Complete Solutions to Exercises 1.1

1. (a) $x-y-z=3$ is linear because the index of $x,\ y$ and z is one.

(b) $\sqrt{x}+y+z=6$ is **not** linear because $\sqrt{x}=x^{1/2}$ that is the index of x is ½.

(c) $\cos(x)+\sin(y)=1$ is **not** linear because x and y are arguments of trigonometric functions.

(d) $e^{x+y+z}=1$ is **not** linear because $x,\ y$ and z are arguments of the exponential function.

(e) $x-2y+5z=\sqrt{3}$ is linear because the index of $x,\ y$ and z is one.

(f) $x=-3y$ is linear because the index of x and y is one.

(g) $x=\dfrac{-b\pm\sqrt{b^2-4ac}}{2a}$ is linear because the place-holder x has index 1 and the Right Hand Side, $\dfrac{-b\pm\sqrt{b^2-4ac}}{2a}$, is a constant.

(h) $\pi x+y+ez=5$ is linear because the index of $x,\ y$ and z is one.

(i) $\sqrt{2}x+\dfrac{1}{2}y+z=0$ is linear because the index of $x,\ y$ and z is one.

(j) $\sinh^{-1}(x)=\ln\left|x+\sqrt{x^2+1}\right|$ is **not** linear because x is an argument of a hyperbolic and logarithmic functions.

(k) $\dfrac{\pi}{2}x-\sqrt{2}y+z\sin(\pi)=0$ is linear because the index of $x,\ y$ and z is one.

(l) $x^{2^0}+y^{3^0}+z^{3^0}=0$ is linear because $2^0=3^0=1$ so the index of each unknown is 1.

(m) $y^{\cos^2(x)+\sin^2(x)}+x-z=9$ is linear because the index of $x,\ y$ and z is one. Remember from trigonometry that $\cos^2(x)+\sin^2(x)=1$.

2. (a) We add the two given simultaneous equations:

$$\begin{array}{rr} & x+y=2 \\ + & (x-y=0) \\ \hline & 2x+0=2 \end{array}$$

We have $2x=2$ which gives $x=1$. *How do we find the value of the other unknown, y?*
Substitute $x=1$ into the first equation $x+y=2$:

$$1+y=2 \text{ which gives } y=1$$

Hence $x=1$ and $y=1$ is the solution.

(b) We can label each linear equation

$$\begin{array}{ccccccc} 2x & - & 3y & = & 5 & & (*) \\ x & - & y & = & 2 & & (**) \end{array}$$

How do we eliminate one of the unknowns?
Eliminate x by multiplying equation (**) by 2 and subtracting from equation (*):

$$\begin{array}{rll} & 2x-3y=5 & (*) \\ - & (2x-2y=4) & \left[\text{Multiplying } (**) \text{ by } 2\right] \\ \hline & 0-y=1 & \end{array}$$

Hence $y=-1$. *How do we determine x?*

By substituting $y=-1$ into the given equation $x-y=2 \quad (**)$:

$$x-(-1)=2$$

$$x+1=2 \text{ which gives } x=1$$

The solution is $x=1$ and $y=-1$.

(c) We label each equation:

$$\begin{array}{cccccc} 2x & - & 3y & = & 35 & (\square) \\ x & - & y & = & 2 & (\square\square) \end{array}$$

We need to eliminate one of the unknowns. *Which one?*
To make life easier it is better to eliminate *x*. *How?*
Multiply equation $(\square\square)$ by 2 and subtract from equation $(\square)$:

$$\begin{array}{rll} & 2x-3y=35 & (\square) \\ - & (2x-2y=4\ ) & \left[\text{Multiplying } (\square\square) \text{ by } 2\right] \\ \hline & 0-y=31 & \end{array}$$

We have $y=-31$. *How do we find x?*

Substitute $y=-31$ into the given equation $x \quad - \quad y \quad = \quad 2 \quad (\square\square)$:

$$x-(-31)=2$$

$$x+31=2 \quad \text{which gives } x=-29$$

The solution is $x=-29$ and $y=-31$.

