmatrix}5&1&4\\8&2&7\\1&4&9\end{pmatrix}+\begin{pmatrix}7&9&4\\1&3&-5\\2&-1&-3\end{pmatrix}$$

$$=\begin{pmatrix}5+7&1+9&4+4\\8+1&2+3&7+(-5)\\1+2&4+(-1)&9+(-3)\end{pmatrix}=\begin{pmatrix}12&10&8\\9&5&2\\3&3&6\end{pmatrix}$$

(f) Clearly $\mathbf{D}+\mathbf{B}$ gives the same result as part (e) because the same elements are added but in different order.

(g) $\mathbf{A}-\mathbf{C}$ is impossible because matrices are of **different** size.

(h) $$\frac{1}{2}\mathbf{C}=\frac{1}{2}\begin{pmatrix}2\\4\\7\end{pmatrix}=\begin{pmatrix}\frac{1}{2}\times 2\\ \frac{1}{2}\times 4\\ \frac{1}{2}\times 7\end{pmatrix}=\begin{pmatrix}1\\2\\3.5\end{pmatrix}$$

(i) Working out **BD** gives

$$\mathbf{BD}=\begin{pmatrix}5&1&4\\8&2&7\\1&4&9\end{pmatrix}\begin{pmatrix}7&9&4\\1&3&-5\\2&-1&-3\end{pmatrix}$$

$$=\begin{pmatrix}(5\times 7)+(1\times 1)+(4\times 2)&(5\times 9)+(1\times 3)+(4\times -1)&(5\times 4)+(1\times -5)+(4\times -3)\\(8\times 7)+(2\times 1)+(7\times 2)&(8\times 9)+(2\times 3)+(7\times -1)&(8\times 4)+(2\times -5)+(7\times -3)\\(1\times 7)+(4\times 1)+(9\times 2)&(1\times 9)+(4\times 3)+(9\times -1)&(1\times 4)+(4\times -5)+(9\times -3)\end{pmatrix}$$

$$=\begin{pmatrix}44&44&3\\72&71&1\\29&12&-43\end{pmatrix}$$

(j) **DB** is evaluated row by column:

$$\mathbf{DB}=\begin{pmatrix}7&9&4\\1&3&-5\\2&-1&-3\end{pmatrix}\begin{pmatrix}5&1&4\\8&2&7\\1&4&9\end{pmatrix}$$

$$=\begin{pmatrix}(7\times 5)+(9\times 8)+(4\times 1)&(7\times 1)+(9\times 2)+(4\times 4)&(7\times 4)+(9\times 7)+(4\times 9)\\(1\times 5)+(3\times 8)+(-5\times 1)&(1\times 1)+(3\times 2)+(-5\times 4)&(1\times 4)+(3\times 7)+(-5\times 9)\\(2\times 5)+(-1\times 8)+(-3\times 1)&(2\times 1)+(-1\times 2)+(-3\times 4)&(2\times 4)+(-1\times 7)+(-3\times 9)\end{pmatrix}$$

$$=\begin{pmatrix}111&41&127\\24&-13&-20\\-1&-12&-26\end{pmatrix}$$

(k) By parts (i) and (j) above we have $\mathbf{BD}-\mathbf{DB}$ given by

$$\mathbf{BD}-\mathbf{DB}=\begin{pmatrix}44&44&3\\72&71&1\\29&12&-43\end{pmatrix}-\begin{pmatrix}111&41&127\\24&-13&-20\\-1&-12&-26\end{pmatrix}$$

$$=\begin{pmatrix}44-111&44-41&3-127\\72-24&71-(-13)&1-(-20)\\29-(-1)&12-(-12)&-43-(-26)\end{pmatrix}$$

$$=\begin{pmatrix}-67&3&-124\\48&84&21\\30&24&-17\end{pmatrix}$$

(l) **CD** is impossible because vector **C** has only one column but matrix **D** has 3 rows.

