Complete Solutions to Exercises 2.4

1. To show that the given vectors span $\mathbb{R}^2$ we prove that an arbitrary vector $\mathbf{w}=\begin{pmatrix}a\\b\end{pmatrix}$ is a linear combination of the given vectors.

(a) The scalars k_1 and k_2 need to satisfy $k_1\mathbf{e}_1+k_2\mathbf{e}_2=\mathbf{w}$:

$$k_1\begin{pmatrix}1\\0\end{pmatrix}+k_2\begin{pmatrix}0\\1\end{pmatrix}=\begin{pmatrix}k_1\\0\end{pmatrix}+\begin{pmatrix}0\\k_2\end{pmatrix}=\begin{pmatrix}k_1\\k_2\end{pmatrix}=\begin{pmatrix}a\\b\end{pmatrix}$$

Therefore $k_1=a$ and $k_2=b$ which means the given vectors $\mathbf{e}_1$ and $\mathbf{e}_2$ span $\mathbb{R}^2$.

(b) Similarly we need to find scalars k_1 and k_2 which satisfy $k_1\mathbf{u}+k_2\mathbf{v}=\mathbf{w}$:

$$k_1\mathbf{u}+k_2\mathbf{v}=k_1\begin{pmatrix}1\\1\end{pmatrix}+k_2\begin{pmatrix}-1\\1\end{pmatrix}$$

$$=\begin{pmatrix}k_1\\k_1\end{pmatrix}+\begin{pmatrix}-k_2\\k_2\end{pmatrix}=\begin{pmatrix}k_1-k_2\\k_1+k_2\end{pmatrix}=\begin{pmatrix}a\\b\end{pmatrix}$$

We need to solve the linear simultaneous equations:

$$k_1-k_2=a \qquad (*)$$

$$k_1+k_2=b \qquad (**)$$

for k_1 and k_2. Adding these equations (*) and (**) gives

$$2k_1=a+b \text{ yields } k_1=\frac{a+b}{2}$$

Subtracting (*) from (**) gives

$$2k_2=b-a \text{ implies } k_2=\frac{b-a}{2}$$

Since we have found the scalars, $k_1=\frac{a+b}{2}$ and $k_2=\frac{b-a}{2}$, for any real numbers a and b therefore the given vectors **u** and **v** span $\mathbb{R}^2$.

(c) Again we need to find scalars k_1 and k_2 which satisfy $k_1\mathbf{u}+k_2\mathbf{v}=\mathbf{w}$:

$$k_1\mathbf{u}+k_2\mathbf{v}=k_1\begin{pmatrix}2\\2\end{pmatrix}+k_2\begin{pmatrix}-1\\-1\end{pmatrix}$$

$$=\begin{pmatrix}2k_1\\2k_1\end{pmatrix}+\begin{pmatrix}-k_2\\-k_2\end{pmatrix}=\begin{pmatrix}2k_1-k_2\\2k_1-k_2\end{pmatrix}=\begin{pmatrix}a\\b\end{pmatrix}$$

We need to solve the linear simultaneous equations

$$2k_1-k_2=a \qquad (*)$$

$$2k_1-k_2=b \qquad (**)$$

for k_1 and k_2. Subtracting these equations (*) and (**) gives

$$0=a-b \text{ or } a=b$$

Since $a=b$ the given vectors span $\begin{pmatrix}a\\a\end{pmatrix}=a\begin{pmatrix}1\\1\end{pmatrix}$ and not $\begin{pmatrix}a\\b\end{pmatrix}$.

Hence the given vectors **u** and **v** **do not** span $\mathbb{R}^2$. [Note that vectors **u** and **v** are linearly dependent.]

(d) Similarly we need to find scalars k_1 and k_2 which satisfy $k_1\mathbf{u}+k_2\mathbf{v}=\mathbf{w}$:

$$k_1\mathbf{u}+k_2\mathbf{v}=k_1\begin{pmatrix}1\\2\end{pmatrix}+k_2\begin{pmatrix}-1\\10\end{pmatrix}=\begin{pmatrix}a\\b\end{pmatrix}$$

The augmented matrix is

$$\begin{array}{c}\mathrm{R}_1\\\mathrm{R}_2\end{array}\left(\begin{array}{cc|c}1 & -1 & a\\2 & 10 & b\end{array}\right)$$

Carrying out the row operation $\mathrm{R}_2-2\mathrm{R}_1$ gives

$$\begin{array}{cc}& \begin{array}{cc}k_1 & k_2\end{array}\\\begin{array}{c}\mathrm{R}_1\\\mathrm{R}_2-2\mathrm{R}_1\end{array} & \left(\begin{array}{cc|c}1 & -1 & a\\0 & 12 & b-2a\end{array}\right)\end{array}$$

From the bottom row we have

$$12k_2=b-2a \quad \text{gives} \quad k_2=\frac{b-2a}{12}$$

Substituting this $k_2=\dfrac{b-2a}{12}$ into the top row $k_1=k_2+a$ yields

$$k_1=\frac{b-2a}{12}+a=\frac{b-2a}{12}+\frac{12a}{12}=\frac{b-2a+12a}{12}=\frac{b+10a}{12}$$

Since we have found scalars, $k_1=\dfrac{b+10a}{12}$ and $k_2=\dfrac{b-2a}{12}$, for any real numbers a and b therefore the given vectors **u** and **v** span $\mathbb{R}^2$.

2. Generally similar to question 1 but we have 3 scalars k_1, k_2 and k_3, and we need to find them for any real numbers a, b and c in an arbitrary vector $\mathbf{x}=\begin{pmatrix}a\\b\\c\end{pmatrix}$.

