Complete Solutions to Exercises 5.5

1. Read off the coefficients of x, y, z and w. The coefficients of x are the entries in the first column of standard matrix $\mathbf{A}$, coefficients of y are the entries in the second column of standard matrix $\mathbf{A}$, etc.

(a) $\mathbf{A} = \begin{pmatrix} 1 & 1 \\ 2 & 2 \end{pmatrix}$ (b) $\mathbf{A} = \begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix}$ (c) $\mathbf{A} = \begin{pmatrix} 2 & 3 \\ 1 & -5 \end{pmatrix}$

(d) $\mathbf{A} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & -1 & -1 \\ 2 & 1 & -1 \end{pmatrix}$ (e) $\mathbf{A} = \begin{pmatrix} 2 & -1 & 0 \\ 4 & -1 & 3 \\ 7 & -1 & -1 \end{pmatrix}$

(f) $\mathbf{A} = \begin{pmatrix} 1 & -1 & 1 & -3 \\ -1 & 3 & -7 & -1 \\ 9 & 5 & 6 & 12 \\ 1 & 0 & 0 & 0 \end{pmatrix}$ (g) $\mathbf{A} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = \mathbf{O}$

2. We use the standard basis $\mathbf{e}_1,\ \mathbf{e}_2,\ \mathbf{e}_3,\ \cdots,\ \mathbf{e}_n$ in each case because we need to find the **standard** matrix.

(a) We are given $T\left(\begin{pmatrix} x \\ y \end{pmatrix}\right) = \begin{pmatrix} x-y \\ x+2y \end{pmatrix}$. Applying this linear transformation to $\mathbf{e}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\mathbf{e}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ gives

$$T(\mathbf{e}_1) = T\left(\begin{pmatrix} 1 \\ 0 \end{pmatrix}\right) = \begin{pmatrix} 1-0 \\ 1+2(0) \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$T(\mathbf{e}_2) = T\left(\begin{pmatrix} 0 \\ 1 \end{pmatrix}\right) = \begin{pmatrix} 0-1 \\ 0+2(1) \end{pmatrix} = \begin{pmatrix} -1 \\ 2 \end{pmatrix}$$

What is the standard matrix $\mathbf{A}$ *equal to in this case?*

By Proposition (5-17) we have $\mathbf{A} = \big(T(\mathbf{e}_1) \mid T(\mathbf{e}_2)\big)$ which is $\mathbf{A} = \begin{pmatrix} 1 & -1 \\ 1 & 2 \end{pmatrix}$.

(b) The given transformation is $T\left(\begin{pmatrix} x \\ y \end{pmatrix}\right) = \begin{pmatrix} 3x-2y \\ 5y-x \end{pmatrix}$. Applying this linear transformation to $\mathbf{e}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\mathbf{e}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ gives

$$T(\mathbf{e}_1) = T\left(\begin{pmatrix} 1 \\ 0 \end{pmatrix}\right) = \begin{pmatrix} 3(1)-2(0) \\ 5(0)-1 \end{pmatrix} = \begin{pmatrix} 3 \\ -1 \end{pmatrix}$$

$$T(\mathbf{e}_2) = T\left(\begin{pmatrix} 0 \\ 1 \end{pmatrix}\right) = \begin{pmatrix} 3(0)-2(1) \\ 5(1)-0 \end{pmatrix} = \begin{pmatrix} -2 \\ 5 \end{pmatrix}$$

By Proposition (5-17) we have $\mathbf{A} = \big(T(\mathbf{e}_1) \mid T(\mathbf{e}_2)\big)$ which is $\mathbf{A} = \begin{pmatrix} 3 & -2 \\ -1 & 5 \end{pmatrix}$.

(c) We are given the transformation $T\left(\begin{pmatrix} x \\ y \\ z \end{pmatrix}\right)=\begin{pmatrix} x-y-z \\ x+y+z \end{pmatrix}$. The standard basis in $\mathbb{R}^3$ is $\mathbf{e}_1=\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$, $\mathbf{e}_2=\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ and $\mathbf{e}_3=\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$. Applying the given linear transformation to these vectors gives

$$T(\mathbf{e}_1)=T\left(\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}\right)=\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$T(\mathbf{e}_2)=T\left(\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}\right)=\begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

$$T(\mathbf{e}_3)=T\left(\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}\right)=\begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

What is the standard matrix ***A*** *equal to?*

By Proposition (5-17) we write these as the first, second and last columns of matrix **A**:

$$\mathbf{A}=\big(T(\mathbf{e}_1)\ \big|\ T(\mathbf{e}_2)\ \big|\ T(\mathbf{e}_3)\big) \text{ gives } \mathbf{A}=\begin{pmatrix} 1 & -1 & -1 \\ 1 & 1 & 1 \end{pmatrix}$$

(d) We are given $T\left(\begin{pmatrix} x \\ y \\ z \end{pmatrix}\right)=\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$. The standard basis in $\mathbb{R}^3$ is

$$\mathbf{e}_1=\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix},\ \mathbf{e}_2=\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \text{ and } \mathbf{e}_3=\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

Applying the given linear transformation to these vectors gives

$$T(\mathbf{e}_1)=T\left(\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}\right)=\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$T(\mathbf{e}_2)=T\left(\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}\right)=\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$T(\mathbf{e}_3)=T\left(\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}\right)=\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

What is the standard matrix ***A*** *equal to?*

By Proposition (5-17) we write these as the first, second and last columns of matrix **A**:

$$\mathbf{A} = \left(T(\mathbf{e}_1) \;\middle|\; T(\mathbf{e}_2) \;\middle|\; T(\mathbf{e}_3)\right) \text{ gives } \mathbf{A} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \mathbf{O}$$

(e) We are given the linear transformation $T\left(\begin{pmatrix} x \\ y \\ z \end{pmatrix}\right) = \begin{pmatrix} -3x-5y-6z \\ -2x+7y+5z \\ 0 \end{pmatrix}$. Applying the given linear transformation to the standard basis for $\mathbb{R}^3$ which are $\{\mathbf{e}_1,\ \mathbf{e}_2,\ \mathbf{e}_3\}$:

$$T(\mathbf{e}_1) = T\left(\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}\right) = \begin{pmatrix} -3(1)-5(0)-6(0) \\ -2(1)+7(0)+5(0) \\ 0 \end{pmatrix} = \begin{pmatrix} -3 \\ -2 \\ 0 \end{pmatrix}$$

$$T(\mathbf{e}_2) = T\left(\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}\right) = \begin{pmatrix} -3(0)-5(1)-6(0) \\ -2(0)+7(1)+5(0) \\ 0 \end{pmatrix} = \begin{pmatrix} -5 \\ 7 \\ 0 \end{pmatrix}$$

$$T(\mathbf{e}_3) = T\left(\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}\right) = \begin{pmatrix} -3(0)-5(0)-6(1) \\ -2(0)+7(0)+5(1) \\ 0 \end{pmatrix} = \begin{pmatrix} -6 \\ 5 \\ 0 \end{pmatrix}$$