3. (a) We have

$$\begin{pmatrix}2&4\\3&9\end{pmatrix}\begin{pmatrix}1&0\\0&1\end{pmatrix}=\begin{pmatrix}(2\times1)+(4\times0)&(2\times0)+(4\times1)\\(3\times1)+(9\times0)&(3\times0)+(9\times1)\end{pmatrix}$$
$$=\begin{pmatrix}2&4\\3&9\end{pmatrix}$$

(b) Similarly we have

$$\begin{pmatrix}6&7\\2&3\end{pmatrix}\begin{pmatrix}1&0\\0&1\end{pmatrix}=\begin{pmatrix}(6\times1)+(7\times0)&(6\times0)+(7\times1)\\(2\times1)+(3\times0)&(2\times0)+(3\times1)\end{pmatrix}$$
$$=\begin{pmatrix}6&7\\2&3\end{pmatrix}$$

(c) Again we have

$$\begin{pmatrix}a&b\\c&d\end{pmatrix}\begin{pmatrix}1&0\\0&1\end{pmatrix}=\begin{pmatrix}(a\times1)+(b\times0)&(a\times0)+(b\times1)\\(c\times1)+(d\times0)&(c\times0)+(d\times1)\end{pmatrix}$$
$$=\begin{pmatrix}a&b\\c&d\end{pmatrix}$$

(d) '3 by 3' matrices involve a long multiplication process:

$$\begin{pmatrix}2&3&6\\1&4&5\\0&9&7\end{pmatrix}\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}=\begin{pmatrix}(2\times1)+(3\times0)+(6\times0)&(2\times0)+(3\times1)+(6\times0)&(2\times0)+(3\times0)+(6\times1)\\(1\times1)+(4\times0)+(5\times0)&(1\times0)+(4\times1)+(5\times0)&(1\times0)+(4\times0)+(5\times1)\\(0\times1)+(9\times0)+(7\times0)&(0\times0)+(9\times1)+(7\times0)&(0\times0)+(9\times0)+(7\times1)\end{pmatrix}$$
$$=\begin{pmatrix}2&3&6\\1&4&5\\0&9&7\end{pmatrix}$$

The result is always the first (Left Hand) matrix. In arithmetic or algebra when you multiply a real number such as x by 1 your result is always x. Similarly when you multiply a matrix by the above Right Hand matrix called the **identity matrix** the result is always what you started with.

4. (a) $\begin{pmatrix}3&7\\2&5\end{pmatrix}\begin{pmatrix}5&-7\\-2&3\end{pmatrix}=\begin{pmatrix}15-14&-21+21\\10-10&-14+15\end{pmatrix}=\begin{pmatrix}1&0\\0&1\end{pmatrix}$

(b) We can evaluate this matrix multiplication in a similar way:

$$\frac{1}{5}\begin{pmatrix}3&-4\\-7&11\end{pmatrix}\begin{pmatrix}11&4\\7&3\end{pmatrix}=\frac{1}{5}\begin{pmatrix}33-28&12-12\\-77+77&-28+33\end{pmatrix}$$
$$=\frac{1}{5}\begin{pmatrix}5&0\\0&5\end{pmatrix}=\begin{pmatrix}1&0\\0&1\end{pmatrix}$$

(c) Similarly we have

$$\frac{1}{4}\begin{pmatrix}7&-9\\-5&7\end{pmatrix}\begin{pmatrix}7&9\\5&7\end{pmatrix}=\frac{1}{4}\begin{pmatrix}49-45&63-63\\-35+35&-45+49\end{pmatrix}$$
$$=\frac{1}{4}\begin{pmatrix}4&0\\0&4\end{pmatrix}=\begin{pmatrix}1&0\\0&1\end{pmatrix}$$

5. $\mathbf{A}^2=\mathbf{A}\times\mathbf{A}=\begin{pmatrix}1&-1\\1&-1\end{pmatrix}\times\begin{pmatrix}1&-1\\1&-1\end{pmatrix}=\begin{pmatrix}0&0\\0&0\end{pmatrix}$. Notice that multiplying two non-zero matrices gives all zero entries which is called a **zero matrix**.