(d) We label each equation:

$$\begin{array}{cccccc} 5x & - & 7y & = & 2 & (\dagger) \\ 9x & - & 3y & = & 6 & (\dagger\dagger) \end{array}$$

We need to eliminate one of the unknowns. *Which one?*
Eliminate *y*. *How?*
Multiply equation (†) by 3 and multiply (††) by 7:

$$15x-21y=6 \qquad \left[\text{Multiplying } (\dagger) \text{ by } 3\right]$$

$$63x-21y=42 \qquad \left[\text{Multiplying } (\dagger\dagger) \text{ by } 7\right]$$

To eliminate *y* we subtract these equations

$$\begin{array}{rll} & 63x-21y=42 & \\ - & (15x-21y=6) & \\ \hline & 48x-0=36 & [\text{Subtracting}] \end{array}$$

From $48x=36$ we have $x=\dfrac{36}{48}=\dfrac{3}{4}$. *How do we find y?*

Substitute $x=\dfrac{3}{4}$ into the given equation $5x-7y=2 \quad$ (†):

$$5\left(\frac{3}{4}\right)-7y=2$$
$$\frac{15}{4}-7y=2$$
$$7y=\frac{15}{4}-2=\frac{7}{4}$$

How do find y from $7y=\frac{7}{4}$ *?*

Dividing both sides by 7:

$$y=\frac{\cancel{7}}{4(\cancel{7})}=\frac{1}{4} \qquad \text{[Cancelling]}$$

The solution is $x=\frac{3}{4}$ and $y=\frac{1}{4}$.

(e) We are given the equations

$$\begin{array}{ccccc} \pi x & - & 5y & = & 2 \qquad (*) \\ \pi x & - & y & = & 1 \qquad (**) \end{array}$$

Subtracting these equations we have

$$\begin{array}{rl} & \pi x-5y=2 \\ - & (\pi x-y=1) \\ \hline & 0-4y=1 \end{array}$$

From the last line $-4y=1$ we have $y=-\frac{1}{4}$. *How do we determine x?*

Substitute $y=-\frac{1}{4}$ into (**):

$$\pi x-\left(-\frac{1}{4}\right)=1$$
$$\pi x+\frac{1}{4}=1 \quad \Rightarrow \quad \pi x=\frac{3}{4} \quad \text{which gives } x=\frac{3}{4\pi}$$

The solution is $x=\frac{3}{4\pi}$ and $y=-\frac{1}{4}$.

(f) We add the given equations

$$\begin{array}{rl} & ex-ey=2 \\ + & ex+ey=0 \\ \hline & 2ex+0=2 \end{array}$$

Transposing $2ex=2$ gives $x=\frac{1}{e}$. *What else do we need to find?*

The value of y. *How?*

By substituting $x=\frac{1}{e}$ into the given second equation $ex+ey=0$:

$$e\left(\frac{1}{e}\right)+ey=0$$

$$1+ey=0$$

$$ey=-1 \text{ we have } y=-\frac{1}{e}$$

The solution is $x=\frac{1}{e}$ and $y=-\frac{1}{e}$.

3. (a) We can label the given equations:

$$\begin{array}{ccccccccc} x & + & y & + & z & = & 3 & & (*) \\ x & - & y & - & z & = & -1 & & (**) \\ 2x & + & y & + & 5z & = & 8 & & (***) \end{array}$$

What do we need to find?