(a) We need to determine scalars k_1, k_2 and k_3 in the linear combination $k_1\mathbf{u}+k_2\mathbf{v}+k_3\mathbf{w}=\mathbf{x}$:

$$k_1\mathbf{u}+k_2\mathbf{v}+k_3\mathbf{w}=k_1\begin{pmatrix}2\\0\\0\end{pmatrix}+k_2\begin{pmatrix}0\\2\\0\end{pmatrix}+k_3\begin{pmatrix}0\\0\\2\end{pmatrix}$$

$$=\begin{pmatrix}2k_1\\0\\0\end{pmatrix}+\begin{pmatrix}0\\2k_2\\0\end{pmatrix}+\begin{pmatrix}0\\0\\2k_3\end{pmatrix}=\begin{pmatrix}2k_1\\2k_2\\2k_3\end{pmatrix}=\begin{pmatrix}a\\b\\c\end{pmatrix}$$

We have the simultaneous linear equations

$$2k_1=a,\ 2k_2=b \ \text{ and } \ 2k_3=c$$

Dividing each equation by 2 gives

$$k_1=\frac{a}{2},\ k_2=\frac{b}{2} \ \text{ and } \ k_3=\frac{c}{2}$$

Since we have scalars, $k_1=\dfrac{a}{2}$, $k_2=\dfrac{b}{2}$ and $k_3=\dfrac{c}{2}$, for any real numbers a, b and c therefore the given vectors **u**, **v** and **w** span $\mathbb{R}^3$.

(b) Consider the linear combination $k_1\mathbf{u}+k_2\mathbf{v}+k_3\mathbf{w}=\mathbf{x}$:

$$k_1\mathbf{u}+k_2\mathbf{v}+k_3\mathbf{w}=k_1\begin{pmatrix}1\\1\\1\end{pmatrix}+k_2\begin{pmatrix}2\\2\\2\end{pmatrix}+k_3\begin{pmatrix}1\\2\\3\end{pmatrix}=\begin{pmatrix}a\\b\\c\end{pmatrix}$$

We use elementary row operations to determine the scalars k_1, k_2 and k_3. Writing out the augmented matrix we have

$$\begin{array}{c}\mathrm{R}_1\\\mathrm{R}_2\\\mathrm{R}_3\end{array}\left(\begin{array}{ccc|c}1&2&1&a\\1&2&2&b\\1&2&3&c\end{array}\right)$$

Carrying out the row operation $\mathrm{R}_3-\mathrm{R}_2$ gives

$$\begin{array}{c}\mathrm{R}_1\\\mathrm{R}_2\\\mathrm{R}_3^*=\mathrm{R}_3-\mathrm{R}_2\end{array}\left(\begin{array}{ccc|c}1&2&1&a\\1&2&2&b\\0&0&1&c-b\end{array}\right)$$

Executing $\mathrm{R}_2-\mathrm{R}_1$ gives

$$\begin{array}{c}\mathrm{R}_1\\\mathrm{R}_2^*=\mathrm{R}_2-\mathrm{R}_1\\\mathrm{R}_3^*\end{array}\left(\begin{array}{ccc|c}1&2&1&a\\0&0&1&b-a\\0&0&1&c-b\end{array}\right)$$

Subtracting the last two rows $\mathrm{R}_3^*-\mathrm{R}_2^*$ gives

$$\begin{array}{c}\mathrm{R}_1\\\mathrm{R}_2^*=\mathrm{R}_2-\mathrm{R}_1\\\mathrm{R}_3^{**}=\mathrm{R}_3^*-\mathrm{R}_2^*\end{array}\left(\begin{array}{ccc|c}1&2&1&a\\0&0&1&b-a\\0&0&0&c-b-(b-a)\end{array}\right)$$

From the bottom row we have

$$0=c-b-(b-a)=c-2b+a$$

This result is only true when $c-2b+a=0$ or $c=2b-a$ but for the vectors to span $\mathbb{R}^3$ we need to show that the result is true for **all** values of *a*, *b* and *c* and not only when $c=2b-a$.

Hence the given vectors **u**, **v** and **w** **do not** span $\mathbb{R}^3$.

(c) We need to determine the scalars k_1, k_2 and k_3 in the linear combination $k_1\mathbf{u}+k_2\mathbf{v}+k_3\mathbf{w}=\mathbf{x}$:

$$k_1\mathbf{u}+k_2\mathbf{v}+k_3\mathbf{w}=k_1\begin{pmatrix}1\\1\\1\end{pmatrix}+k_2\begin{pmatrix}1\\1\\0\end{pmatrix}+k_3\begin{pmatrix}1\\0\\0\end{pmatrix}$$

$$=\begin{pmatrix}k_1\\k_1\\k_1\end{pmatrix}+\begin{pmatrix}k_2\\k_2\\0\end{pmatrix}+\begin{pmatrix}k_3\\0\\0\end{pmatrix}=\begin{pmatrix}k_1+k_2+k_3\\k_1+k_2\\k_1\end{pmatrix}=\begin{pmatrix}a\\b\\c\end{pmatrix}$$

From the last row we have $k_1=c$ and substituting this into the second row gives

$$k_1+k_2=c+k_2=b \text{ which yields } k_2=b-c$$

Substituting $k_1=c$ and $k_2=b-c$ into the first row gives

$$\begin{aligned} k_1 + k_2 + k_3 &= c + (b - c) + k_3 \\ &= b + k_3 = a \quad \text{which gives } k_3 = a - b \end{aligned}$$

Hence we have scalars $k_1 = c$, $k_2 = b - c$ and $k_3 = a - b$. These are valid for any real numbers $a,\ b$ and c so therefore the given vectors **u**, **v** and **w** span $\mathbb{R}^3$.