What is the standard matrix **A** *equal to?*

By Proposition (5-17) we write these as the first, second and last columns of matrix **A**:

$$\mathbf{A} = \left(T(\mathbf{e}_1) \;\middle|\; T(\mathbf{e}_2) \;\middle|\; T(\mathbf{e}_3)\right) \text{ gives } \mathbf{A} = \begin{pmatrix} -3 & -5 & -6 \\ -2 & 7 & 5 \\ 0 & 0 & 0 \end{pmatrix}$$

3. Since $T: \mathbb{R}^2 \to \mathbb{R}^3$ so matrix **A** must be of size 3 by 2. Let $\mathbf{A} = \begin{pmatrix} a & b \\ c & d \\ f & g \end{pmatrix}$. We have

$$T\left(\begin{pmatrix} 1 \\ 0 \end{pmatrix}\right) = \begin{pmatrix} a & b \\ c & d \\ f & g \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} a \\ c \\ f \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \quad \Rightarrow \quad a = 1,\ c = 2,\ f = 3$$

$$T\left(\begin{pmatrix} 0 \\ 1 \end{pmatrix}\right) = \begin{pmatrix} a & b \\ c & d \\ f & g \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} b \\ d \\ g \end{pmatrix} = \begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix} \quad \Rightarrow \quad b = 4,\ d = 5,\ g = 6$$

Hence $\mathbf{A} = \begin{pmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{pmatrix}$.

4. We have the basis $B = \left\{1,\ 1-x,\ (1-x)^2\right\}$. Applying the linear transformation to these basis vectors and expressing the answers in terms of the $C = \{1,\ x\}$ axes:

$$T(1) = (1)' = 0 = 0(1) + 0(x)$$

$$T(1-x) = (1-x)' = -1 = -1(1) + 0(x)$$

$$T\left([1-x]^2\right) = \left([1-x]^2\right)' = -2(1-x) = -2 + 2x = -2(1) + 2(x)$$

The matrix **A** is given by $\mathbf{A} = \begin{pmatrix} 0 & -1 & -2 \\ 0 & 0 & 2 \end{pmatrix}$. Next we write $2x^2 + 3x + 1$ in terms of $B = \left\{1,\ 1-x,\ (1-x)^2\right\}$:

$$2x^2 + 3x + 1 = 2(1-x)^2 - 7(1-x) + 6(1)$$

Hence we have

$$\begin{pmatrix} 0 & -1 & -2 \\ 0 & 0 & 2 \end{pmatrix}\begin{pmatrix} 6 \\ -7 \\ 2 \end{pmatrix} = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$$

Using these results we have

$$T\left(2x^2 + 3x + 1\right) = 3 + 4x$$

5. (a) We are given the bases $B = \left\{1,\ x,\ x^2,\ x^3\right\}$ and $C = \left\{1,\ x,\ x^2\right\}$ for the transformation $T(\mathbf{p}) = \mathbf{p}'$. Applying the given linear transformation to the vectors in basis $B = \left\{1,\ x,\ x^2,\ x^3\right\}$ we have

$$T(1) = 1' = 0$$

$$T(x) = x' = 1$$

$$T\left(x^2\right) = \left(x^2\right)' = 2x$$

$$T\left(x^3\right) = \left(x^3\right)' = 3x^2$$

We need to write each of these above as the coordinates of the basis $C = \left\{1,\ x,\ x^2\right\}$:

$$T(1) = 0 = a(1) + b(x) + c\left(x^2\right) \text{ gives } a = 0\ \ b = 0 \text{ and } c = 0$$

$$T(x) = 1 = d(1) + e(x) + f\left(x^2\right) \text{ gives } d = 1,\ e = 0 \text{ and } f = 0$$

$$T\left(x^2\right) = 2x = g(1) + h(x) + i\left(x^2\right) \text{ gives } g = 0,\ h = 2 \text{ and } i = 0$$

$$T\left(x^3\right) = 3x^2 = j(1) + k(x) + l\left(x^2\right) \text{ gives } j = 0,\ k = 0 \text{ and } l = 3$$

*What is our matrix **A** equal to?*

$$\mathbf{A} = \left(\left[T(1)\right]_C \ \middle|\ \left[T(x)\right]_C \ \middle|\ \left[T\left(x^2\right)\right]_C \ \middle|\ \left[T\left(x^3\right)\right]_C\right)$$

$$= \begin{pmatrix} a & d & g & j \\ b & e & h & k \\ c & f & i & l \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix}$$

Using this matrix **A** to find $T(\mathbf{p})$ where $\mathbf{p} = -1+3x-7x^2-2x^3$ we have coefficients of the basis $B=\{1,\ x,\ x^2,\ x^3\}$ and are $-1,\ 3,\ -7$ and -2 respectively therefore

$$[\mathbf{p}]_B = \begin{pmatrix} -1 \\ 3 \\ -7 \\ -2 \end{pmatrix}$$

$$[T(\mathbf{p})]_C = \mathbf{A}[\mathbf{p}]_B = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix}\begin{pmatrix} -1 \\ 3 \\ -7 \\ -2 \end{pmatrix} = \begin{pmatrix} 3 \\ -14 \\ -6 \end{pmatrix}$$

The entries $3,\ -14$ and -6 in the Right Hand column vector are the coefficients of the basis $C=\{1,\ x,\ x^2\}$ which means that we have

$$T\left(-1+3x-7x^2-2x^3\right) = 3(1)-14(x)-6\left(x^2\right) = 3-14x-6x^2$$

Thus $T\left(-1+3x-7x^2-2x^3\right)=3-14x-6x^2$. You may check this by differentiating the given function, that is $\left(-1+3x-7x^2-2x^3\right)' = 3-14x-6x^2$.

(b) We are given the bases $B=\{1,\ x,\ x^2,\ x^3\}$ and $C=\{x^2,\ x,\ 1\}$ for the transformation $T(\mathbf{p})=\mathbf{p}'$. Applying the given linear transformation to the vectors in basis $B=\{1,\ x,\ x^2,\ x^3\}$which is identical to part (a) so we have the same answers:

$$T(1)=1'=0,\ T(x)=x'=1,\ T\left(x^2\right)=\left(x^2\right)'=2x \text{ and } T\left(x^3\right)=\left(x^3\right)'=3x^2$$

We need to write each of these above as the coordinates of the basis $C=\{x^2,\ x,\ 1\}$:

$$T(1)=0=a\left(x^2\right)+b(x)+c(1) \text{ gives } a=0 \ \ b=0 \text{ and } c=0$$

$$T(x)=1=d\left(x^2\right)+e(x)+f(1) \text{ gives } d=0,\ e=0 \text{ and } f=1$$

$$T\left(x^2\right)=2x=g\left(x^2\right)+h(x)+i(1) \text{ gives } g=0,\ h=2 \text{ and } i=0$$

$$T\left(x^3\right)=3x^2=j\left(x^2\right)+k(x)+l(1) \text{ gives } j=3,\ k=0 \text{ and } l=0$$