The values of x, y and z that satisfy the linear equations (*), (**) and (***). *How?*

By elimination. If we add the given equations (*) and (**) we get:

$$\begin{array}{rl} x+y+z=3 & (*) \\ +\quad x-y-z=-1 & (**) \\ \hline 2x+0+0=2 & \end{array}$$

From the last line $2x=2$ we have $x=1$. *What else do we need to find?*

The values of the place-holders y and z. *How?*

By substituting $x=1$ into the given equations (*) and (***):

$$1+y+z=3 \qquad (\square)$$

$$2+y+5z=8 \qquad (\square\square)$$

We can subtract these equations, that is $(\square\square)-(\square)$:

$$\begin{array}{r} 2+y+5z=8 \\ -\quad (1+y+z=3\ ) \\ \hline 1+0+4z=5 \end{array}$$

From the last line we have $1+4z=5$. Transposing gives

$$4z=4 \quad \text{implies that } z=1$$

So far we have $x=1$ and $z=1$. *How can we find y?*

Substitute these values $x=1$ and $z=1$ into the given equation $x+y+z=3 \quad (*)$.

$$1+y+1=3$$

$$y=1$$

We have the solution $x=1$, $y=1$ and $z=1$.

(b) We can label the given equations:

$$\begin{array}{ccccccccc} x & + & 2y & - & 2z & = & 6 & & (\Diamond) \\ 2x & - & 3y & + & z & = & -10 & & (\Diamond\Diamond) \\ 3x & - & y & + & 3z & = & -16 & & (\Diamond\Diamond\Diamond) \end{array}$$

What do we need to find?

The values of x, y and z that satisfy the given linear equations $(\Diamond), (\Diamond\Diamond)$ and $(\Diamond\Diamond\Diamond)$. *How?*

By elimination. If we multiply $(\lozenge)$ by 2 and then subtract $(\lozenge\lozenge)$ we get:

$$\begin{array}{rll} & 2x+4y-4z=12 & \left[\text{Multiplying } (\lozenge) \text{ by } 2\right] \\ - & (2x-3y+z=-10) & (\lozenge\lozenge) \\ \hline & 0+7y-5z=22 & \end{array}$$

We need another linear equation involving y and z so that we have two simultaneous equations with unknowns y and z. *How can we get such an equation?*

By multiplying equation $(\lozenge)$ by 3 and then subtracting equation $(\lozenge\lozenge\lozenge)$:

$$\begin{array}{rll} & 3x+6y-6z=18 & \left[\text{Multiplying } (\lozenge) \text{ by } 3\right] \\ - & (3x-y+3z=-16) & (\lozenge\lozenge\lozenge) \\ \hline & 0+7y-9z=34 & \end{array}$$

Hence we have obtained two simultaneous linear equations with unknowns y and z only:

$$7y-5z=22 \qquad (\dagger)$$
$$7y-9z=34 \qquad (\dagger\dagger)$$

Subtracting these equations, $(\dagger)-(\dagger\dagger)$, we have

$$\begin{array}{rl} & 7y-5z=22 \\ - & (7y-9z=34) \\ \hline & 0+4z=-12 \end{array}$$

From the last line $4z=-12$ we have $z=-3$. *What else do we need to find?*

The values of y and x. *How?*

By substituting $z=-3$ into equation $(\dagger)$:

$$\begin{aligned} 7y-5(-3)&=22 \\ 7y+15&=22 \\ 7y&=7 \quad \text{which gives } y=1 \end{aligned}$$

We have found $y=1$ and $z=-3$. *How can we find x?*

Substitute these values $y=1$ and $z=-3$ into the equation $x + 2y - 2z = 6 \quad (\Box)$.

$$\begin{aligned} x+2(1)-2(-3)&=6 \\ x+2+6&=6 \\ x&=-2 \end{aligned}$$

We have the solution $x=-2$, $y=1$ and $z=-3$ that satisfy the given linear system.

(c) We can label the given equations:

$$\begin{array}{rcrcrcrl} 3x & + & y & - & 2z & = & 4 & \quad (\bullet) \\ 5x & - & 3y & + & 10z & = & 32 & \quad (\bullet\bullet) \\ 7x & + & 4y & + & 16z & = & 13 & \quad (\bullet\bullet\bullet) \end{array}$$

What do we need to find?

