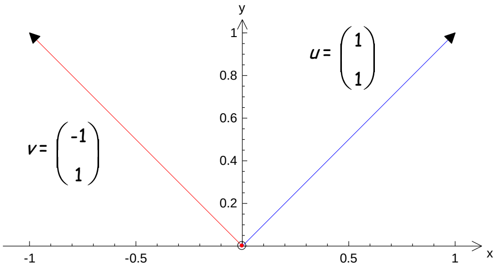
Complete Solutions to Exercises 4.2

1. (a) We have

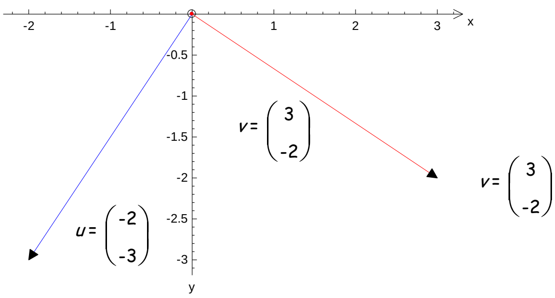


Evaluating the inner (dot) product of these gives

$$\langle \mathbf{u},\ \mathbf{v} \rangle = \mathbf{u} \cdot \mathbf{v} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} -1 \\ 1 \end{pmatrix} = \left(1 \times (-1)\right) + (1 \times 1) = -1 + 1 = 0$$

Thus vectors **u** and **v** are orthogonal.

(b) We have $\mathbf{u} = \begin{pmatrix} -2 \\ -3 \end{pmatrix}$ and $\mathbf{v} = \begin{pmatrix} 3 \\ -2 \end{pmatrix}$:

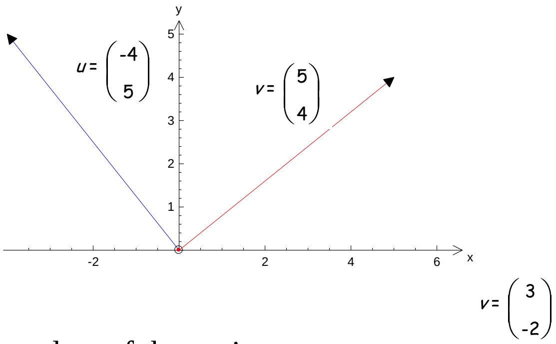


Evaluating the inner (dot) product of these gives

$$\langle \mathbf{u},\ \mathbf{v} \rangle = \mathbf{u} \cdot \mathbf{v} = \begin{pmatrix} -2 \\ -3 \end{pmatrix} \cdot \begin{pmatrix} 3 \\ -2 \end{pmatrix} = (-2 \times 3) + \left(-3 \times (-2)\right) = -6 + 6 = 0$$

Thus vectors **u** and **v** are orthogonal.

(c) We are given $\mathbf{u} = \begin{pmatrix} -4 \\ 5 \end{pmatrix}$ and $\mathbf{v} = \begin{pmatrix} 5 \\ 4 \end{pmatrix}$:

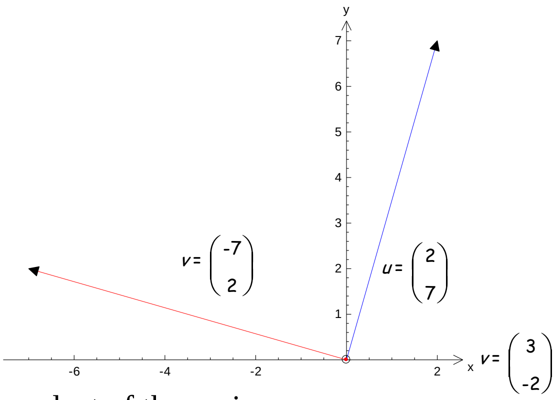


Evaluating the inner (dot) product of these gives

$$\langle \mathbf{u},\ \mathbf{v}\rangle = \mathbf{u}\cdot\mathbf{v} = \begin{pmatrix} -4 \\ 5 \end{pmatrix}\cdot\begin{pmatrix} 5 \\ 4 \end{pmatrix} = (-4\times 5)+(5\times 4) = -20+20 = 0$$

Thus vectors **u** and **v** are orthogonal.