(d) Similarly we have scalars $k_1,\ k_2$ and k_3 in the linear combination $k_1\mathbf{u} + k_2\mathbf{v} + k_3\mathbf{w} = \mathbf{x}$ where **x** is an arbitrary vector - $\mathbf{x} = \begin{pmatrix} a & b & c \end{pmatrix}^T$:

$$k_1\mathbf{u} + k_2\mathbf{v} + k_3\mathbf{w} = k_1\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} + k_2\begin{pmatrix} 2 \\ 4 \\ 0 \end{pmatrix} + k_3\begin{pmatrix} -2 \\ -2 \\ 3 \end{pmatrix} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$$

We use elementary row operations to determine the scalars $k_1,\ k_2$ and k_3. Writing out the augmented matrix we have

$$\begin{array}{c} \mathrm{R}_1 \\ \mathrm{R}_2 \\ \mathrm{R}_3 \end{array} \left(\begin{array}{ccc|c} 1 & 2 & -2 & a \\ 2 & 4 & -2 & b \\ 1 & 0 & 3 & c \end{array}\right)$$

Carrying out the row operation $\mathrm{R}_2 - \mathrm{R}_1$:

$$\begin{array}{c} \\ \mathrm{R}_1 \\ \mathrm{R}_2 - 2\mathrm{R}_1 \\ \mathrm{R}_3 \end{array} \begin{array}{c} \begin{array}{ccc} k_1 & k_2 & k_3 \end{array} \\ \left(\begin{array}{ccc|c} 1 & 2 & -2 & a \\ 0 & 0 & 2 & b - 2a \\ 1 & 0 & 3 & c \end{array}\right) \end{array}$$

From the middle row we have

$$2k_3 = b - 2a \quad \text{gives} \quad k_3 = \frac{b - 2a}{2}$$

Substituting this $k_3 = \dfrac{b - 2a}{2}$ into the bottom row gives

$$k_1 + 3\left(\frac{b - 2a}{2}\right) = c$$

$$k_1 = c - 3\left(\frac{b - 2a}{2}\right) = \frac{2c}{2} - \frac{3b - 6a}{2} = \frac{2c - 3b + 6a}{2}$$

What else do we need to find?

The last scalar k_2. *How?*

By substituting $k_1 = \dfrac{2c - 3b + 6a}{2}$ and $k_3 = \dfrac{b - 2a}{2}$ into the top row

$$k_1 + 2k_2 - 2k_3 = a$$

$$\underbrace{\frac{2c-3b+6a}{2}}_{=k_1}+2k_2-2\underbrace{\left(\frac{b-2a}{2}\right)}_{=k_3}=a$$

$$2c-3b+6a+4k_2-2(b-2a)=2a \qquad \text{[Multiplying by 2]}$$

$$4k_2=2a+2(b-2a)-2c+3b-6a \qquad \text{[Transposing]}$$

$$4k_2=2a+2b-4a-2c+3b-6a$$

$$4k_2=5b-2c-8a \qquad \text{[Simplifying]}$$

$$k_2=\frac{5b-2c-8a}{4}$$

We have $k_1=\dfrac{2c-3b+6a}{2}$, $k_2=\dfrac{5b-2c-8a}{4}$ and $k_3=\dfrac{b-2a}{2}$ for any real values of *a, b* and *c*. Hence the given vectors **u**, **v** and **w** span $\mathbb{R}^3$.

3. To find whether the 2 given vectors form a basis for $\mathbb{R}^2$ we need to check only one of the following:
(i) The vectors span $\mathbb{R}^2$.
Or (ii) The vectors are linearly independent.

(a) We show that the given 2 vectors $\mathbf{u}=\begin{pmatrix}1\\2\end{pmatrix}$ and $\mathbf{v}=\begin{pmatrix}0\\1\end{pmatrix}$ are linearly independent.

Since these vectors are not scalar multiples of each other so they are linearly independent.
Hence by:

Proposition (2-16). Any *n* linearly independent vectors in $\mathbb{R}^n$ form a basis for $\mathbb{R}^n$.

The given vectors $\mathbf{u}=\begin{pmatrix}1\\2\end{pmatrix}$ and $\mathbf{v}=\begin{pmatrix}0\\1\end{pmatrix}$ form a basis for $\mathbb{R}^2$.

(b) *What do you notice when examining the given vectors*

$$\mathbf{u}=\begin{pmatrix}-2\\-4\end{pmatrix} \textit{ and } \mathbf{v}=\begin{pmatrix}1\\2\end{pmatrix}?$$

The vector **u** is -2 times the vector **v** or in mathematical notation we have

$$\mathbf{u}=-2\mathbf{v}$$

This means that $\mathbf{u}+2\mathbf{v}=\mathbf{O}$, that is there are non - zero scalars which produce the zero vector. Hence $k_1\mathbf{u}+k_2\mathbf{v}=\mathbf{O}$ where $k_1=1$ and $k_2=2$ which means that the vectors **u** and **v** are linearly **dependent**. The given vectors **cannot** form a basis for $\mathbb{R}^2$.

(c) Since the given vectors $\begin{pmatrix}4\\1\end{pmatrix}\neq m\begin{pmatrix}1\\3\end{pmatrix}$ (*m* is a scalar) are not scalar multiples of each other so they are linearly independent. Again by

Proposition (2-16). Any *n* linearly independent vectors in $\mathbb{R}^n$ form a basis for $\mathbb{R}^n$.

The given vectors form a basis for $\mathbb{R}^2$.

(d) We know that

$$\begin{pmatrix}3\\5\end{pmatrix}\neq m\begin{pmatrix}2\\3\end{pmatrix} \quad \text{where } m \text{ is a scalar}$$

Hence the given vectors are linearly independent so they form a basis for $\mathbb{R}^2$.

4. (a) The given set of vectors $\mathbf{u}=\begin{pmatrix}1\\2\end{pmatrix}$, $\mathbf{v}=\begin{pmatrix}0\\1\end{pmatrix}$ and $\mathbf{w}=\begin{pmatrix}1\\0\end{pmatrix}$ do **not** form a basis for $\mathbb{R}^2$ because by Proposition (2-20) the basis of $\mathbb{R}^n$ has exactly *n* vectors but in this case we have 3 vectors in $\mathbb{R}^2$.