The values of x, y and z that satisfy all the linear equations $(\bullet)$, $(\bullet\bullet)$ and $(\bullet\bullet\bullet)$. *How?*

By elimination. If we multiply equation $(\bullet)$ by 3 and then add equation $(\bullet\bullet)$ we get:

$$\begin{array}{rrl} & 9x+3y-6z=12 & \left[\text{Multiplying } (\bullet) \text{ by } 3\right] \\ + & 5x-3y+10z=32 & (\bullet\bullet) \\ \hline & 14x+0+4z=44 & \end{array}$$

We need another linear equation involving x and z so that we have two simultaneous equations with unknowns x and z only. *How can we get such an equation?*
By multiplying the given equation ($\bullet$) by 4 and then subtract equation ($\bullet\bullet\bullet$):

$$\begin{array}{rrl} & 12x+4y-8z=16 & \left[\text{Multiplying } (\bullet) \text{ by } 4\right] \\ - & (7x+4y+16z=13) & (\bullet\bullet\bullet) \\ \hline & 5x+0-24z=3 & [\text{Subtracting}] \end{array}$$

Hence we have two simultaneous linear equations with unknowns x and z only:

$$14x+4z=44 \qquad (\dagger)$$
$$5x-24z=3 \qquad (\dagger\dagger)$$

How can we determine x or z?
By multiplying (†) by 6 and then adding (††):

$$\begin{array}{rrl} & 84x+24z=264 & \left[\text{Multiplying } (\dagger) \text{ by } 6\right] \\ + & 5x-24z=3 & (\dagger\dagger) \\ \hline & 89x-0=267 & \end{array}$$

From the last line $89x=267$ we have $x=\frac{267}{89}=3$. *What else do we need to find?*
The values of y and z. *How?*
By substituting $x=3$ into equation $14x+4z=44 \quad (\dagger)$:

$$14(3)+4z=44$$
$$42+4z=44$$
$$4z=44-42=2$$
$$z=\frac{2}{4}=\frac{1}{2}$$

So far we have found $x=3$ and $z=\frac{1}{2}$. *How can we find y?*

Substitute these values $x=3$ and $z=\frac{1}{2}$ into the given equation $3x+y-2z=4 \quad (\bullet)$.

$$3(3)+y-2\left(\frac{1}{2}\right)=4$$
$$9+y-1=4$$
$$y+8=4 \text{ which gives } y=-4$$

We have the solution $x=3$, $y=-4$ and $z=\frac{1}{2}$.

(d) We can label the given equations:

$$\begin{array}{rcrcrcrl} 6x & - & 3y & + & 2z & = & 31 & (*) \\ 5x & + & y & + & 12z & = & 36 & (**) \\ 8x & + & 5y & + & z & = & 11 & (***) \end{array}$$

What do we need to find?
The values of x, y and z that satisfy all the linear equations (*), (**) and (***). *How?*
By elimination. If we multiply equation (**) by 3 and add equation (*) we get:

$$\begin{array}{rll} & 6x-3y+2z=31 & (*) \\ + & 15x+3y+36z=108 & \left[\text{Multiplying } (**) \text{ by } 3\right] \\ \hline & 21x+0+38z=139 & \end{array}$$

We need another linear equation involving x and z so that we have two simultaneous equations with two unknowns x and z only. *How can we get such an equation?*
By multiplying the given equation (**) by 5 and then subtract equation (***):

$$\begin{array}{rll} & 25x+5y+60z=180 & \left[\text{Multiplying } (**) \text{ by } 5\right] \\ - & (8x+5y+z=11) & (***) \\ \hline & 17x+0+59z=169 & \end{array}$$