(d) We are given $\mathbf{u} = \begin{pmatrix} 2 \\ 7 \end{pmatrix}$ and $\mathbf{v} = \begin{pmatrix} -7 \\ 2 \end{pmatrix}$:



Evaluating the inner (dot) product of these gives

$$\langle \mathbf{u},\ \mathbf{v}\rangle = \mathbf{u}\cdot\mathbf{v} = \begin{pmatrix} 2 \\ 7 \end{pmatrix}\cdot\begin{pmatrix} -7 \\ 2 \end{pmatrix} = (2\times(-7))+(7\times 2) = -14+14 = 0$$

Thus vectors **u** and **v** are orthogonal.

2. We need to show that the inner product for $\mathbf{u} = \begin{pmatrix} a \\ b \end{pmatrix}$ and $\mathbf{v} = \begin{pmatrix} -b \\ a \end{pmatrix}$ is zero.

We have

$$\langle \mathbf{u},\ \mathbf{v}\rangle = \mathbf{u}\cdot\mathbf{v} = \begin{pmatrix} a \\ b \end{pmatrix}\cdot\begin{pmatrix} -b \\ a \end{pmatrix}$$
$$= -ab+ab = 0$$

Hence the vectors **u** and **v** are orthogonal.

3. (a) We are given $\mathbf{u} = \begin{pmatrix} 5 \\ 0 \\ 0 \end{pmatrix}$, $\mathbf{v} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ and $\mathbf{w} = \begin{pmatrix} 0 \\ 0 \\ 10 \end{pmatrix}$. To check orthogonality:

$$\langle \mathbf{u},\ \mathbf{v}\rangle = \mathbf{u}\cdot\mathbf{v} = \begin{pmatrix} 5 \\ 0 \\ 0 \end{pmatrix}\cdot\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = (5\times 0)+(0\times 1)+(0\times 0) = 0$$

$$\langle \mathbf{u},\ \mathbf{w}\rangle = \mathbf{u}\cdot\mathbf{w} = \begin{pmatrix} 5 \\ 0 \\ 0 \end{pmatrix}\cdot\begin{pmatrix} 0 \\ 0 \\ 10 \end{pmatrix} = (5\times 0)+(0\times 0)+(0\times 10) = 0$$

$$\langle \mathbf{v},\ \mathbf{w}\rangle = \mathbf{v}\cdot\mathbf{w} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}\cdot\begin{pmatrix} 0 \\ 0 \\ 10 \end{pmatrix} = (0\times 0)+(1\times 0)+(0\times 10) = 0$$

All 3 vectors **u**, **v** and **w** are orthogonal.

(b) We are given the vectors $\mathbf{u}=\begin{pmatrix}0\\0\\0\end{pmatrix}$, $\mathbf{v}=\begin{pmatrix}1\\1\\-1\end{pmatrix}$ and $\mathbf{w}=\begin{pmatrix}-3\\10\\7\end{pmatrix}$. *What do you notice about the vector **u**?*

$\mathbf{u}=\mathbf{O}$ [Zero Vector]. *What do we know about the orthogonality of the zero vector?*
By Proposition (4-6) which says every vector is orthogonal to the zero vector. We only need to check orthogonality of vectors **v** and **w**:

$$\langle \mathbf{v},\ \mathbf{w}\rangle=\mathbf{v}\cdot\mathbf{w}=\begin{pmatrix}1\\1\\-1\end{pmatrix}\cdot\begin{pmatrix}-3\\10\\7\end{pmatrix}=\left(1\times(-3)\right)+\left(1\times 10\right)+\left(-1\times 7\right)=0$$

4. (a) We are given $\mathbf{A}=\begin{pmatrix}3&7\\5&4\end{pmatrix}$ and $\mathbf{B}=\begin{pmatrix}2&1\\7&-12\end{pmatrix}$. Applying $\langle \mathbf{A},\ \mathbf{B}\rangle=tr\left(\mathbf{B}^T\mathbf{A}\right)$ gives

$$\begin{aligned}\langle \mathbf{A},\ \mathbf{B}\rangle=tr\left(\mathbf{B}^T\mathbf{A}\right)&=tr\left[\begin{pmatrix}2&1\\7&-12\end{pmatrix}^T\begin{pmatrix}3&7\\5&4\end{pmatrix}\right]\\&=tr\left[\begin{pmatrix}2&7\\1&-12\end{pmatrix}\begin{pmatrix}3&7\\5&4\end{pmatrix}\right]\\&=tr\begin{pmatrix}41&*\\ *&-41\end{pmatrix}=41-41=0\end{aligned}$$