6. Multiplying the two matrices gives

$$\begin{pmatrix}5&-1&-2\\10&-2&-4\\15&-3&-6\end{pmatrix}\begin{pmatrix}1&1&3\\1&-1&-1\\2&3&8\end{pmatrix}=\begin{pmatrix}0&0&0\\0&0&0\\0&0&0\end{pmatrix}$$

Notice that multiplying two non-zero matrices gives a zero matrix.
By solutions to questions 5 and 6 we conclude that it is possible that multiplying matrices with non- zero entries gives a matrix with **all zero** entries called a zero matrix.
In real numbers you have the following:

$$ab=0 \quad\Rightarrow\quad a=0 \quad\text{or}\quad b=0$$

However in matrices this is **not** the case as you can observe in answers to questions 5 and 6.

7. Remember in the subscript the row number goes first and then the column number:

$$\begin{array}{cc} & \begin{array}{cccc}\text{Col. 1} & \text{Col. 2} & \text{Col. 3} & \text{Col. 4}\end{array}\\ \begin{array}{c}\text{Row 1}\\ \text{Row 2}\end{array} & \begin{pmatrix}a_{\boxed{11}} & a_{\boxed{12}} & a_{\boxed{13}} & a_{\boxed{14}}\\ a_{\boxed{21}} & a_{\boxed{22}} & a_{\boxed{23}} & a_{\boxed{24}}\end{pmatrix}\end{array}$$

8. (a) We have

$$\mathbf{A}^2=\mathbf{A}\times\mathbf{A}=\begin{pmatrix}1&0\\0&1\end{pmatrix}\times\begin{pmatrix}1&0\\0&1\end{pmatrix}=\begin{pmatrix}1&0\\0&1\end{pmatrix}$$

$$\mathbf{A}^3=\mathbf{A}\times\mathbf{A}\times\mathbf{A}=\underbrace{\begin{pmatrix}1&0\\0&1\end{pmatrix}\times\begin{pmatrix}1&0\\0&1\end{pmatrix}}_{=\mathbf{A}^2}\times\begin{pmatrix}1&0\\0&1\end{pmatrix}$$

$$=\underbrace{\begin{pmatrix}1&0\\0&1\end{pmatrix}}_{\text{By Above}}\times\begin{pmatrix}1&0\\0&1\end{pmatrix}=\begin{pmatrix}1&0\\0&1\end{pmatrix}$$

Similarly $\mathbf{A}^4=\begin{pmatrix}1&0\\0&1\end{pmatrix}$. In general $\mathbf{A}^n=\begin{pmatrix}1&0\\0&1\end{pmatrix}$. Hence the formula for $\mathbf{x}_n=\mathbf{A}^n\mathbf{x}$ is

$$\mathbf{x}_n=\mathbf{A}^n\mathbf{x}=\begin{pmatrix}1&0\\0&1\end{pmatrix}\mathbf{x}$$

(b) Similarly we have

$$\mathbf{A}^2=\mathbf{A}\times\mathbf{A}=\begin{pmatrix}0&1\\-1&0\end{pmatrix}\times\begin{pmatrix}0&1\\-1&0\end{pmatrix}=\begin{pmatrix}-1&0\\0&-1\end{pmatrix}$$

$$\mathbf{A}^3=\mathbf{A}\times\mathbf{A}\times\mathbf{A}=\begin{pmatrix}0&1\\-1&0\end{pmatrix}\times\begin{pmatrix}0&1\\-1&0\end{pmatrix}\times\begin{pmatrix}0&1\\-1&0\end{pmatrix}$$

$$=\underbrace{\begin{pmatrix}-1&0\\0&-1\end{pmatrix}}_{=\mathbf{A}^2}\times\begin{pmatrix}0&1\\-1&0\end{pmatrix}=\begin{pmatrix}0&-1\\1&0\end{pmatrix}$$

$$\mathbf{A}^4 = \mathbf{A}\times\mathbf{A}\times\mathbf{A}\times\mathbf{A} = \underbrace{\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\times\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\times\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}}_{=\mathbf{A}^3}\times\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

$$= \underbrace{\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}}_{=\mathbf{A}^3 \text{ By Above}}\times\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