Hence we have two simultaneous linear equations with two unknowns x and z only:

$$21x+38z=139$$

$$17x+59z=169$$

How can we determine x or z?
By multiplying the first equation, $21x+38z=139$, by 17

$$(21\times17)x+(38\times17)z=(139\times17)$$

$$357x+646z=2363 \qquad (\dagger)$$

and multiplying the second equation, $17x+59z=169$, by 21

$$(17\times21)x+(59\times21)z=(169\times21)$$

$$357x+1239z=3549 \qquad (\dagger\dagger)$$

How can we eliminate x from these two equations, (†) and (††)?
By subtracting:

$$\begin{array}{rll} & 357x+1239z=3549 & (\dagger\dagger) \\ - & (357x+646z=2363) & (\dagger) \\ \hline & 0+593z=1186 & \end{array}$$

From the last line $593z=1186$ we have $z=\frac{1186}{593}=2$. *What else do we need to find?*
The values of x and y. *How?*
By substituting $z=2$ into $357x+646z=2363$ $(\dagger)$:

$$357x+646(2)=2363$$

$$357x+1292=2363$$

$$357x=2363-1292=1071$$

$$x=\frac{1071}{357}=3$$

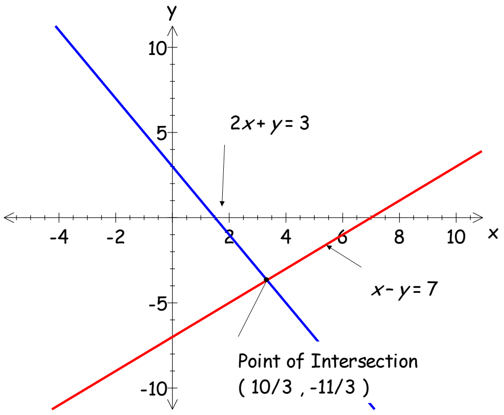
We have found $x=3$ and $z=2$. *How can we find y?*
Substitute these values $x=3$ and $z=2$ into the given equation $6x-3y+2z=31$ (*).

$$\begin{aligned} 6(3)-3y+2(2)&=31\\ 18-3y+4&=31\\ 22-3y&=31\\ -3y&=31-22=9\\ y&=-\frac{9}{3}=-3 \end{aligned}$$

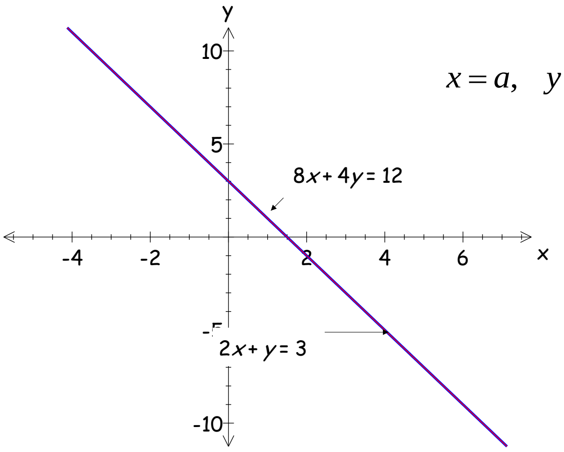
We have the solution $x=3$, $y=-3$ and $z=2$.