What is our matrix ***A*** *equal to?*

$$\mathbf{A} = \left(\left[T(1)\right]_C \ \middle|\ \left[T(x)\right]_C \ \middle|\ \left[T\left(x^2\right)\right]_C \ \middle|\ \left[T\left(x^3\right)\right]_C\right)$$

$$= \begin{pmatrix} a & d & g & j \\ b & e & h & k \\ c & f & i & l \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 3 \\ 0 & 0 & 2 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

As part (a) we have $[\mathbf{p}]_B = \begin{pmatrix} -1 \\ 3 \\ -7 \\ -2 \end{pmatrix}$.

$$\left[T(\mathbf{p})\right]_C = \mathbf{A}[\mathbf{p}]_B = \begin{pmatrix} 0 & 0 & 0 & 3 \\ 0 & 0 & 2 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}\begin{pmatrix} -1 \\ 3 \\ -7 \\ -2 \end{pmatrix} = \begin{pmatrix} -6 \\ -14 \\ 3 \end{pmatrix}$$

The entries -6, -14 and 3 in the Right Hand column vector are the coefficients of the basis $C = \{x^2,\ x,\ 1\}$ which means that we have

$$T\left(-1+3x-7x^2-2x^3\right) = -6\left(x^2\right)-14(x)+3(1) = -6x^2-14x+3$$

Thus $T\left(-1+3x-7x^2-2x^3\right) = -6x^2-14x+3$.

(c) We are given the bases $B = \{x^3,\ x^2,\ x,\ 1\}$ and $C = \{1,\ x,\ x^2\}$ for the transformation $T(\mathbf{p}) = \mathbf{p}'$. Applying the given linear transformation to the vectors in basis $B = \{x^3,\ x^2,\ x,\ 1\}$ we have

$$T\left(x^3\right) = \left(x^3\right)' = 3x^2,\ T\left(x^2\right) = \left(x^2\right)' = 2x,\ T(x) = x' = 1 \text{ and } T(1) = 1' = 0$$

We need to write each of these above as the coordinates of the basis $C = \{1,\ x,\ x^2\}$:

$$T\left(x^3\right) = 3x^2 = a(1)+b(x)+c\left(x^2\right) \text{ gives } a=0\ \ b=0 \text{ and } c=3$$

$$T\left(x^2\right) = 2x = d(1)+e(x)+f\left(x^2\right) \text{ gives } d=0,\ e=2 \text{ and } f=0$$

$$T(x) = 1 = g(1)+h(x)+i\left(x^2\right) \text{ gives } g=1,\ h=0 \text{ and } i=0$$

$$T(1) = 0 = j(1)+k(x)+l\left(x^2\right) \text{ gives } j=0,\ k=0 \text{ and } l=0$$

What is our matrix **A** *equal to?*

$$\mathbf{A} = \left(\left[T\left(x^3\right)\right]_C \ \middle|\ \left[T\left(x^2\right)\right]_C \ \middle|\ \left[T(x)\right]_C \ \middle|\ \left[T(1)\right]_C\right)$$

$$= \begin{pmatrix} a & d & g & j \\ b & e & h & k \\ c & f & i & l \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 2 & 0 & 0 \\ 3 & 0 & 0 & 0 \end{pmatrix}$$

Using this matrix **A** to find $T\left(-1+3x-7x^2-2x^3\right)$ we have coefficients of the basis $B = \{x^3,\ x^2,\ x,\ 1\}$ are $-2,\ -7,\ 3$ and -1 respectively therefore $[\mathbf{p}]_B = \begin{pmatrix} -2 \\ -7 \\ 3 \\ -1 \end{pmatrix}$.

$$\left[T(\mathbf{p})\right]_C = \mathbf{A}[\mathbf{p}]_B = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 2 & 0 & 0 \\ 3 & 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} -2 \\ -7 \\ 3 \\ -1 \end{pmatrix} = \begin{pmatrix} 3 \\ -14 \\ -6 \end{pmatrix}$$

The entries $3,\ -14$ and -6 in the Right Hand column vector are the coefficients of the basis $C = \{1,\ x,\ x^2\}$ which means that we have

$$T\left(-1+3x-7x^2-2x^3\right) = 3(1)-14(x)-6\left(x^2\right) = 3-14x-6x^2$$

Thus $T\left(-1+3x-7x^2-2x^3\right)=3-14x-6x^2$.

Note that in each case we have a different matrix. This means that even changing the order of the basis vectors gives a different matrix.

6. We are given $T\left(\begin{pmatrix} x \\ y \end{pmatrix}\right)=\begin{pmatrix} x+y \\ x-y \end{pmatrix}$ and we need to find the transformation matrix **A** with respect to the basis $B=\left\{\mathbf{v}_1=\begin{pmatrix} 1 \\ 2 \end{pmatrix},\ \mathbf{v}_2=\begin{pmatrix} 0 \\ -1 \end{pmatrix}\right\}$:

$$T\left(\mathbf{v}_1\right)=T\left(\begin{pmatrix} 1 \\ 2 \end{pmatrix}\right)=\begin{pmatrix} 1+2 \\ 1-2 \end{pmatrix}=\begin{pmatrix} 3 \\ -1 \end{pmatrix}$$

$$T\left(\mathbf{v}_2\right)=T\left(\begin{pmatrix} 0 \\ -1 \end{pmatrix}\right)=\begin{pmatrix} 0-1 \\ 0-(-1) \end{pmatrix}=\begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

We need to write these in terms of the basis vectors $B=\left\{\mathbf{v}_1=\begin{pmatrix} 1 \\ 2 \end{pmatrix},\ \mathbf{v}_2=\begin{pmatrix} 0 \\ -1 \end{pmatrix}\right\}$:

$$T\left(\mathbf{v}_1\right)=\begin{pmatrix} 3 \\ -1 \end{pmatrix}=a\,\mathbf{v}_1+b\,\mathbf{v}_2=a\begin{pmatrix} 1 \\ 2 \end{pmatrix}+b\begin{pmatrix} 0 \\ -1 \end{pmatrix} \qquad (*)$$

$$T\left(\mathbf{v}_2\right)=\begin{pmatrix} -1 \\ 1 \end{pmatrix}=c\,\mathbf{v}_1+d\,\mathbf{v}_2=c\begin{pmatrix} 1 \\ 2 \end{pmatrix}+d\begin{pmatrix} 0 \\ -1 \end{pmatrix} \qquad (**)$$

Solving (*) yields

$$\left.\begin{aligned} a \quad\quad &= 3 \\ 2a-b &= -1 \end{aligned}\right\} \text{ gives } a=3 \text{ and } b=7$$