You may have noticed by the results of question 3 that multiplying by this last matrix $\mathbf{A}^4 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ does not change the matrix. This means that

$$\mathbf{A}^5 = \mathbf{A}^4\times\mathbf{A} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \mathbf{A}$$

Similarly $\mathbf{A}^6 = \underbrace{\mathbf{A}^5}_{=\mathbf{A}}\times\mathbf{A} = \mathbf{A}\times\mathbf{A} = \mathbf{A}^2$. Expanding in this way yields

$$\mathbf{A}^7 = \mathbf{A}^3,\ \mathbf{A}^8 = \mathbf{A}^4 \text{ and } \mathbf{A}^9 = \mathbf{A}^5 = \mathbf{A}$$

Also with $\mathbf{A}^4\mathbf{x} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\mathbf{x} = \mathbf{x}$.

The formula for $\mathbf{x}_n = \mathbf{A}^n\mathbf{x}$ is given by

$$\mathbf{x}_n = \mathbf{A}^r\mathbf{x}$$

where r is the reminder after dividing n by 4. If the reminder $r = 0$ then $\mathbf{A}^r\mathbf{x} = \mathbf{x}$.

(c) We have

$$\mathbf{A}^2 = \begin{pmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{pmatrix}\begin{pmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{pmatrix} = \begin{pmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{pmatrix} = \mathbf{A}$$

Also $\mathbf{A}^3 = \underbrace{\mathbf{A}^2}_{=\mathbf{A}}\mathbf{A} = \mathbf{A}^2 = \mathbf{A}$ and $\mathbf{A}^4 = \underbrace{\mathbf{A}^3}_{=\mathbf{A}}\mathbf{A} = \mathbf{A}^2 = \mathbf{A}$. We have $\mathbf{A}^n = \mathbf{A}$ so the formula for the discrete dynamic system is $\mathbf{x}_n = \mathbf{A}^n\mathbf{x} = \mathbf{A}\mathbf{x}$.

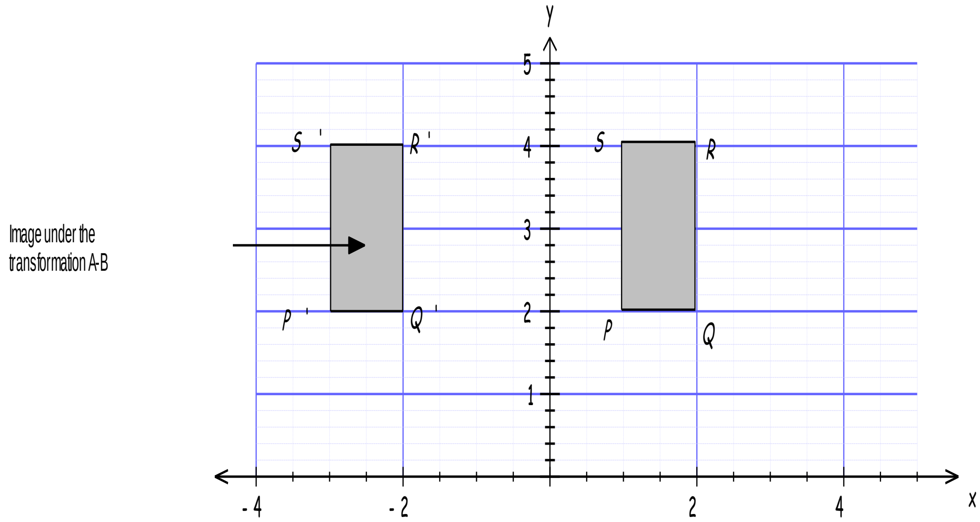
9. The matrix representing the points $P(1,\ 2)$, $Q(2,\ 2)$, $R(2,\ 4)$ and $S(1,\ 4)$ is

$$\begin{array}{cc} & \begin{array}{cccc} P & Q & R & S \end{array} \\ \mathbf{A} = & \begin{pmatrix} 1 & 2 & 2 & 1 \\ 2 & 2 & 4 & 4 \end{pmatrix} \end{array}$$