4. The graphs are as follows:

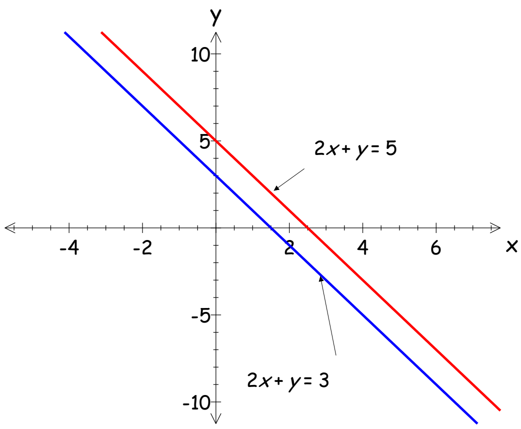
(a) One Solution



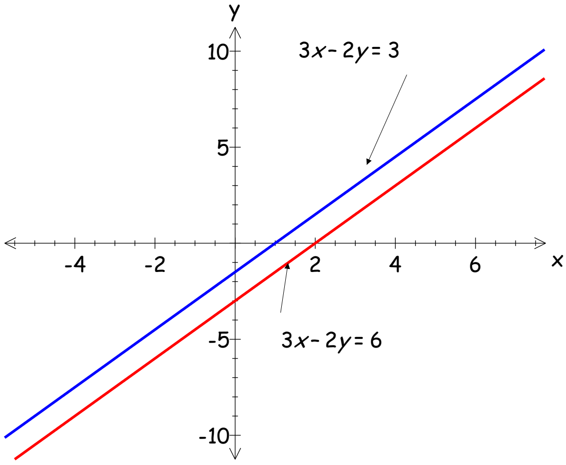
(b) Infinite number of solutions.



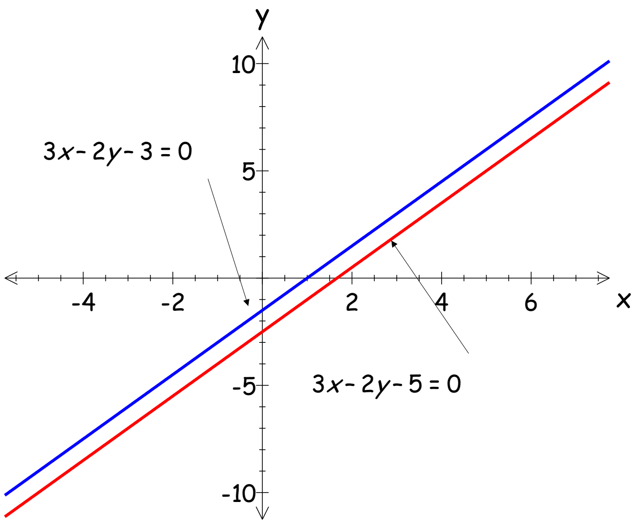
(c) No solution.



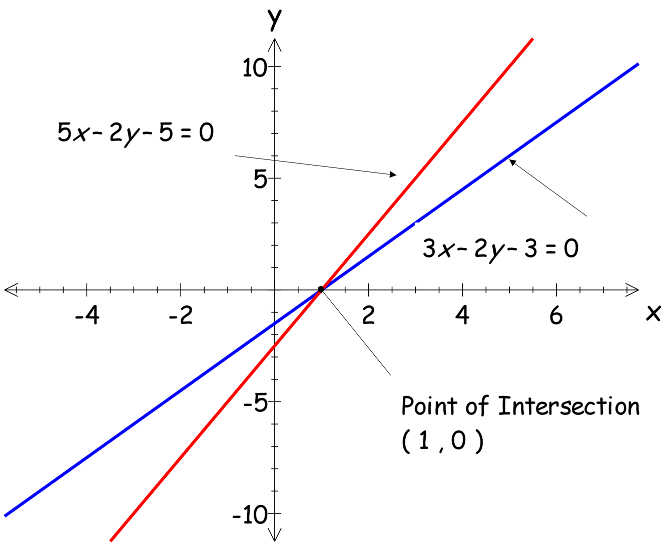
(d) No solution.



(e) No solution.



(f) Unique solution.



5. (a) If we had the given equations:

$$\begin{aligned} 7x &+ y &= 10 \\ x &- y &= 7 \end{aligned}$$

we will find a value of *x*. This means the solution is unique.

(b) *What do you notice about the given equations?*

$$\begin{aligned} 12x &+ 4y &= 16 \\ 8x &+ 4y &= 16 \end{aligned}$$

Different coefficients of *x* but the same *y* give 16. Let $x = 0$ then we have the same equation, $4y = 16$.

(c) What do you notice about these equations:

$$\begin{aligned} 2x &- y &- z &= 3 \\ 4x &- 2y &- 2z &= 3 \end{aligned}$$

Different coefficients of *x*, *y* and *z* give the same answer, 3, therefore the system has no solution.