Thus matrices **A** and **B** are orthogonal.
(b) Since matrices **A** and **B** are orthogonal we can apply Pythagoras's Theorem (4-5):

$$\|\mathbf{A}+\mathbf{B}\|^2=\|\mathbf{A}\|^2+\|\mathbf{B}\|^2 \qquad (*)$$

We can evaluate $\|\mathbf{A}\|^2$ and $\|\mathbf{B}\|^2$ by the inner products:

$$\begin{aligned}\|\mathbf{A}\|^2=\langle \mathbf{A},\ \mathbf{A}\rangle=tr\left(\mathbf{A}^T\mathbf{A}\right)&=tr\left[\begin{pmatrix}3&7\\5&4\end{pmatrix}^T\begin{pmatrix}3&7\\5&4\end{pmatrix}\right]\\&=tr\left[\begin{pmatrix}3&5\\7&4\end{pmatrix}\begin{pmatrix}3&7\\5&4\end{pmatrix}\right]\\&=tr\begin{pmatrix}34&*\\ *&65\end{pmatrix}=34+65=99\end{aligned}$$

Similarly

$$\begin{aligned}\|\mathbf{B}\|^2=\langle \mathbf{B},\ \mathbf{B}\rangle=tr\left(\mathbf{B}^T\mathbf{B}\right)&=tr\left[\begin{pmatrix}2&1\\7&-12\end{pmatrix}^T\begin{pmatrix}2&1\\7&-12\end{pmatrix}\right]\\&=tr\left[\begin{pmatrix}2&7\\1&-12\end{pmatrix}\begin{pmatrix}2&1\\7&-12\end{pmatrix}\right]\\&=tr\begin{pmatrix}53&*\\ *&145\end{pmatrix}=53+145=198\end{aligned}$$

Substituting $\|\mathbf{A}\|^2 = 99$ and $\|\mathbf{B}\|^2 = 198$ into (*) and taking the square root gives:

$$\|\mathbf{A}+\mathbf{B}\| = \sqrt{\|\mathbf{A}\|^2 + \|\mathbf{B}\|^2} = \sqrt{99+198} = 17.23 \text{ (2dp)}$$

5. (a) We have

$$\langle \mathbf{u},\ \mathbf{v}\rangle = \mathbf{u}\cdot\mathbf{v} = \begin{pmatrix}1\\2\\3\\4\end{pmatrix}\cdot\begin{pmatrix}-2\\3\\k\\5\end{pmatrix} = \left(1\times(-2)\right)+(2\times 3)+(3\times k)+(4\times 5)$$

$$= -2+6+3k+20 = 24+3k = 0$$

Solving $24+3k=0$ gives $3k=-24 \quad\Rightarrow\quad k=-8$.

(b) Similarly we have

$$\langle \mathbf{u},\ \mathbf{v}\rangle = \mathbf{u}\cdot\mathbf{v} = \begin{pmatrix}k\\-1\\k\\1\end{pmatrix}\cdot\begin{pmatrix}2\\4\\k\\5\end{pmatrix} = (k\times 2)+(-1\times 4)+(k\times k)+(1\times 5)$$

$$= 2k-4+k^2+5 = k^2+2k+1 = 0$$

Solving the quadratic gives

$$k^2+2k+1=0$$

$$(k+1)^2 = 0 \quad\Rightarrow\quad k=-1$$

6. Cauchy Schwarz Inequality is $|\langle \mathbf{u},\ \mathbf{v}\rangle| \le \|\mathbf{u}\|\|\mathbf{v}\|$.

(a) We need to verify $|\langle \mathbf{f},\ \mathbf{g}\rangle| \le \|\mathbf{f}\|\|\mathbf{g}\|$. Evaluating the inner product and the norms:

$$\langle \mathbf{f},\ \mathbf{g}\rangle = \langle x,\ x-1\rangle = \int_0^1 x(x-1)\,\mathrm{d}x$$

$$= \int_0^1 \left(x^2-x\right)\mathrm{d}x \qquad \left[\text{Expanding } x(x-1)\right]$$

$$= \left[\frac{x^3}{3}-\frac{x^2}{2}\right]_0^1 \qquad [\text{Integrating}]$$

$$= \left[\frac{1}{3}-\frac{1}{2}\right] = -\frac{1}{6}$$

$|\langle \mathbf{f},\ \mathbf{g}\rangle| = \left|-\frac{1}{6}\right| = \frac{1}{6}$.