Solving (**) yields

$$\left.\begin{aligned} c \quad\quad &= -1 \\ 2c-d &= 1 \end{aligned}\right\} \text{ gives } c=-1 \text{ and } d=-3$$

The transformation matrix **A** is given by $\mathbf{A}=\left(\left[T\left(\mathbf{v}_1\right)\right]_B \;\middle|\; \left[T\left(\mathbf{v}_2\right)\right]_B\right)$ therefore

$$\mathbf{A}=\begin{pmatrix} a & c \\ b & d \end{pmatrix}=\begin{pmatrix} 3 & -1 \\ 7 & -3 \end{pmatrix}$$

By using this matrix we need to find $T\left(\begin{pmatrix} -3 \\ 1 \end{pmatrix}\right)$. *How?*

We need to write the vector $\begin{pmatrix} -3 \\ 1 \end{pmatrix}$ with respect to the basis $B=\left\{\mathbf{v}_1=\begin{pmatrix} 1 \\ 2 \end{pmatrix},\ \mathbf{v}_2=\begin{pmatrix} 0 \\ -1 \end{pmatrix}\right\}$:

$$\begin{pmatrix} -3 \\ 1 \end{pmatrix}=k_1\begin{pmatrix} 1 \\ 2 \end{pmatrix}+k_2\begin{pmatrix} 0 \\ -1 \end{pmatrix}$$

This gives $k_1=-3$ and $k_2=-7$. Thus

$$\left[T\left(\begin{pmatrix} -3 \\ 1 \end{pmatrix}\right)\right]_B=\begin{pmatrix} 3 & -1 \\ 7 & -3 \end{pmatrix}\begin{pmatrix} -3 \\ -7 \end{pmatrix}=\begin{pmatrix} -2 \\ 0 \end{pmatrix}$$

We have

$$T\left(\begin{pmatrix}-3\\1\end{pmatrix}\right)=-2\mathbf{v}_1+0\mathbf{v}_2=-2\begin{pmatrix}1\\2\end{pmatrix}+0\begin{pmatrix}0\\-1\end{pmatrix}=\begin{pmatrix}-2\\-4\end{pmatrix}$$

Working out this directly we have $T\left(\begin{pmatrix}-3\\1\end{pmatrix}\right)=\begin{pmatrix}-3+1\\-3-1\end{pmatrix}=\begin{pmatrix}-2\\-4\end{pmatrix}$.

7. First we apply the matrix $\mathbf{A}$ to the standard basis vectors $\mathbf{e}_1=\begin{pmatrix}1\\0\end{pmatrix}$ and $\mathbf{e}_2=\begin{pmatrix}0\\1\end{pmatrix}$:

$$\mathbf{v}_1=\mathbf{A}\mathbf{e}_1=\begin{pmatrix}\cos(45°) & -\sin(45°)\\ \sin(45°) & \cos(45°)\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix}=\begin{pmatrix}\cos(45°)\\ \sin(45°)\end{pmatrix}=\frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix}$$

$$\mathbf{v}_2=\mathbf{A}\mathbf{e}_2=\begin{pmatrix}\cos(45°) & -\sin(45°)\\ \sin(45°) & \cos(45°)\end{pmatrix}\begin{pmatrix}0\\1\end{pmatrix}=\begin{pmatrix}-\sin(45°)\\ \cos(45°)\end{pmatrix}=\frac{1}{\sqrt{2}}\begin{pmatrix}-1\\1\end{pmatrix}$$

Our new basis is $B=\left\{\mathbf{v}_1=\frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix},\ \mathbf{v}_2=\frac{1}{\sqrt{2}}\begin{pmatrix}-1\\1\end{pmatrix}\right\}$. We need to write $\begin{pmatrix}2\\1\end{pmatrix}$ in terms of this B new basis. *How?*

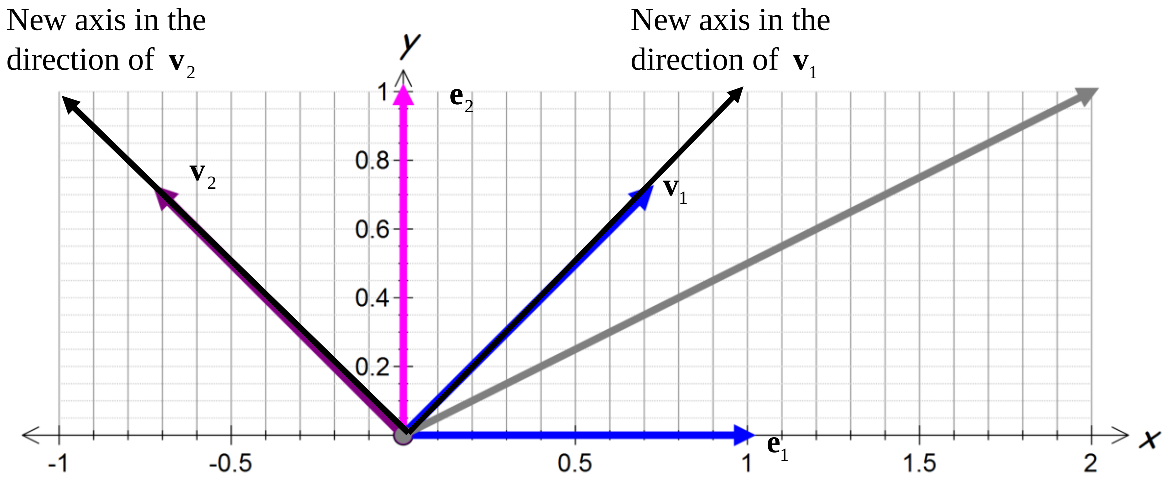
$$\begin{pmatrix}2\\1\end{pmatrix}=a\frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix}+b\frac{1}{\sqrt{2}}\begin{pmatrix}-1\\1\end{pmatrix}\quad\Rightarrow\quad \begin{aligned}2&=\frac{a}{\sqrt{2}}-\frac{b}{\sqrt{2}}=\frac{a-b}{\sqrt{2}}\\ 1&=\frac{a}{\sqrt{2}}+\frac{b}{\sqrt{2}}=\frac{a+b}{\sqrt{2}}\end{aligned}$$

Transposing these equations gives

$$\begin{aligned}2\sqrt{2}&=a-b &&(1)\\ \sqrt{2}&=a+b &&(2)\\ \hline 3\sqrt{2}&=2a &&\Rightarrow\quad a=\frac{3\sqrt{2}}{2}\end{aligned}$$

Substituting $a=\frac{3\sqrt{2}}{2}=\frac{3}{\sqrt{2}}$ into (1) gives $b=\frac{3\sqrt{2}}{2}-2\sqrt{2}=-\frac{1}{\sqrt{2}}$.