(b) The given set of vectors $\mathbf{u}=\begin{pmatrix}1\\1\\1\end{pmatrix}$, $\mathbf{v}=\begin{pmatrix}1\\1\\0\end{pmatrix}$ and $\mathbf{w}=\begin{pmatrix}0\\0\\0\end{pmatrix}$ do **not** form a basis for $\mathbb{R}^3$ because we have the zero vector **w** which means that these vectors are linearly **dependent**.

(c) The given set of vectors $\mathbf{u}=\begin{pmatrix}1\\1\\1\\1\end{pmatrix}$, $\mathbf{v}=\begin{pmatrix}1\\1\\0\\-1\end{pmatrix}$, $\mathbf{w}=\begin{pmatrix}-1\\-2\\-3\\4\end{pmatrix}$ and $\mathbf{x}=\begin{pmatrix}2\\2\\2\\2\end{pmatrix}$ do **not** form a basis for $\mathbb{R}^4$ because the vectors **u** and **x** are multiplies of each other, $\mathbf{u}=2\mathbf{x}$, which means that these vectors are linearly **dependent**.

(d) The given set of vectors $\mathbf{u}=\begin{pmatrix}1\\2\\3\\4\\5\end{pmatrix}$, $\mathbf{v}=\begin{pmatrix}1\\5\\0\\2\\4\end{pmatrix}$, $\mathbf{w}=\begin{pmatrix}3\\2\\-5\\4\\-9\end{pmatrix}$ and $\mathbf{x}=\begin{pmatrix}2\\9\\2\\7\\7\end{pmatrix}$ do **not** form a basis for $\mathbb{R}^5$ because by Proposition (2-20) the basis of $\mathbb{R}^n$ has exactly *n* vectors but in this case we have 4 vectors in $\mathbb{R}^5$.

5. (a) We are given that $\mathbf{A}=\begin{pmatrix}1&1&0\\2&5&0\\3&4&9\end{pmatrix}$, $\mathbf{b}=\begin{pmatrix}1\\3\\4\end{pmatrix}$. Let the scalars be $k_1,\ k_2,\ k_3$ which we can write as entries in the vector $\mathbf{x}=\begin{pmatrix}k_1\\k_2\\k_3\end{pmatrix}$. The vector **b** is in the space spanned by the columns of matrix **A** if we can find values for the scalars *k*'s such that $\mathbf{Ax}=\mathbf{b}$. The augmented matrix is:

$$\begin{array}{c}\mathrm{R}_1\\\mathrm{R}_2\\\mathrm{R}_3\end{array}\left(\begin{array}{ccc|c}1&1&0&1\\2&5&0&3\\3&4&9&4\end{array}\right)$$

Carrying out the row operations $\mathrm{R}_2-2\mathrm{R}_1$ and $\mathrm{R}_3-3\mathrm{R}_1$:

$$\begin{array}{c}\\\mathrm{R}_1\\\mathrm{R}_2-2\mathrm{R}_1\\\mathrm{R}_3-3\mathrm{R}_1\end{array}\begin{array}{c}\begin{array}{ccc}k_1&k_2&k_3\end{array}\\\left(\begin{array}{ccc|c}1&1&0&1\\0&3&0&1\\0&1&9&1\end{array}\right)\end{array}$$

Expanding the middle row gives:

$$3k_2 = 1 \quad \Rightarrow \quad k_2 = \frac{1}{3}$$

Using the top row we have

$$k_1 + k_2 = 1 \quad \Rightarrow \quad k_1 = 1 - k_2 = 1 - \frac{1}{3} = \frac{2}{3}$$

Using the bottom row:

$$k_2 + 9k_3 = 1 \quad \Rightarrow \quad k_3 = \frac{1-k_2}{9} = \frac{1-1/3}{9} = \frac{2/3}{9} = \frac{2}{27}$$

This means that

$$\frac{2}{3}\begin{pmatrix}1\\2\\3\end{pmatrix} + \frac{1}{3}\begin{pmatrix}1\\5\\4\end{pmatrix} + \frac{2}{27}\begin{pmatrix}0\\0\\9\end{pmatrix} = \begin{pmatrix}1\\3\\4\end{pmatrix} = \mathbf{b}$$

Hence the vector **b** is in the space spanned by the columns of matrix **A**.

(b) Let $\mathbf{x} = \begin{pmatrix}k_1\\k_2\\k_3\end{pmatrix}$ where $k_1,\ k_2,\ k_3$ are the scalars associated with the columns of matrix **A**. We have the augmented matrix $(\mathbf{A} \mid \mathbf{b})$ given by:

$$\begin{array}{c} \mathrm{R}_1 \\ \mathrm{R}_2 \\ \mathrm{R}_3 \end{array} \left(\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 4 & 5 & 6 & 3 \\ 7 & 8 & 9 & 4 \end{array}\right)$$

Carrying out row operations $\mathrm{R}_2 - \mathrm{R}_1$ and $\mathrm{R}_3 - \mathrm{R}_2$:

$$\begin{array}{c} \mathrm{R}_1 \\ \mathrm{R}_2{*} = \mathrm{R}_2 - \mathrm{R}_1 \\ \mathrm{R}_3{*} = \mathrm{R}_3 - \mathrm{R}_2 \end{array} \left(\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 3 & 3 & 3 & 2 \\ 3 & 3 & 3 & 1 \end{array}\right)$$

Subtracting the bottom two rows $\mathrm{R}_3{*} - \mathrm{R}_2{*}$ gives:

$$\begin{array}{cc} & \begin{array}{ccc} k_1 & k_2 & k_3 \end{array} \\ \begin{array}{c} \mathrm{R}_1 \\ \mathrm{R}_2{*} \\ \mathrm{R}_3{*} - \mathrm{R}_2{*} \end{array} & \left(\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 3 & 3 & 3 & 2 \\ 0 & 0 & 0 & -1 \end{array}\right) \end{array}$$

Expanding the bottom row we have

$$0k_1 + 0k_2 + 0k_3 = 0 = -1$$

This means the system $\mathbf{Ax} = \mathbf{b}$ is inconsistent, so there is **no** vector **x** which satisfies this linear system. Hence the vector **b** is not in the space spanned by the columns of matrix **A**.