Initially we determine $\|\mathbf{f}\|^2$ and $\|\mathbf{g}\|^2$ and then take the square root:

$$\|\mathbf{f}\|^2 = \langle \mathbf{f},\ \mathbf{f}\rangle = \int_0^1 f(x)f(x)\,\mathrm{d}x$$

$$= \int_0^1 xx\,\mathrm{d}x = \int_0^1 x^2\,\mathrm{d}x = \left[\frac{x^3}{3}\right]_0^1 = \frac{1}{3}$$

$$\|\mathbf{g}\|^2 = \langle \mathbf{g},\ \mathbf{g}\rangle = \int_0^1 g(x)g(x)\,\mathrm{d}x$$

$$= \int_0^1 (x-1)(x-1)\,\mathrm{d}x$$

$$= \int_0^1 \left(x^2-2x+1\right)\mathrm{d}x = \left[\frac{x^3}{3} - 2\frac{x^2}{2} + x\right]_0^1 = \frac{1}{3} - 1 + 1 = \frac{1}{3}$$

Taking square root of $\|\mathbf{f}\|^2 = \frac{1}{3}$ and $\|\mathbf{g}\|^2 = \frac{1}{3}$ gives

$$\|\mathbf{f}\| = \frac{1}{\sqrt{3}} \quad \text{and} \quad \|\mathbf{g}\| = \frac{1}{\sqrt{3}}$$

Hence $\|\mathbf{f}\|\|\mathbf{g}\| = \frac{1}{\sqrt{3}}\frac{1}{\sqrt{3}} = \frac{1}{3}$ and this shows Cauchy Schwarz Inequality:

$$\left|\langle \mathbf{f},\ \mathbf{g}\rangle\right| = \frac{1}{6} < \frac{1}{3} = \|\mathbf{f}\|\|\mathbf{g}\|$$

(b) We need to verify $\left|\langle \mathbf{f},\ \mathbf{g}\rangle\right| \le \|\mathbf{f}\|\|\mathbf{g}\|$ for $\mathbf{f} = f(x) = 1$ and $\mathbf{g} = g(x) = e^x$. Evaluating the inner product and the norms for $\mathbf{f} = 1$ and $\mathbf{g} = e^x$:

$$\langle \mathbf{f},\ \mathbf{g}\rangle = \langle 1,\ e^x\rangle = \int_0^1 e^x\,\mathrm{d}x$$

$$= \left[e^x\right]_0^1 \qquad \left[\text{Because} \int e^x\,dx = e^x\right]$$

$$= \left[e^1 - 1\right] = e - 1 = 1.72\ \ (2\text{ dp})$$

$\left|\langle \mathbf{f},\ \mathbf{g}\rangle\right| = |1.72| = 1.72$.

First we determine $\|\mathbf{f}\|^2$ and $\|\mathbf{g}\|^2$ and then take the square root:

$$\|\mathbf{f}\|^2 = \langle \mathbf{f},\ \mathbf{f}\rangle = \int_0^1 1\,\mathrm{d}x = [x]_0^1 = 1$$

$$\|\mathbf{g}\|^2 = \langle \mathbf{g},\ \mathbf{g}\rangle = \int_0^1 e^x e^x\,\mathrm{d}x$$

$$= \int_0^1 e^{2x}\,\mathrm{d}x = \left[\frac{e^{2x}}{2}\right]_0^1 = \frac{e^2-1}{2} = 3.19\ \ (2\text{ dp})$$

Taking square root gives

$$\|\mathbf{f}\| = \sqrt{1} = 1 \quad \text{and} \quad \|\mathbf{g}\| = \sqrt{3.19} = 1.79$$

Hence $\|\mathbf{f}\|\|\mathbf{g}\| = 1 \times 1.79 = 1.79$ and this shows Cauchy Schwarz Inequality

$$\left|\langle \mathbf{f},\ \mathbf{g}\rangle\right| = 1.72 < 1.79 = \|\mathbf{f}\|\|\mathbf{g}\|$$