Hence the coordinates of $\begin{pmatrix}2\\1\end{pmatrix}$ under the new basis B is $\frac{1}{\sqrt{2}}\begin{pmatrix}3\\-1\end{pmatrix}$. This is illustrated below:



8. We are given the linear transformation $T\left(\begin{pmatrix} x \\ y \end{pmatrix}\right)=\begin{pmatrix} -x \\ -y \\ x+3y \end{pmatrix}$. *What do we need to find first?*

The transformation of the basis vectors $T(\mathbf{v}_1)$ and $T(\mathbf{v}_2)$ where $\mathbf{v}_1=\begin{pmatrix} 1 \\ 2 \end{pmatrix}$, $\mathbf{v}_2=\begin{pmatrix} 1 \\ 1 \end{pmatrix}$:

$$T(\mathbf{v}_1)=T\left(\begin{pmatrix} 1 \\ 2 \end{pmatrix}\right)=\begin{pmatrix} -1 \\ -2 \\ 1+3(2) \end{pmatrix}=\begin{pmatrix} -1 \\ -2 \\ 7 \end{pmatrix}$$

$$T(\mathbf{v}_2)=T\left(\begin{pmatrix} 1 \\ 1 \end{pmatrix}\right)=\begin{pmatrix} -1 \\ -1 \\ 1+3(1) \end{pmatrix}=\begin{pmatrix} -1 \\ -1 \\ 4 \end{pmatrix}$$

What else do we need to find?

We need to write each of these above vectors $T(\mathbf{v}_1)$ and $T(\mathbf{v}_2)$ as the coordinates of the basis $C=\left\{\mathbf{w}_1=\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \mathbf{w}_2=\begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}, \mathbf{w}_3=\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}\right\}$:

$$T(\mathbf{v}_1)=\begin{pmatrix} -1 \\ -2 \\ 7 \end{pmatrix}=a\,\mathbf{w}_1+b\,\mathbf{w}_2+c\,\mathbf{w}_3=a\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}+b\begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}+c\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \qquad (\dagger)$$

$$T(\mathbf{v}_2)=\begin{pmatrix} -1 \\ -1 \\ 4 \end{pmatrix}=d\,\mathbf{w}_1+e\,\mathbf{w}_2+f\,\mathbf{w}_3=d\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}+e\begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}+f\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \qquad (\dagger\dagger)$$

How can we find the matrix $\mathbf{A}$?

By Proposition (5-18) we have $\mathbf{A}=\left(\left[T\left(\mathbf{v}_1\right)\right]_C \;\middle|\; \left[T\left(\mathbf{v}_2\right)\right]_C\right)$ which in this case is

$$\mathbf{A}=\begin{pmatrix} a & d \\ b & e \\ c & f \end{pmatrix} \text{ because } \left[T\left(\mathbf{v}_1\right)\right]_C=\begin{pmatrix} a \\ b \\ c \end{pmatrix} \text{ and } \left[T\left(\mathbf{v}_2\right)\right]_C=\begin{pmatrix} d \\ e \\ f \end{pmatrix}$$

How can we determine $a,\ b,\ c,\ \cdots,\ f$ *?*

We need to solve the 2 pairs of simultaneous equations (†) and (††) in the above. Consider the first pair (†):

$$\begin{pmatrix} -1 \\ -2 \\ 7 \end{pmatrix}=a\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}+b\begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}+c\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}=\begin{pmatrix} a \\ 0 \\ a \end{pmatrix}+\begin{pmatrix} b \\ 2b \\ 0 \end{pmatrix}+\begin{pmatrix} 0 \\ c \\ c \end{pmatrix}=\begin{pmatrix} a+b \\ 2b+c \\ a+c \end{pmatrix}$$

We have

$$\left.\begin{aligned} a+b&=-1 \\ 2b+c&=-2 \\ a+c&=7 \end{aligned}\right\} \text{ gives } a=\frac{7}{3},\ b=-\frac{10}{3} \text{ and } c=\frac{14}{3}$$

Similarly we can find the solution of the other simultaneous equation (††):

$$\begin{pmatrix} -1 \\ -1 \\ 4 \end{pmatrix}=d\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}+e\begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}+f\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}=\begin{pmatrix} d \\ 0 \\ d \end{pmatrix}+\begin{pmatrix} e \\ 2e \\ 0 \end{pmatrix}+\begin{pmatrix} 0 \\ f \\ f \end{pmatrix}=\begin{pmatrix} d+e \\ 2e+f \\ d+f \end{pmatrix}$$

$$\left.\begin{aligned} d+e&=-1 \\ 2e+f&=-1 \\ d+f&=4 \end{aligned}\right\} \text{ gives } d=1,\ e=-2 \text{ and } f=3$$

What is the matrix $\mathbf{A}$ *equal to?*

$$\mathbf{A}=\left(\left[T\left(\mathbf{v}_1\right)\right]_C \;\middle|\; \left[T\left(\mathbf{v}_2\right)\right]_C\right)=\begin{pmatrix} a & d \\ b & e \\ c & f \end{pmatrix}=\begin{pmatrix} 7/3 & 1 \\ -10/3 & -2 \\ 14/3 & 3 \end{pmatrix}$$

Remember $\left[T\left(\mathbf{u}\right)\right]_C=\mathbf{A}\left[\mathbf{u}\right]_B$ so we have

$$\left[T\left(\mathbf{u}\right)\right]_C=\begin{pmatrix} 7/3 & 1 \\ -10/3 & -2 \\ 14/3 & 3 \end{pmatrix}\left[\mathbf{u}\right]_B \qquad (*)$$

We need to find $T\left(\mathbf{u}\right)$ where $\mathbf{u}=\begin{pmatrix} 2 \\ 1 \end{pmatrix}$. We first determine $\left[\mathbf{u}\right]_B$ which means we have to write the vector $\mathbf{u}$ in terms of the basis vectors $\mathbf{v}_1=\begin{pmatrix} 1 \\ 2 \end{pmatrix}$ and $\mathbf{v}_2=\begin{pmatrix} 1 \\ 1 \end{pmatrix}$:

$$\mathbf{u}=\begin{pmatrix} 2 \\ 1 \end{pmatrix}=k_1\mathbf{v}_1+k_2\mathbf{v}_2=k_1\begin{pmatrix} 1 \\ 2 \end{pmatrix}+k_2\begin{pmatrix} 1 \\ 1 \end{pmatrix}=\begin{pmatrix} k_1+k_2 \\ 2k_1+k_2 \end{pmatrix}$$

Thus we need to solve the simultaneous equations:

$$\left.\begin{aligned} k_1+k_2&=2 \\ 2k_1+k_2&=1 \end{aligned}\right\} \text{ gives } k_1=-1 \text{ and } k_2=3$$

Thus $[\mathbf{u}]_B = \begin{pmatrix} -1 \\ 3 \end{pmatrix}$. Substituting this into (*) gives

$$[T(\mathbf{u})]_C = \begin{pmatrix} 7/3 & 1 \\ -10/3 & -2 \\ 14/3 & 3 \end{pmatrix}\begin{pmatrix} -1 \\ 3 \end{pmatrix} = \begin{pmatrix} 2/3 \\ -8/3 \\ 13/3 \end{pmatrix} = \frac{1}{3}\begin{pmatrix} 2 \\ -8 \\ 13 \end{pmatrix}$$