6. (a) Since we are given 3 vectors $\mathbf{u} = \begin{pmatrix}5\\0\\0\end{pmatrix}$, $\mathbf{v} = \begin{pmatrix}0\\6\\0\end{pmatrix}$ and $\mathbf{w} = \begin{pmatrix}0\\0\\7\end{pmatrix}$ so it is enough to show that these vectors are linearly independent for them to form a basis for $\mathbb{R}^3$. Consider the linear combination $k_1\mathbf{u} + k_2\mathbf{v} + k_3\mathbf{w} = \mathbf{O}$ where $k_1,\ k_2$ and k_3 are scalars:

$$k_1\mathbf{u}+k_2\mathbf{v}+k_3\mathbf{w}=k_1\begin{pmatrix}5\\0\\0\end{pmatrix}+k_2\begin{pmatrix}0\\6\\0\end{pmatrix}+k_3\begin{pmatrix}0\\0\\7\end{pmatrix}$$
$$=\begin{pmatrix}5k_1\\0\\0\end{pmatrix}+\begin{pmatrix}0\\6k_2\\0\end{pmatrix}+\begin{pmatrix}0\\0\\7k_3\end{pmatrix}=\begin{pmatrix}5k_1\\6k_2\\7k_3\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix}$$

Solving these we have

$$k_1=0,\ k_2=0,\ k_3=0$$

Since $k_1\mathbf{u}+k_2\mathbf{v}+k_3\mathbf{w}=\mathbf{O}$ only has the trivial solution $k_1=k_2=k_3=0$ so the given vectors are linearly independent.

Hence the vectors **u**, **v** and **w** form a basis for $\mathbb{R}^3$.

(b) Similarly consider the linear combination $k_1\mathbf{u}+k_2\mathbf{v}+k_3\mathbf{w}=\mathbf{O}$ where $k_1,\ k_2$ and k_3 are scalars:

$$k_1\mathbf{u}+k_2\mathbf{v}+k_3\mathbf{w}=k_1\begin{pmatrix}\sim\\0\\0\end{pmatrix}+k_2\begin{pmatrix}0\\ \\0\end{pmatrix}+k_3\begin{pmatrix}0\\0\\\text{a}\end{pmatrix}$$
$$=\begin{pmatrix}k_1\sim\\0\\0\end{pmatrix}+\begin{pmatrix}0\\k_2\\0\end{pmatrix}+\begin{pmatrix}0\\0\\k_3\text{a}\end{pmatrix}=\begin{pmatrix}k_1\sim\\k_2\\k_3\text{a}\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix}$$

We are given that $\sim\neq 0,\quad \neq 0$ and $\text{a}\neq 0$. Solving the above gives $k_1=k_2=k_3=0$.

We have $k_1\mathbf{u}+k_2\mathbf{v}+k_3\mathbf{w}=\mathbf{O}$ with only the trivial solution $k_1=k_2=k_3=0$ therefore the given vectors are linearly independent.

Hence the vectors **u**, **v** and **w** form a basis for $\mathbb{R}^3$.

7. To show that given vectors **do not** form a basis we can show it does not span $\mathbb{R}^3$ or they are linearly dependent. Writing out the linear combination of the vectors **u**, **v**, **w** and **x** where **x** is an arbitrary vector in $\mathbb{R}^3$:

$$k_1\mathbf{u}+k_2\mathbf{v}+k_3\mathbf{w}=k_1\begin{pmatrix}1\\1\\2\end{pmatrix}+k_2\begin{pmatrix}-1\\1\\-2\end{pmatrix}+k_3\begin{pmatrix}1\\-5\\2\end{pmatrix}=\begin{pmatrix}a\\b\\c\end{pmatrix}$$

We have the augmented matrix

$$\begin{array}{c}\mathrm{R}_1\\\mathrm{R}_2\\\mathrm{R}_3\end{array}\left(\begin{array}{ccc|c}1&-1&1&a\\1&1&-5&b\\2&-2&2&c\end{array}\right)$$

Carrying out the row operation $\mathrm{R}_2-\mathrm{R}_1$ and $\mathrm{R}_3-2\mathrm{R}_1$:

$$\begin{array}{c}\mathrm{R}_1\\\mathrm{R}_2^*=\mathrm{R}_2-\mathrm{R}_1\\\mathrm{R}_3^*=\mathrm{R}_3-2\mathrm{R}_1\end{array}\left(\begin{array}{ccc|c}1&-1&1&a\\0&2&-6&b-a\\0&0&0&c-2a\end{array}\right)$$

From the last row we have $c-2a=0$ or $c=2a$. This means that the given vectors $\mathbf{u}=\begin{pmatrix}1 & 1 & 2\end{pmatrix}^T$, $\mathbf{v}=\begin{pmatrix}-1 & 1 & -2\end{pmatrix}^T$, $\mathbf{w}=\begin{pmatrix}1 & -5 & 2\end{pmatrix}^T$ only span vectors of the form:

$$\begin{pmatrix}a\\b\\2a\end{pmatrix} \text{ and not } \begin{pmatrix}a\\b\\c\end{pmatrix}$$

Hence vectors **u**, **v** and **w** do not span $\mathbb{R}^3$ so they cannot form a basis for $\mathbb{R}^3$.