(a) The matrix subtraction $\mathbf{A}-\mathbf{B}$ is

$$\begin{array}{cccccc} & \begin{array}{cccc} P & Q & R & S \end{array} & & & & \begin{array}{cccc} P' & Q' & R' & S' \end{array} \\ \mathbf{A}-\mathbf{B} = & \begin{pmatrix} 1 & 2 & 2 & 1 \\ 2 & 2 & 4 & 4 \end{pmatrix} & - & \begin{pmatrix} 4 & 4 & 4 & 4 \\ 0 & 0 & 0 & 0 \end{pmatrix} & = & \begin{pmatrix} -3 & -2 & -2 & -3 \\ 2 & 2 & 4 & 4 \end{pmatrix} \end{array}$$

This can be illustrated as:

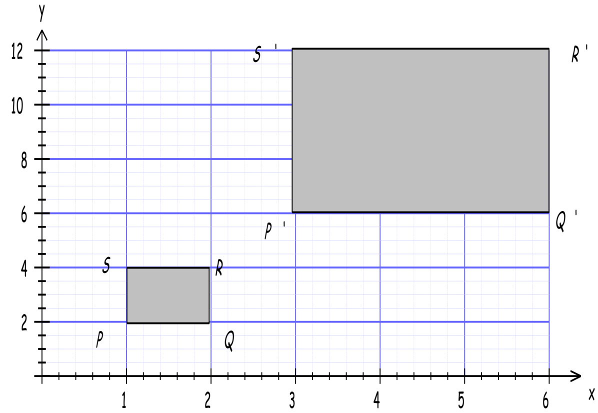


The operation $\mathbf{A}-\mathbf{B}$ translates the table top $PQRS$ four units to the left.

(b) The scalar multiplication 3**A** is given by

$$3\mathbf{A}=3\begin{pmatrix}1&2&2&1\\2&2&4&4\end{pmatrix}=\begin{pmatrix}3&6&6&3\\6&6&12&12\end{pmatrix}$$

Plotting this on the plane gives

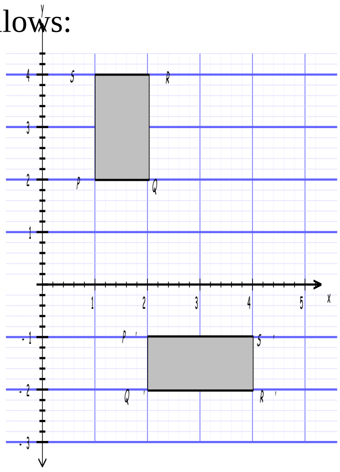


The matrix transformation represented by 3**A** increases each side of the rectangle by 3.

(c) The matrix transformation represented by **CA** is

$$\mathbf{CA}=\begin{pmatrix}0&1\\-1&0\end{pmatrix}\begin{array}{c}\begin{array}{cccc}P&Q&R&S\end{array}\\\begin{pmatrix}1&2&2&1\\2&2&4&4\end{pmatrix}\end{array}=\begin{array}{c}\begin{array}{cccc}P'&Q'&R'&S'\end{array}\\\begin{pmatrix}2&2&4&4\\-1&-2&-2&-1\end{pmatrix}\end{array}$$

We can represent this on the plan as follows:

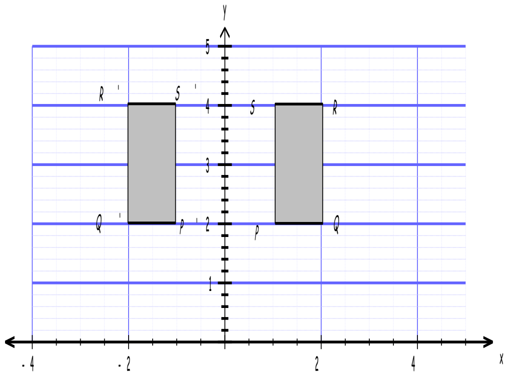


The transformation **CA** represents a rotation of $90°$ clockwise with the origin as the centre.