We have

$$T\left(\begin{pmatrix} 2 \\ 1 \end{pmatrix}\right) = \frac{1}{3}[2\mathbf{w}_1 - 8\mathbf{w}_2 + 13\mathbf{w}_3]$$

$$= \frac{1}{3}\left[2\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} - 8\begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} + 13\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}\right] = \frac{1}{3}\begin{pmatrix} -6 \\ -3 \\ 15 \end{pmatrix} = \begin{pmatrix} -2 \\ -1 \\ 5 \end{pmatrix}$$

As a check we can evaluate the transformation directly by using $T\left(\begin{pmatrix} x \\ y \end{pmatrix}\right) = \begin{pmatrix} -x \\ -y \\ x+3y \end{pmatrix}$:

$$T\left(\begin{pmatrix} 2 \\ 1 \end{pmatrix}\right) = \begin{pmatrix} -2 \\ -1 \\ 2+3(1) \end{pmatrix} = \begin{pmatrix} -2 \\ -1 \\ 5 \end{pmatrix}$$

Note that both our answers are identical.

9. Applying the given linear transformation to the matrices in basis B we have

$$T(\mathbf{m}_1) = T\left(\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\right) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}^T = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

$$T(\mathbf{m}_2) = T\left(\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\right) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}^T = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

$$T(\mathbf{m}_3) = T\left(\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\right) = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}^T = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

$$T(\mathbf{m}_4) = T\left(\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\right) = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}^T = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

What else do we need to find?

We need to write each of these above matrices $T(\mathbf{m}_1)$, $T(\mathbf{m}_2)$, $T(\mathbf{m}_3)$ and $T(\mathbf{m}_4)$ as the coordinates of the basis:

$$C = \left\{\mathbf{m}_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix},\ \mathbf{m}_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix},\ \mathbf{m}_3 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix},\ \mathbf{m}_4 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\right\}$$

We have

$$T(\mathbf{m}_1) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = a\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + b\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + c\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} + d\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

This gives $a = 1,\ b = 0,\ c = 0$ and $d = 0$. Similarly we have

$$T(\mathbf{m}_2)=\begin{pmatrix}0&0\\1&0\end{pmatrix}=e\begin{pmatrix}1&0\\0&0\end{pmatrix}+f\begin{pmatrix}0&1\\0&0\end{pmatrix}+g\begin{pmatrix}0&0\\1&0\end{pmatrix}+h\begin{pmatrix}0&0\\0&1\end{pmatrix}$$

This gives $e=0,\ f=0,\ g=1$ and $h=0$. In the same manner we have

$$T(\mathbf{m}_3)=\begin{pmatrix}0&1\\0&0\end{pmatrix}=i\begin{pmatrix}1&0\\0&0\end{pmatrix}+j\begin{pmatrix}0&1\\0&0\end{pmatrix}+k\begin{pmatrix}0&0\\1&0\end{pmatrix}+l\begin{pmatrix}0&0\\0&1\end{pmatrix}$$

This yields $i=0,\ j=1,\ k=0$ and $l=0$. Evaluating

$$T(\mathbf{m}_4)=\begin{pmatrix}0&0\\0&1\end{pmatrix}=m\begin{pmatrix}1&0\\0&0\end{pmatrix}+n\begin{pmatrix}0&1\\0&0\end{pmatrix}+p\begin{pmatrix}0&0\\1&0\end{pmatrix}+q\begin{pmatrix}0&0\\0&1\end{pmatrix}$$

Thus $m=0,\ n=0,\ p=0$ and $q=1$.

How can we find the transformation matrix $\mathbf{A}$*?*

By Proposition (5-18) we have

$$\mathbf{A}=\left(\left[T(\mathbf{m}_1)\right]_C \,\middle|\, \left[T(\mathbf{m}_2)\right]_C \,\middle|\, \left[T(\mathbf{m}_3)\right]_C \,\middle|\, \left[T(\mathbf{m}_4)\right]_C\right)$$

which in this case is $\mathbf{A}=\begin{pmatrix}a&e&i&m\\b&f&j&n\\c&g&k&p\\d&h&l&q\end{pmatrix}$. *Why?*

Because $\left[T(\mathbf{m}_1)\right]_C=\begin{pmatrix}a\\b\\c\\d\end{pmatrix}$, $\left[T(\mathbf{m}_2)\right]_C=\begin{pmatrix}e\\f\\g\\h\end{pmatrix}$, $\left[T(\mathbf{m}_3)\right]_C=\begin{pmatrix}i\\j\\k\\l\end{pmatrix}$, $\left[T(\mathbf{m}_4)\right]_C=\begin{pmatrix}m\\n\\p\\q\end{pmatrix}$.

We can substitute the above values of $a, b, c, d, \cdots$

What is the matrix $\mathbf{A}$ *equal to?*

$$\mathbf{A}=\begin{pmatrix}a&e&i&m\\b&f&j&n\\c&g&k&p\\d&h&l&q\end{pmatrix}=\begin{pmatrix}1&0&0&0\\0&0&1&0\\0&1&0&0\\0&0&0&1\end{pmatrix}$$

This means that $\left[T(\mathbf{X})\right]_C=\mathbf{A}[\mathbf{X}]_B$ we have

$$\left[T(\mathbf{X})\right]_C=\begin{pmatrix}1&0&0&0\\0&0&1&0\\0&1&0&0\\0&0&0&1\end{pmatrix}[\mathbf{X}]_B$$

We need to evaluate $T\left(\begin{pmatrix}1&2\\3&4\end{pmatrix}\right)$ by using the matrix $\mathbf{A}$ above. Let $\mathbf{X}=\begin{pmatrix}1&2\\3&4\end{pmatrix}$

$$\begin{pmatrix}1&2\\3&4\end{pmatrix}=a\begin{pmatrix}1&0\\0&0\end{pmatrix}+b\begin{pmatrix}0&1\\0&0\end{pmatrix}+c\begin{pmatrix}0&0\\1&0\end{pmatrix}+d\begin{pmatrix}0&0\\0&1\end{pmatrix}\qquad (*)$$

These are **not** the same a, b, c and d values found above. From this (*) we have

$$a=1,\ b=2,\ c=3 \text{ and } d=4$$

What is $[\mathbf{X}]_B$ *equal to?*

$$[\mathbf{X}]_B = \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix}$$

From the above we have $\left[T(\mathbf{X})\right]_C = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}[\mathbf{X}]_B$ and evaluating this Right Hand Side gives:

$$\left[T(\mathbf{X})\right]_C = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \\ 2 \\ 4 \end{pmatrix}$$

We have

$$T\left(\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}\right) = 1\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + 3\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + 2\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} + 4\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$
$$= \begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix}$$

We can check this by evaluating the transformation directly:

$$T\left(\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}\right) = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}^t = \begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix}$$

The matrix $\mathbf{A} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$ represents the transformation defined by $T(\mathbf{X}) = \mathbf{X}^T$ with respect to the given basis.

