Complete Solutions to Exercises 6.1

1. (a) $\det(\mathbf{A}) = \det\begin{pmatrix} 7 & 9 \\ 5 & 7 \end{pmatrix} = (7\times 7) - (5\times 9) = 4$

(b) $\det(\mathbf{B}) = \det\begin{pmatrix} 9 & 2 \\ 13 & 3 \end{pmatrix} = (9\times 3) - (13\times 2) = 1$

(c) $\det(\mathbf{C}) = \det\begin{pmatrix} 17 & 7 \\ 12 & 5 \end{pmatrix} = (17\times 5) - (12\times 7) = 1$

(d) $\det(\mathbf{D}) = \det\begin{pmatrix} 7 & 1 \\ 14 & 2 \end{pmatrix} = (7\times 2) - (14\times 1) = 0$

2. Use (6.1) to find the determinants.

(a) $\det\begin{pmatrix} 1 & 3 \\ 5 & 7 \end{pmatrix} = (1\times 7) - (5\times 3) = -8$ and $\det\begin{pmatrix} 1 & 5 \\ 3 & 7 \end{pmatrix} = (1\times 7) - (3\times 5) = -8$

(b) $\det\begin{pmatrix} -1 & 2 \\ 5 & 3 \end{pmatrix} = (-1\times 3) - (5\times 2) = -13$ and $\det\begin{pmatrix} -1 & 5 \\ 2 & 3 \end{pmatrix} = (-1\times 3) - (2\times 5) = -13$

(c) $\mathbf{A} = \begin{pmatrix} -1 & -1 \\ 0 & 0 \end{pmatrix}$, $\mathbf{B} = \begin{pmatrix} -1 & 0 \\ -1 & 0 \end{pmatrix}$ and evaluating the determinants by using (6.1):

$$\det(\mathbf{A}) = \det(\mathbf{B}) = 0$$

The matrix **A** is transposed (rows $\to$ columns) to give matrix **B**. The same numbers on each of the diagonals, so the determinant is the same,

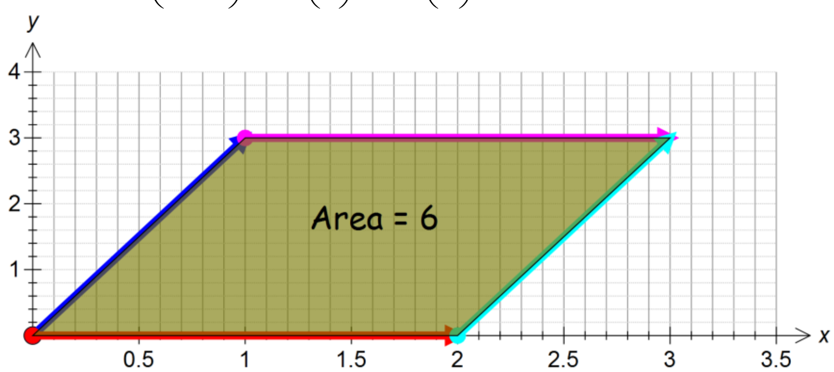
$$\det(\mathbf{A}) = \det(\mathbf{B})$$

3. By (6.1) we have

$$\det(\mathbf{A}) = \det\begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - cb \quad \text{and} \quad \det(\mathbf{A}^T) = \det\begin{pmatrix} a & c \\ b & d \end{pmatrix} = ad - bc = ad - cb = \det\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

Hence $\det(\mathbf{A}) = \det(\mathbf{A}^T)$.