(d) Similarly we have the matrix transformation represented by **DA** is

$$\mathbf{DA} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \begin{array}{c} \begin{array}{cccc} P & Q & R & S \end{array} \\ \begin{pmatrix} 1 & 2 & 2 & 1 \\ 2 & 2 & 4 & 4 \end{pmatrix} \end{array} = \begin{array}{c} \begin{array}{cccc} P' & Q' & R' & S' \end{array} \\ \begin{pmatrix} -1 & -2 & -2 & -1 \\ 2 & 2 & 4 & 4 \end{pmatrix} \end{array}$$

Plotting this transformation on the plane we have



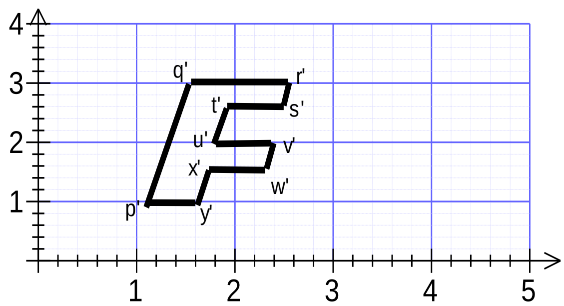
The transformation **DA** represents reflection in the vertical axis.

10. Multiplying the given matrices we have

$$\mathbf{AF} = \begin{pmatrix} 1 & 0.2 \\ 0 & 1 \end{pmatrix} \begin{array}{c} \begin{array}{cccccccccc} p & q & r & s & t & u & v & w & x & y \end{array} \\ \begin{pmatrix} 1 & 1 & 2 & 2 & 1.4 & 1.4 & 2 & 2 & 1.4 & 1.4 \\ 1 & 3 & 3 & 2.6 & 2.6 & 2 & 2 & 1.6 & 1.6 & 1 \end{pmatrix} \end{array}$$

$$= \begin{array}{c} \begin{array}{cccccccccc} p' & q' & r' & s' & t' & u' & v' & w' & x' & y' \end{array} \\ \begin{pmatrix} 1.2 & 1.6 & 2.6 & 2.52 & 1.92 & 1.8 & 2.4 & 2.32 & 1.72 & 1.6 \\ 1 & 3 & 3 & 2.6 & 2.6 & 2 & 2 & 1.6 & 1.6 & 1 \end{pmatrix} \end{array}$$

We can illustrate these vertices (vectors) as follows:



Applying transformation **A** results in italicizing the given letter F.

11. (a) We have $\mathbf{Ax} = \mathbf{O}$ which means that

$$\begin{pmatrix} 1 & 2 \\ 3 & 5 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Multiplying the matrix **A** with the vector **x** gives

$$\left. \begin{array}{rcrcl} x & + & 2y & = & 0 \\ 3x & + & 5y & = & 0 \end{array} \right\} \text{ implies that } x = 0,\ \ y = 0$$

(b) Similarly we have $\mathbf{Ax} = \mathbf{O}$ means

$$\begin{pmatrix} 2 & 7 \\ 3 & 15 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Applying the given matrix **A** to the vector **x** we obtain:

$$\left.\begin{array}{rcrcl} 2x & + & 7y & = & 0 \\ 3x & + & 15y & = & 0 \end{array}\right\} \text{ implies } x = 0,\ y = 0$$

(c) With the given matrix **A** you need to be a bit more careful. Of course $x = 0,\ y = 0$ which means that vector $\mathbf{x} = \mathbf{O}$ is going to work because

$$\mathbf{Ax} = \begin{pmatrix} 1 & 4 \\ 3 & 12 \end{pmatrix}\begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

However there may be other values of x and y for which $\mathbf{Ax} = \mathbf{O}$. We have

$$\mathbf{Ax} = \begin{pmatrix} 1 & 4 \\ 3 & 12 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

The simultaneous equations are

$$\begin{array}{rcrcll} x & + & 4y & = & 0 & \quad (1) \\ 3x & + & 12y & = & 0 & \quad (2) \end{array}$$

Transposing the first equation we have $x = -4y$. If we let y equal any real number r say then $x = -4r$. We can check that $x = -4r$ and $y = r$ satisfies **both** the above equations (1) and (2). Hence $x = -4r$ and $y = r$ where r is any real number is a solution. Hence it will work with $r = 0,\ 1,\ \frac{1}{2},\ \ \pi,\ \sqrt{2},\ e,\ e^{\pi},\cdots$. Try checking by substituting any real number for r into the above equations (1) and (2).