10. We are given the transformation $T : V \to V$ given by

$$T(\mathbf{f}) = \mathbf{f}'$$

and we need to find the transformation matrix **A** with respect to the bases

$$B = \{\sin(x),\ \cos(x)\}$$

The transformation of the basis vectors is given by

$$T(\sin(x)) = (\sin(x))' = \cos(x)$$
$$T(\cos(x)) = (\cos(x))' = -\sin(x)$$

We need to write each of these above as the coordinates of the basis $B = \{\sin(x),\ \cos(x)\}$:

$$T(\sin(x)) = \cos(x) = a\sin(x) + b\cos(x) \text{ gives } a = 0 \text{ and } b = 1$$
$$T(\cos(x)) = -\sin(x) = c\sin(x) + d\cos(x) \text{ gives } c = -1 \text{ and } d = 0$$

*What is our matrix **A** equal to?*

$$\mathbf{A}=\begin{pmatrix} a & c \\ b & d \end{pmatrix}=\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

(a)Using this matrix **A** to find $T(\mathbf{g})$ where $\mathbf{g}=5\sin(x)+2\cos(x)$ we have $[\mathbf{g}]_B=\begin{pmatrix} 5 \\ 2 \end{pmatrix}$ because these are the coefficients of $\sin(x)$ and $\cos(x)$ respectively:

$$[T(\mathbf{g})]_B=\mathbf{A}[\mathbf{g}]_B=\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 5 \\ 2 \end{pmatrix}=\begin{pmatrix} -2 \\ 5 \end{pmatrix}$$

The entries -2 and 5 in the Right Hand column vector are the coefficients of the basis $B=\{\sin(x),\ \cos(x)\}$ which means that we have

$$T[5\sin(x)+2\cos(x)]=-2\sin(x)+5\cos(x)$$

(b) In this case we have $\mathbf{g}=m\sin(x)+n\cos(x)$ so we can write the coefficients of $\sin(x)$ and $\cos(x)$ as $[\mathbf{g}]_B=\begin{pmatrix} m \\ n \end{pmatrix}$. We have

$$[T(\mathbf{g})]_B=\mathbf{A}[\mathbf{g}]_B=\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} m \\ n \end{pmatrix}=\begin{pmatrix} -n \\ m \end{pmatrix}$$

The entries $-n$ and m in the Right Hand column vector are the coefficients of the basis $B=\{\sin(x),\ \cos(x)\}$ which means that we have

$$T(m\sin(x)+n\cos(x))=-n\sin(x)+m\cos(x)$$

11. We are given that $T(\mathbf{p})=\mathbf{p}(x+3)$ and we need to find the transformation matrix **A** with respect to the basis $B=\{1,\ x,\ x^2\}$. The transformation of these basis vectors is given by

$$T(1)=1$$
$$T(x)=x+3$$
$$T(x^2)=(x+3)^2=x^2+6x+9 \qquad \text{[Expanding]}$$

We need to write each of these as the coordinates of the basis $B=\{1,\ x,\ x^2\}$:

$$T(1)=1=a(1)+b(x)+c(x^2) \text{ gives } a=1,\ b=c=0$$
$$T(x)=x+3=e(1)+f(x)+g(x^2) \text{ gives } e=3,\ f=1 \text{ and } g=0$$
$$T(x^2)=x^2+6x+9=h(1)+i(x)+j(x^2) \text{ gives } h=9,\ i=6 \text{ and } j=1$$

We have $[T(1)]_B=\begin{pmatrix} a \\ b \\ c \end{pmatrix}=\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$, $[T(x)]_B=\begin{pmatrix} e \\ f \\ g \end{pmatrix}=\begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix}$ and $[T(x^2)]_B=\begin{pmatrix} h \\ i \\ j \end{pmatrix}=\begin{pmatrix} 9 \\ 6 \\ 1 \end{pmatrix}$.

Our transformation matrix **A** is given by

$$\mathbf{A}=\left([T(1)]_B \;\middle|\; [T(x)]_B \;\middle|\; [T(x^2)]_B\right)=\begin{pmatrix} 1 & 3 & 9 \\ 0 & 1 & 6 \\ 0 & 0 & 1 \end{pmatrix}$$

The coordinates of the given vector $\mathbf{p} = q + nx + mx^2$ with respect to the basis $B = \{1,\ x,\ x^2\}$ is

$$[\mathbf{p}]_B = \begin{pmatrix} q \\ n \\ m \end{pmatrix}$$

Thus we have

$$[T(\mathbf{p})]_B = \begin{pmatrix} 1 & 3 & 9 \\ 0 & 1 & 6 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} q \\ n \\ m \end{pmatrix} = \begin{pmatrix} q+3n+9m \\ n+6m \\ m \end{pmatrix}$$

The entries in the Right Hand Column vector $q+3n+9m,\ n+6m$ and m are the coefficients of 1, x and x^2 because $B = \{1,\ x,\ x^2\}$.

$$\begin{aligned} T(\mathbf{p}) &= T\left(q + nx + mx^2\right) \\ &= q + 3n + 9m + (n+6m)x + mx^2 \end{aligned}$$

Evaluating this transformation directly gives

$$\begin{aligned} T\left(q + nx + mx^2\right) &= q + n(x+3) + m(x+3)^2 \\ &= q + nx + 3n + m\left(x^2 + 6x + 9\right) \\ &= q + nx + 3n + mx^2 + 6mx + 9m \\ &= q + 9m + 3n + (n+6m)x + mx^2 \end{aligned}$$

This is identical to the above therefore matrix **A** is the transformation matrix of the given transformation.