8. It is enough to show that the given vectors are linearly independent.
Consider the linear combination $k_1\mathbf{u}+k_2\mathbf{v}+k_3\mathbf{w}+k_4\mathbf{x}=\mathbf{O}$:

$$k_1\mathbf{u}+k_2\mathbf{v}+k_3\mathbf{w}+k_4\mathbf{x}=k_1\begin{pmatrix}1\\1\\1\\1\end{pmatrix}+k_2\begin{pmatrix}0\\1\\1\\1\end{pmatrix}+k_3\begin{pmatrix}0\\0\\1\\1\end{pmatrix}+k_4\begin{pmatrix}0\\0\\0\\1\end{pmatrix}$$

$$=\begin{pmatrix}k_1\\k_1\\k_1\\k_1\end{pmatrix}+\begin{pmatrix}0\\k_2\\k_2\\k_2\end{pmatrix}+\begin{pmatrix}0\\0\\k_3\\k_3\end{pmatrix}+\begin{pmatrix}0\\0\\0\\k_4\end{pmatrix}=\begin{pmatrix}k_1\\k_1+k_2\\k_1+k_2+k_3\\k_1+k_2+k_3+k_4\end{pmatrix}=\begin{pmatrix}0\\0\\0\\0\end{pmatrix}$$

Solving this only gives the trivial solution $k_1=k_2=k_3=k_4=0$.
This means that the given vectors **u**, **v**, **w** and **x** are linearly **independent**.
Since the 4 vectors **u**, **v**, **w** and **x** are linearly independent so we conclude that these vectors form a basis for $\mathbb{R}^4$.

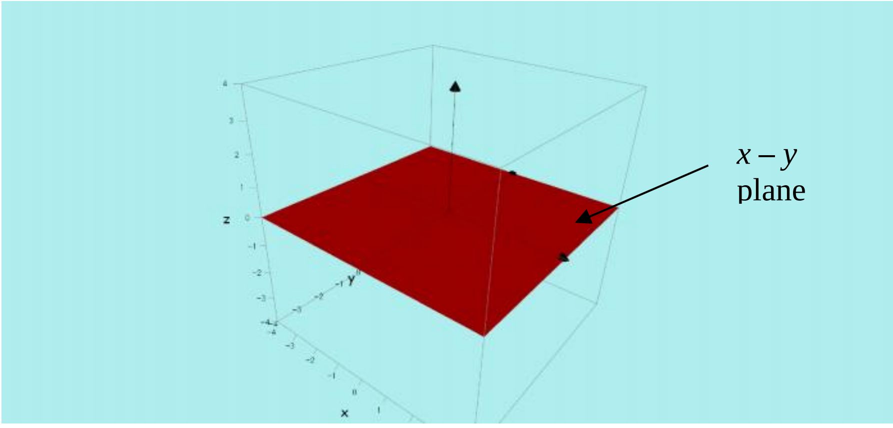
9. We are given the vectors $\mathbf{u}=\begin{pmatrix}1 & 5 & 0\end{pmatrix}^T=\begin{pmatrix}1\\5\\0\end{pmatrix}$, $\mathbf{v}=\begin{pmatrix}0 & 1 & 0\end{pmatrix}^T=\begin{pmatrix}0\\1\\0\end{pmatrix}$.

(a) Note that $\begin{pmatrix}1\\5\\0\end{pmatrix}\neq m\begin{pmatrix}0\\1\\0\end{pmatrix}$ where m is a scalar. Hence vectors **u** and **v** are not scalar multiples of each other so they are linearly independent.

(b) The space spanned by vectors **u** and **v** is every linear combination of these vectors, that is

$$k\mathbf{u}+c\mathbf{v}=k\begin{pmatrix}1\\5\\0\end{pmatrix}+c\begin{pmatrix}0\\1\\0\end{pmatrix}=\begin{pmatrix}k\\5k+c\\0\end{pmatrix}=\begin{pmatrix}x\\y\\0\end{pmatrix} \text{ where } x \text{ and } y \text{ are any real numbers}$$

The vectors **u** and **v** span the $x-y$ plane:



The dark plane shown is the space spanned by vectors **u** and **v**. This is all the vectors of the form $\begin{pmatrix} x \\ y \\ 0 \end{pmatrix}$ where x and y are any real numbers.

(c) We need a vector **w** which is linearly independent of $\mathbf{u} = \begin{pmatrix} 1 \\ 5 \\ 0 \end{pmatrix}$, $\mathbf{v} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ and all 3 vectors span $\mathbb{R}^3$. Since **u** and **v** span $\begin{pmatrix} x \\ y \\ 0 \end{pmatrix}$ so we let $\mathbf{w} = \begin{pmatrix} 0 \\ 0 \\ z \end{pmatrix}$ where $z \neq 0$.

10. Need to show that if $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \cdots, \mathbf{v}_n$ span $\mathbb{R}^n$ then $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \cdots, \mathbf{v}_n$ and $\mathbf{w}$ also span $\mathbb{R}^n$.

Proof.

Since $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \cdots, \mathbf{v}_n$ span $\mathbb{R}^n$ so for any vector $\mathbf{u}$ in $\mathbb{R}^n$ we have

$$\begin{aligned} \mathbf{u} &= k_1\mathbf{v}_1 + k_2\mathbf{v}_2 + k_3\mathbf{v}_3 + \cdots + k_n\mathbf{v}_n \\ &= k_1\mathbf{v}_1 + k_2\mathbf{v}_2 + k_3\mathbf{v}_3 + \cdots + k_n\mathbf{v}_n + 0\mathbf{w} \end{aligned}$$

Hence $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \cdots, \mathbf{v}_n$ and $\mathbf{w}$ also span $\mathbb{R}^n$.

■

11. Need to prove that

$$T = \{k_1\mathbf{v}_1,\ k_2\mathbf{v}_2,\ k_3\mathbf{v}_3,\ \cdots,\ k_n\mathbf{v}_n\}$$

where none of the k's are zero is also a basis for $\mathbb{R}^n$.