12. (a) Let k and c be scalars such that $k\mathbf{u} + c\mathbf{v} = \mathbf{w}$ which means

$$k\mathbf{u} + c\mathbf{v} = k\begin{pmatrix} 5 \\ 8 \end{pmatrix} + c\begin{pmatrix} 2 \\ 4 \end{pmatrix} = \begin{pmatrix} 5k + 2c \\ 8k + 4c \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad (*)$$

From the bottom row we have

$$8k + 4c = 0 \quad \Rightarrow \quad c = -2k$$

Substituting this $c = -2k$ into the top row gives

$$5k + 2c = 5k + 2(-2k) = k = 1$$

Substituting $k = 1$ into $c = -2k$ gives $c = -2$. Putting $k = 1$ and $c = -2$ into $k\mathbf{u} + c\mathbf{v} = \mathbf{w}$ gives

$$(1)\mathbf{u} + (-2)\mathbf{v} = \mathbf{u} - 2\mathbf{v} = \mathbf{w}$$

You may check that this result is correct by substituting these values into (*). Hence **w** is a linear combination of the vectors **u** and **v**.

(b) Similarly we have

$$k\mathbf{u} + c\mathbf{v} = k\begin{pmatrix} 5 \\ 8 \end{pmatrix} + c\begin{pmatrix} 2 \\ 4 \end{pmatrix} = \begin{pmatrix} 5k + 2c \\ 8k + 4c \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

From the top row we have the equation

$$5k + 2c = 0 \quad \Rightarrow \quad c = -\frac{5}{2}k$$

Putting this into the bottom equation gives

$$8k+4c=8k+4\left(-\frac{5}{2}k\right)=8k-10k=-2k=1$$

Dividing by -2 gives $k=-\frac{1}{2}$. Substituting this $k=-\frac{1}{2}$ into $c=-\frac{5}{2}k$ yields

$$c=-\frac{5}{2}\left(-\frac{1}{2}\right)=\frac{5}{4}$$

Hence we have $k\mathbf{u}+c\mathbf{v}=-\frac{1}{2}\mathbf{u}+\frac{5}{4}\mathbf{v} \underset{\text{Factorizing}}{=} \frac{1}{4}(5\mathbf{v}-2\mathbf{u})=\mathbf{w}$.

(c) We are given $\mathbf{w}=\begin{pmatrix}1\\2\\3\end{pmatrix}$, $\mathbf{u}=\begin{pmatrix}1\\0\\0\end{pmatrix}$ and $\mathbf{v}=\begin{pmatrix}0\\1\\0\end{pmatrix}$.

The bottom entries of the vectors **u** and **v** is zero so whatever scalar multiple of vector **u** added to the scalar multiple of vector **v** it is impossible to get 3 which is the bottom entry of vector **w**. Hence the vector **w** is **not** a linear combination of the vectors **u** and **v**.

(d) Let *k* and *c* be scalars such that

$$k\mathbf{u}+c\mathbf{v}=k\begin{pmatrix}4\\8\\0\end{pmatrix}+c\begin{pmatrix}1\\2\\-3/7\end{pmatrix}=\begin{pmatrix}4k+c\\8k+2c\\-3c/7\end{pmatrix}=\begin{pmatrix}1\\2\\3\end{pmatrix}=\mathbf{w}$$

From the bottom row we have $-\frac{3c}{7}=3 \;\Rightarrow\; c=-7$. Substituting $c=-7$ into the top row gives

$$4k+c=4k-7=1 \quad\Rightarrow\quad k=2$$

Do these values $c=-7$ and $k=2$ work for the middle row:

$$8k+2c=8(2)+2(-7)=2$$

Hence we have $k\mathbf{u}+c\mathbf{v}=2\mathbf{u}-7\mathbf{v}=\mathbf{w}$ which means that the vector **w** is a linear combination of the vectors **u** and **v**.