12. We are given the transformation $T: V \to V$ given by

$$T(\mathbf{f}) = \mathbf{f}'$$

and we need to find the transformation matrix **A** with respect to the basis

$$B = \{\sin(x),\ \cos(x),\ e^x\}$$

The transformation of the basis vectors is given by

$$T(\sin(x)) = (\sin(x))' = \cos(x)$$

$$T(\cos(x)) = (\cos(x))' = -\sin(x)$$

$$T\left(e^x\right) = e^x$$

We need to write each of these above as the coordinates of the basis $B = \{\sin(x),\ \cos(x),\ e^x\}$:

$T(\sin(x)) = \cos(x) = a\sin(x) + b\cos(x) + ce^x$ gives $a = 0,\ b = 1$ and $c = 0$

$T(\cos(x)) = -\sin(x) = m\sin(x) + n\cos(x) + le^x$ gives $m = -1,\ n = 0$ and $l = 0$

$T\left(e^x\right) = e^x = p\sin(x) + q\cos(x) + re^x$ gives $p = q = 0$ and $r = 1$

What is our matrix ***A*** *equal to?*

$$\mathbf{A}=\begin{pmatrix} a & m & p \\ b & n & q \\ c & l & r \end{pmatrix}=\begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

(a) Using this matrix **A** to find $T(\mathbf{g})$ where $\mathbf{g}=-\sin(x)+4\cos(x)-2e^x$ we have

$[\mathbf{g}]_B=\begin{pmatrix} -1 \\ 4 \\ -2 \end{pmatrix}$ because these are the coefficients of $\sin(x)$, $\cos(x)$ and e^x:

$$[T(\mathbf{g})]_B=\mathbf{A}[\mathbf{g}]_B=\begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} -1 \\ 4 \\ -2 \end{pmatrix}=\begin{pmatrix} -4 \\ -1 \\ -2 \end{pmatrix}$$

The entries -4, -1 and -2 in the Right Hand column vector are the coefficients of $\sin(x)$, $\cos(x)$ and e^x respectively because the basis $B=\{\sin(x),\ \cos(x),\ e^x\}$ which means that we have

$$T\left(-\sin(x)+4\cos(x)-2e^x\right)=-4\sin(x)-\cos(x)-2e^x$$

(b) In this case we have $\mathbf{g}=m\sin(x)+n\cos(x)+pe^x$ so we can write the coefficients of $\sin(x)$, $\cos(x)$ and e^x as $[\mathbf{g}]_B=\begin{pmatrix} m \\ n \\ p \end{pmatrix}$. We have

$$[T(\mathbf{g})]_B=\mathbf{A}[\mathbf{g}]_B=\begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} m \\ n \\ p \end{pmatrix}=\begin{pmatrix} -n \\ m \\ p \end{pmatrix}$$

The entries $-n$, m and p in the Right Hand column vector are the coefficients of the basis $B=\{\sin(x),\ \cos(x),\ e^x\}$ which means that we have

$$T\left(m\sin(x)+n\cos(x)+pe^x\right)=-n\sin(x)+m\cos(x)+pe^x$$

In general if we apply the transformation directly we have

$$\begin{aligned} T(\mathbf{g}) &= \left[m\sin(x)+n\cos(x)+pe^x\right]' \\ &= m\cos(x)-n\sin(x)+pe^x \end{aligned}$$

This is identical to the above result achieved by using the transformation matrix **A**. Thus the matrix **A** represents the differential transformation.

13. We need to find the matrix **A** which represents

$$T(\mathbf{f})=\mathbf{f}' \quad \text{where } \mathbf{f}' \text{ is the derivative of } \mathbf{f}$$

with respect to the bases $B=\{e^{2x},\ xe^{2x},\ x^2e^{2x}\}$.

The transformation of the basis vectors is given by

$$T\left(e^{2x}\right)=\left(e^{2x}\right)'=2e^{2x}$$

$$T\left(xe^{x}\right)=\left(xe^{2x}\right)'=e^{2x}+2xe^{2x}$$

$$T\left(x^2e^{2x}\right)=\left(x^2e^{2x}\right)'=2xe^{2x}+2x^2e^{2x}$$

We need to write each of these above as the coordinates of the basis $B=\left\{e^{2x},\ xe^{2x},\ x^2e^{2x}\right\}$:

$$T\left(e^{2x}\right)=2e^{2x}=ae^{2x}+bxe^{2x}+cx^2e^{2x}\ \text{ gives }\ a=2,\ b=0\ \text{ and }\ c=0$$

$$T\left(xe^{x}\right)=e^{2x}+2xe^{2x}=de^{2x}+fxe^{2x}+gx^2e^{2x}\ \text{ gives }\ d=1,\ f=2\ \text{ and }\ g=0$$

$$T\left(x^2e^{2x}\right)=2xe^{2x}+2x^2e^{2x}=he^{2x}+ixe^{2x}+jx^2e^{2x}\ \text{ gives }\ h=0,\ i=2\ \text{ and }\ j=2$$

We have

$$\left[T\left(e^{2x}\right)\right]_B=\begin{pmatrix}2\\0\\0\end{pmatrix},\ \left[T\left(xe^{x}\right)\right]_B=\begin{pmatrix}1\\2\\0\end{pmatrix}\text{ and }\left[T\left(x^2e^{2x}\right)\right]_B=\begin{pmatrix}0\\2\\2\end{pmatrix}$$

What is our matrix ***A*** *equal to?*

$$\mathbf{A}=\begin{pmatrix}2&1&0\\0&2&2\\0&0&2\end{pmatrix}$$

Next by using this matrix we need to find $T\left(ae^{2x}+bxe^{2x}+cx^2e^{2x}\right)$. First we write $ae^{2x}+bxe^{2x}+cx^2e^{2x}$ in terms of the basis vectors $B=\left\{e^{2x},\ xe^{2x},\ x^2e^{2x}\right\}$. Clearly we have

$$\left[ae^{2x}+bxe^{2x}+cx^2e^{2x}\right]_B=\begin{pmatrix}a\\b\\c\end{pmatrix}$$

Thus

$$\left[T\left(ae^{2x}+bxe^{2x}+cx^2e^{2x}\right)\right]_B=\begin{pmatrix}2&1&0\\0&2&2\\0&0&2\end{pmatrix}\begin{pmatrix}a\\b\\c\end{pmatrix}=\begin{pmatrix}2a+b\\2b+2c\\2c\end{pmatrix}$$

The entries in the Right Hand column vector are the coefficients of e^{2x}, xe^{2x} and x^2e^{2x}:

$$T\left(ae^{2x}+bxe^{2x}+cx^2e^{2x}\right)=(2a+b)e^{2x}+(2b+2c)xe^{2x}+2cx^2e^{2x}$$

14. Since V is of dimension n so let $B=\left\{\mathbf{b}_1,\ \mathbf{b}_2,\ \cdots,\ \mathbf{b}_n\right\}$ be a basis for V. Applying any identity transformation to these basis vectors gives:

$$\begin{aligned}T\left(\mathbf{b}_1\right)&=\mathbf{b}_1=1\left(\mathbf{b}_1\right)+0\mathbf{b}_2+\cdots+0\mathbf{b}_n\\T\left(\mathbf{b}_2\right)&=\mathbf{b}_2=0\mathbf{b}_1+1\left(\mathbf{b}_2\right)+\cdots+0\mathbf{b}_n\\&\ \ \vdots\qquad\qquad\vdots\qquad\qquad\vdots\\T\left(\mathbf{b}_n\right)&=\mathbf{b}_2=0\mathbf{b}_1+0\mathbf{b}_2+\cdots+1\left(\mathbf{b}_n\right)\end{aligned}$$

Writing the transformation matrix $\mathbf{A}=\begin{pmatrix}1&&0\\&\ddots&\\0&&1\end{pmatrix}=\mathbf{I}$