7. Need to show the following result

$$0 \le \left| \left\langle \frac{\mathbf{x}}{\|\mathbf{x}\|},\ \frac{\mathbf{y}}{\|\mathbf{y}\|} \right\rangle \right| \le 1$$

Since we have the modulus symbol which measures the absolute value so clearly we have the given expression is ≥ 0. Using Cauchy Schwarz inequality:

(4-3) $\quad \left|\langle \mathbf{u},\ \mathbf{v} \rangle\right| \le \|\mathbf{u}\|\|\mathbf{v}\|$

Gives

$$\left| \left\langle \frac{\mathbf{x}}{\|\mathbf{x}\|},\ \frac{\mathbf{y}}{\|\mathbf{y}\|} \right\rangle \right| \le \left\| \frac{\mathbf{x}}{\|\mathbf{x}\|} \right\| \left\| \frac{\mathbf{y}}{\|\mathbf{y}\|} \right\|$$

By Proposition (4-7) we know that $\frac{\mathbf{x}}{\|\mathbf{x}\|}$ and $\frac{\mathbf{y}}{\|\mathbf{y}\|}$ are unit vectors so they have a norm or length of 1. Hence we have our result:

$$\left| \left\langle \frac{\mathbf{x}}{\|\mathbf{x}\|},\ \frac{\mathbf{y}}{\|\mathbf{y}\|} \right\rangle \right| \le \left\| \frac{\mathbf{x}}{\|\mathbf{x}\|} \right\| \left\| \frac{\mathbf{y}}{\|\mathbf{y}\|} \right\| = 1 \times 1 = 1$$

8. (a) We have

$$\begin{aligned}
\langle \mathbf{f},\ \mathbf{g} \rangle &= \langle \cos(x),\ \sin(x) \rangle \\
&= \int_{-\pi}^{\pi} \cos(x)\sin(x)\,\mathrm{d}x \\
&= \frac{1}{2}\int_{-\pi}^{\pi} \sin(2x)\,\mathrm{d}x && \left[\text{Because } \cos(x)\sin(x) = \frac{1}{2}\sin(2x)\right] \\
&= -\frac{1}{2}\left[\frac{\cos(2x)}{2}\right]_{-\pi}^{\pi} && \left[\text{Because } \int \sin(kx)\ \mathrm{d}x = -\frac{\cos(kx)}{k}\right] \\
&= -\frac{1}{4}\left[\underbrace{\cos(2\pi)}_{=1} - \underbrace{\cos(-2\pi)}_{=1}\right] = -\frac{1}{4}[1-1] = 0
\end{aligned}$$

Hence **f** and **g** are orthogonal.

(b) We need to show $\left|\langle \mathbf{f},\ \mathbf{g} \rangle\right| \le \|\mathbf{f}\|\|\mathbf{g}\|$. *What is* $\left|\langle \mathbf{f},\ \mathbf{g} \rangle\right|$ *equal to?*

By part (a) we know **f** and **g** are orthogonal therefore $\left|\langle \mathbf{f},\ \mathbf{g} \rangle\right| = 0$. By properties of a norm, Proposition (4-2) part (a), we know $\|\mathbf{f}\| \ge 0$ and $\|\mathbf{g}\| \ge 0$ therefore we have Cauchy Schwarz Inequality $\left|\langle \mathbf{f},\ \mathbf{g} \rangle\right| \le \|\mathbf{f}\|\|\mathbf{g}\|$.

(c) The Minkowski Inequality is $\|\mathbf{f} + \mathbf{g}\| \le \|\mathbf{f}\| + \|\mathbf{g}\|$. We first find the square of these, that is $\|\mathbf{f} + \mathbf{g}\|^2$, $\|\mathbf{f}\|^2$ and $\|\mathbf{g}\|^2$, then take the square root:

$$\begin{aligned}
\| \mathbf{f}+\mathbf{g}\|^2 &= \langle \mathbf{f}+\mathbf{g},\ \mathbf{f}+\mathbf{g}\rangle \\
&= \int_{-\pi}^{\pi} \left[f(x)+g(x)\right]\left[f(x)+g(x)\right]\,\mathrm{d}x \\
&= \int_{-\pi}^{\pi} \left[\cos(x)+\sin(x)\right]^2\,\mathrm{d}x \\
&= \int_{-\pi}^{\pi} \left[\cos^2(x)+2\cos(x)\sin(x)+\sin^2(x)\right]\,\mathrm{d}x \\
&= \int_{-\pi}^{\pi} \left[1+\sin(2x)\right]\,\mathrm{d}x && \left[\begin{array}{l}\text{Because } \cos^2(x)+\sin^2(x)=1 \\ \text{and } 2\cos(x)\sin(x)=\sin(2x)\end{array}\right] \\
&= \left[x-\frac{\cos(2x)}{2}\right]_{-\pi}^{\pi} && \left[\text{Because } \int \sin(kx)\,\mathrm{d}x = -\frac{\cos(kx)}{k}\right] \\
&= \left[\pi-\frac{\cos(2\pi)}{2}\right]-\left[-\pi-\frac{\cos(-2\pi)}{2}\right] && \left[\text{Substituting Limits}\right] \\
&= 2\pi
\end{aligned}$$

Taking the square root of $\| \mathbf{f}+\mathbf{g}\|^2 = 2\pi$ gives $\|\mathbf{f}+\mathbf{g}\| = \sqrt{2\pi}$.

We need to find $\|\mathbf{f}\|$ and $\|\mathbf{g}\|$:

$$\begin{aligned}
\| \mathbf{f}\|^2 &= \langle \mathbf{f},\ \mathbf{f}\rangle \\
&= \int_{-\pi}^{\pi} \left[\cos^2(x)\right]\,\mathrm{d}x \underset{\text{By Result}}{=} \pi
\end{aligned}$$

Similarly we have

$$\begin{aligned}
\| \mathbf{g}\|^2 &= \langle \mathbf{g},\ \mathbf{g}\rangle \\
&= \int_{-\pi}^{\pi} \left[\sin^2(x)\right]\,\mathrm{d}x \underset{\text{By Result}}{=} \pi
\end{aligned}$$

Taking the square root of these $\| \mathbf{f}\|^2 = \pi$ and $\|\mathbf{g}\|^2 = \pi$ gives $\|\mathbf{f}\| = \sqrt{\pi}$ and $\|\mathbf{g}\| = \sqrt{\pi}$ respectively. Adding these

$$\|\mathbf{f}\|+\|\mathbf{g}\| = \sqrt{\pi}+\sqrt{\pi} = 2\sqrt{\pi}$$

Since $\|\mathbf{f}+\mathbf{g}\| = \sqrt{2\pi} < 2\sqrt{\pi} = \|\mathbf{f}\|+\|\mathbf{g}\|$ therefore we have Minkowski Inequality.

(d) We normalize these vectors **f** and **g** by $\dfrac{\mathbf{f}}{\|\mathbf{f}\|}$ and $\dfrac{\mathbf{g}}{\|\mathbf{g}\|}$ with $\mathbf{f} = \cos(x)$, $\mathbf{g} = \sin(x)$, $\|\mathbf{f}\| = \sqrt{\pi}$ and $\|\mathbf{g}\| = \sqrt{\pi}$:

$$\frac{\mathbf{f}}{\|\mathbf{f}\|} = \frac{\cos(x)}{\sqrt{\pi}} \quad\text{and}\quad \frac{\mathbf{g}}{\|\mathbf{g}\|} = \frac{\sin(x)}{\sqrt{\pi}}$$

9. We use the given inner product $\langle \mathbf{f},\ \mathbf{g}_n\rangle = \int_0^{\pi} f(t)g_n(t)\,\mathrm{d}t$.

First we test $f(x)=1$ and $g_n(t)=\sin(2nt)$:

$$\langle \mathbf{f},\ \mathbf{g}_n \rangle = \int_0^{\pi} 1\sin(2nt)\,\mathrm{d}t$$

$$= \left[-\frac{\cos(2nt)}{2n}\right]_0^{\pi} = -\frac{1}{2n}\left[\underbrace{\cos(2n\pi)}_{=1} - \underbrace{\cos(0)}_{=1}\right] = -\frac{1}{2n}[1-1] = 0$$

Hence **f** and **g** are orthogonal for $n \geq 1$.

Now we test $g_n(t) = \sin(2nt)$ and $g_m(t) = \sin(2mt)$ where $n \neq m$:

$$\langle \sin(2nt),\ \sin(2mt) \rangle = \int_0^{\pi} \sin(2nt)\sin(2mt)\,\mathrm{d}t \underset{\text{By Result}}{=} 0$$

Hence the set $\{1,\ \sin(2t),\ \sin(4t),\ \sin(6t), \cdots\}$ is orthogonal.