Proof.

Let **u** be an arbitrary vector in $\mathbb{R}^n$ then

$$\mathbf{u} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3 + \cdots + c_n\mathbf{v}_n$$

because the **v**'s form a basis for $\mathbb{R}^n$. Since all the scalars are real numbers therefore for each j in $1 \le j \le n$ we let

$$\frac{c_j}{k_j} = a_j \text{ or } c_j = k_j a_j$$

Substituting this, $c_j = k_j a_j$, for each j we have

$$\begin{aligned}\mathbf{u} &= c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3 + \cdots + c_n\mathbf{v}_n \\ &= (k_1a_1)\mathbf{v}_1 + (k_2a_2)\mathbf{v}_2 + (k_3a_3)\mathbf{v}_3 + \cdots + (k_na_n)\mathbf{v}_n \\ &= a_1(k_1\mathbf{v}_1) + a_2(k_2\mathbf{v}_2) + a_3(k_3\mathbf{v}_3) + \cdots + a_n(k_n\mathbf{v}_n)\end{aligned}$$

Hence we have shown that the set of vectors $T = \{k_1\mathbf{v}_1,\ k_2\mathbf{v}_2,\ k_3\mathbf{v}_3,\ \cdots,\ k_n\mathbf{v}_n\}$ span $\mathbb{R}^n$. By

Proposition (2-17). Any *n* vectors which span $\mathbb{R}^n$ form a basis for $\mathbb{R}^n$.

Hence we conclude that the set

$$T = \{k_1\mathbf{v}_1,\ k_2\mathbf{v}_2,\ k_3\mathbf{v}_3,\ \cdots,\ k_n\mathbf{v}_n\}$$

forms a basis for $\mathbb{R}^n$.

■

12. We need to prove:

$\mathbf{A} = (\mathbf{v}_1 \quad \mathbf{v}_2 \quad \cdots \quad \mathbf{v}_n)$ is invertible $\Leftrightarrow$ $\{\mathbf{v}_1, \cdots,\ \mathbf{v}_n\}$ form a basis for $\mathbb{R}^n$.

Proof.

This result follows from the following proposition of the last section:

Proposition (2-14). Let **A** be the *n* by *n* matrix whose columns are given by the vectors $\mathbf{v}_1,\ \mathbf{v}_2,\ \mathbf{v}_3,\ \cdots$ and $\mathbf{v}_n$:

$$\mathbf{A} = (\mathbf{v}_1 \quad \mathbf{v}_2 \quad \cdots \quad \mathbf{v}_n)$$

Then vectors $\mathbf{v}_1,\ \mathbf{v}_2,\ \cdots,\ \mathbf{v}_n$ are linearly independent $\Leftrightarrow$ matrix **A** is invertible.

This is because *n* linearly independent vectors form a basis for $\mathbb{R}^n$.

■

13. We need to prove that *n* linearly independent vectors in $\mathbb{R}^n$ form a basis for $\mathbb{R}^n$.

Proof.

If the set $S = \{\mathbf{v}_1,\ \mathbf{v}_2,\ \mathbf{v}_3,\ \cdots,\ \mathbf{v}_n\}$ of independent vectors does *not* span $\mathbb{R}^n$ then you can find another vector **u**, which is *not* a linear combination of $S = \{\mathbf{v}_1,\ \mathbf{v}_2,\ \cdots,\ \mathbf{v}_n\}$.

Therefore a new set $T = \{\mathbf{v}_1,\ \mathbf{v}_2,\ \cdots,\ \mathbf{v}_n,\ \mathbf{u}\}$ which consists of $n+1$ vectors and so by

Proposition (2-13):

Let $\mathbf{v}_1,\ \mathbf{v}_2,\ \mathbf{v}_3,\ \cdots$ and $\mathbf{v}_m$ be different vectors in $\mathbb{R}^n$. If $n < m$, that is the value of *n* in the n – space is less than the number *m* of vectors, then the vectors $\mathbf{v}_1,\ \mathbf{v}_2,\ \mathbf{v}_3,\ \cdots$ and $\mathbf{v}_m$ are linearly dependent.

This set $T = \{\mathbf{v}_1,\ \mathbf{v}_2,\ \cdots,\ \mathbf{v}_n,\ \mathbf{u}\}$ must be linearly dependent. Hence

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots c_n\mathbf{v}_n + c_{n+1}\mathbf{u} = \mathbf{O} \text{ where all the } c\text{'s are not zero}$$

This implies that **u** can be written as a linear combination of $S = \{\mathbf{v}_1,\ \mathbf{v}_2,\ \mathbf{v}_3,\ \cdots,\ \mathbf{v}_n\}$ so it is in the span of *S*. This is a contradiction because we stated above that **u** could not spanned by the vectors in *S*.

Hence $S = \{\mathbf{v}_1,\ \mathbf{v}_2,\ \mathbf{v}_3,\ \cdots,\ \mathbf{v}_n\}$ is a set of linearly independent vectors which spans $\mathbb{R}^n$ so it is a basis for $\mathbb{R}^n$.

■

14. Need to prove that any *n* vectors which span $\mathbb{R}^n$ form a basis for $\mathbb{R}^n$.
Proof.
Let $S = \{\mathbf{v}_1, \ \mathbf{v}_2, \ \mathbf{v}_3, \ \cdots, \ \mathbf{v}_n\}$ be a set of *n* vectors which span $\mathbb{R}^n$.
Need to show these vectors in *S* are linearly independent for *S* to be a basis for $\mathbb{R}^n$. Suppose the vectors in *S* are linearly dependent. This means one of the vectors can be written as a linear combination of the others. Without Loss of Generality we can select this to be the last vector $\mathbf{v}_n$. Hence the vector $\mathbf{v}_n$ is a linear combination of the remaining vectors $T = \{\mathbf{v}_1, \ \mathbf{v}_2, \ \mathbf{v}_3, \ \cdots, \ \mathbf{v}_{n-1}\}$. This set T still spans $\mathbb{R}^n$. If this set *T* is linearly dependent then keep on removing vectors until we end up with a linearly independent set of vectors. This last set of vectors still spans $\mathbb{R}^n$ and is linearly independent which means it is a basis for $\mathbb{R}^n$. We have less than *n* vectors which are a basis for $\mathbb{R}^n$. This contradicts:

> Proposition (2-20). Every basis of $\mathbb{R}^n$ contains exactly *n* vectors.