4. The column vectors of $\begin{pmatrix} 1 & 2 \\ 3 & 0 \end{pmatrix}$ are $\begin{pmatrix} 1 \\ 3 \end{pmatrix}$ and $\begin{pmatrix} 2 \\ 0 \end{pmatrix}$. Drawing the parallelogram we have:



We have

$$\det\begin{pmatrix} 2 & 1 \\ 0 & 3 \end{pmatrix} = (3\times 2) - (1\times 0) = 6$$

Since we change over the vectors so the determinant will be negative:

$$\det\begin{pmatrix}1 & 2\\ 3 & 0\end{pmatrix}=(1\times 0)-(3\times 2)=-6$$

5. Writing the given linear system in terms of matrices $\mathbf{Ax}=\mathbf{b}$ where:

$$\mathbf{A}=\begin{pmatrix}2 & 3\\ 5 & -2\end{pmatrix},\ \mathbf{x}=\begin{pmatrix}x\\ y\end{pmatrix},\ \mathbf{b}=\begin{pmatrix}-2\\ 14\end{pmatrix}$$

Evaluating the determinant of matrix **A**

$$\det(\mathbf{A})=\det\begin{pmatrix}2 & 3\\ 5 & -2\end{pmatrix}=(-4)-15=-19\neq 0$$

Since $\det(\mathbf{A})=-19\neq 0$ so the linear system has a unique solution which we can find by the inverse matrix $\mathbf{A}^{-1}$ by using:

(6.2) $$\begin{pmatrix}a & b\\ c & d\end{pmatrix}^{-1}=\frac{1}{\det(\mathbf{A})}\begin{pmatrix}d & -b\\ -c & a\end{pmatrix}$$

$$\mathbf{x}=\mathbf{A}^{-1}\mathbf{b}=-\frac{1}{19}\begin{pmatrix}-2 & -3\\ -5 & 2\end{pmatrix}\begin{pmatrix}-2\\ 14\end{pmatrix}$$

$$=-\frac{1}{19}\begin{pmatrix}-38\\ 38\end{pmatrix}=\begin{pmatrix}2\\ -2\end{pmatrix}$$

Hence $x=2,\ y=-2$.

6. By using the properties of linear transformations of the last chapter we have

$$(T\circ T)(\mathbf{x})=\mathbf{A}^2\mathbf{x}$$

The transformation T expands the area of the object which is bounded by the column vectors of **A** and is given by $\det(\mathbf{A})$. Applying the transformation to the result $T(\mathbf{x})$ gives another expansion of the area by $\det(\mathbf{A})$. Therefore

$$\det(\mathbf{A}^2)=\det(\mathbf{A})\det(\mathbf{A})$$

7. We need to calculate the Wronskian determinant $W\left(e^{-x},\ e^{-3x}\right)$:

$$W\left(e^{-x},\ e^{-3x}\right)=\det\begin{pmatrix}e^{-x} & e^{-3x}\\ -e^{-x} & -3e^{-3x}\end{pmatrix}\qquad\left[\text{Because }\left(e^{kx}\right)'=ke^{kx}\right]$$

$$=e^{-x}\left(-3e^{-3x}\right)-\left(-e^{-x}\right)e^{-3x}=-3e^{-4x}+e^{-4x}=-2e^{-4x}\neq 0$$

Hence the two solutions e^{-x} and e^{-3x} are linearly independent because $W\left(e^{-x},\ e^{-3x}\right)\neq 0$.

The exponential function is never zero, that is for any real number x we have $e^{x}\neq 0$.

8. We can check condition (b) of Definition (5-2) first, that is $T(k\,\mathbf{A})=kT(\mathbf{A})$.

Let $\mathbf{A}=\begin{pmatrix}a & b\\ c & d\end{pmatrix}$ then

$$
\begin{aligned}
T(k\,\mathbf{A}) &= T\left(k\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) \\
&= T\left(\begin{pmatrix} ka & kb \\ kc & kd \end{pmatrix}\right) \\
&= k^2ad - k^2bc \qquad [\text{Taking Determinant}] \\
&= k^2(ad - bc) = k^2\det(\mathbf{A}) = k^2T(\mathbf{A}) \neq k\,T(\mathbf{A}) \quad [\text{Not Equal}]
\end{aligned}
$$

The given transformation T does **not** satisfy condition (b) therefore we conclude that $T(\mathbf{A}) = \det(\mathbf{A})$ is **not** linear. Hence the determinant of a 2 by 2 matrix is **not** a linear transformation.