Our supposition that the vectors in $S = \{\mathbf{v}_1, \ \mathbf{v}_2, \ \mathbf{v}_3, \ \cdots, \ \mathbf{v}_n\}$ are linearly dependent must be wrong. Hence any *n* vectors which span $\mathbb{R}^n$ form a basis for $\mathbb{R}^n$.

■

15. We need to prove that any *n* non-zero orthogonal vectors form a basis for $\mathbb{R}^n$.
Proof.
Let $S = \{\mathbf{v}_1, \ \mathbf{v}_2, \ \mathbf{v}_3, \ \cdots, \ \mathbf{v}_n\}$ be a set of non-zero orthogonal vectors.
If we can show that this set *S* of vectors is linearly independent then we are done. *Why?* Because

> Proposition (2-16). Any *n* linearly independent vectors in $\mathbb{R}^n$ form a basis for $\mathbb{R}^n$.

Consider the linear combination

$$k_1\mathbf{v}_1 + k_2\mathbf{v}_2 + k_3\mathbf{v}_3 + \cdots + k_n\mathbf{v}_n = \mathbf{O}$$

where *k*'s scalars. Required to prove that the only solution is $k_1 = k_2 = k_3 = \cdots = k_n = 0$.
The dot product of $\mathbf{O}\cdot\mathbf{v}_1 = 0$. Substituting $k_1\mathbf{v}_1 + k_2\mathbf{v}_2 + k_3\mathbf{v}_3 + \cdots + k_n\mathbf{v}_n = \mathbf{O}$ into $\mathbf{O}\cdot\mathbf{v}_1 = 0$ gives:

$$\begin{aligned}(k_1\mathbf{v}_1 + k_2\mathbf{v}_2 + \cdots + k_n\mathbf{v}_n)\cdot\mathbf{v}_1 &= k_1(\mathbf{v}_1\cdot\mathbf{v}_1) + k_2\underbrace{(\mathbf{v}_2\cdot\mathbf{v}_1)}_{=0} + \cdots + k_n\underbrace{(\mathbf{v}_n\cdot\mathbf{v}_1)}_{=0} \\ &= k_1(\mathbf{v}_1\cdot\mathbf{v}_1) \\ &= k_1\|\mathbf{v}_1\| = 0\end{aligned}$$

$\left[\text{Because the } \mathbf{v}\text{'s are orthogonal}\right]$

We have $k_1\|\mathbf{v}_1\| = 0$ which gives $k_1 = 0$ because $\mathbf{v}_1$ is a non-zero vector.
Similarly by following the above process with

$$\mathbf{O}\cdot\mathbf{v}_2 = 0, \ \mathbf{O}\cdot\mathbf{v}_3 = 0, \ \cdots, \ \mathbf{O}\cdot\mathbf{v}_n = 0$$

we have $k_1 = k_2 = k_3 = \cdots = k_n = 0$. This means that $S = \{\mathbf{v}_1, \ \mathbf{v}_2, \ \mathbf{v}_3, \ \cdots, \ \mathbf{v}_n\}$ is a set of linearly independent vectors.
Hence by Proposition (2-16) we conclude that $S = \{\mathbf{v}_1, \ \mathbf{v}_2, \ \mathbf{v}_3, \ \cdots, \ \mathbf{v}_n\}$ forms a basis for $\mathbb{R}^n$.

■

16. Need to prove that if $\{\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_n\}$ is a basis then $\{\mathbf{Av}_1, \mathbf{Av}_2, \cdots, \mathbf{Av}_n\}$ is also a basis for $\mathbb{R}^n$ provided **A** is an invertible matrix.
Proof.
Required to prove $\{\mathbf{Av}_1, \mathbf{Av}_2, \cdots, \mathbf{Av}_n\}$ are linearly independent. *Why?*
Because of the following:

Proposition (2-16). Any *n* linearly independent vectors in $\mathbb{R}^n$ form a basis for $\mathbb{R}^n$.

Consider the linear combination with *k*'s scalars

$$k_1(\mathbf{Av}_1) + k_2(\mathbf{Av}_2) + \cdots + k_n(\mathbf{Av}_n) = \mathbf{O} \qquad (*)$$

Need to show that the only solution to this (*) is when all the scalars are zero; $k_1 = k_2 = \cdots = k_n = 0$. Factorizing out the matrix **A** gives

$$\mathbf{A}(k_1\mathbf{v}_1 + k_2\mathbf{v}_2 + \cdots + k_n\mathbf{v}_n) = \mathbf{O}$$

We are given that matrix **A** is invertible so $\mathbf{A}^{-1}$ exists and

$$k_1\mathbf{v}_1 + k_2\mathbf{v}_2 + \cdots + k_n\mathbf{v}_n = \mathbf{A}^{-1}\mathbf{O} = \mathbf{O} \qquad (**)$$

We are also given that vectors $\{\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_n\}$ are a basis so they linearly independent which means the only solution to (**) is with $k_1 = k_2 = \cdots = k_n = 0$.
Hence the only solution to (*) is with all the scalars are zero so $\{\mathbf{Av}_1, \mathbf{Av}_2, \cdots, \mathbf{Av}_n\}$ is linearly independent which implies it is a basis for $\mathbb{R}^n$.

■