We need to normalize these vectors by using Proposition (4-7) which is $\dfrac{\mathbf{w}}{\|\mathbf{w}\|}$:

$$\|\mathbf{f}\|^2 = \langle \mathbf{f},\ \mathbf{f} \rangle = \int_0^{\pi} (1\times 1)\,\mathrm{d}t$$

$$= [t]_0^{\pi} = \pi$$

What is $\|\mathbf{f}\|$ equal to?

Taking the square root of $\|\mathbf{f}\|^2 = \pi$ gives $\|\mathbf{f}\| = \sqrt{\pi}$. Similarly we find $\|\mathbf{g}\|$.

$$\|\mathbf{g}\|^2 = \langle \mathbf{g},\ \mathbf{g} \rangle = \int_0^{\pi} \sin(2nt)\times\sin(2nt)\,\mathrm{d}t$$

$$= \int_0^{\pi} \sin^2(2nt)\,\mathrm{d}t \underset{\text{By Result}}{=} \frac{\pi}{2}$$

Taking the square root gives $\|\mathbf{g}\| = \sqrt{\dfrac{\pi}{2}}$. Normalizing these vectors gives

$$\frac{\mathbf{f}}{\|\mathbf{f}\|} = \frac{1}{\sqrt{\pi}} \text{ and } \frac{\mathbf{g}}{\|\mathbf{g}\|} = \frac{\sin(2nt)}{\sqrt{\pi/2}} = \sqrt{\frac{2}{\pi}}\sin(2nt) \quad \left[\text{Because } \frac{1}{\sqrt{\pi/2}} = \sqrt{\frac{2}{\pi}}\right]$$

The orthonormal set is $\left\{\dfrac{1}{\sqrt{\pi}},\ \sqrt{\dfrac{2}{\pi}}\sin(2t),\ \sqrt{\dfrac{2}{\pi}}\sin(4t),\ \sqrt{\dfrac{2}{\pi}}\sin(6t), \cdots\right\}$.

10. Clearly we have

$$\langle \mathbf{A},\ \mathbf{B} \rangle = tr\left(\mathbf{B}^T\mathbf{A}\right)$$

$$= tr\left[\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}^T \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\right] = tr\left[\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\right] = tr\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = 0+0=0$$

Similarly we can show that

$$\langle \mathbf{A},\ \mathbf{C} \rangle = 0,\ \langle \mathbf{A},\ \mathbf{D} \rangle = 0,\ \langle \mathbf{B},\ \mathbf{C} \rangle = 0,\ \langle \mathbf{B},\ \mathbf{D} \rangle = 0 \text{ and } \langle \mathbf{C},\ \mathbf{D} \rangle = 0$$

Hence **all** the matrices are orthogonal. *What else do we need to prove?*

Required to show that $\|\mathbf{A}\| = \|\mathbf{B}\| = \|\mathbf{C}\| = \|\mathbf{D}\| = 1$. *What is $\|\mathbf{A}\|$ equal to?*

First we find $\|\mathbf{A}\|^2$ and then take the square root:

$$\|\mathbf{A}\|^2 = \langle \mathbf{A}, \ \mathbf{A} \rangle = tr\left(\mathbf{A}^T\mathbf{A}\right)$$
$$= tr\left[\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}^T \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\right]$$
$$= tr\left[\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\right] = tr\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = 1 + 0 = 1$$

Taking the square root gives $\|\mathbf{A}\| = 1$. Similarly we have

$$\|\mathbf{B}\|^2 = \langle \mathbf{B}, \ \mathbf{B} \rangle = tr\left(\mathbf{B}^T\mathbf{B}\right)$$
$$= tr\left[\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}^T \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\right]$$
$$= tr\left[\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\right] = tr\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = 0 + 1 = 1$$

Thus $\|\mathbf{B}\| = 1$. Similarly we have

$$\|\mathbf{C}\|^2 = \langle \mathbf{C}, \ \mathbf{C} \rangle = tr\left(\mathbf{C}^T\mathbf{C}\right)$$
$$= tr\left[\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}^T \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\right]$$
$$= tr\left[\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\right] = tr\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = 1 + 0 = 1$$

Hence $\|\mathbf{C}\| = 1$ and similarly we can also show that $\|\mathbf{D}\| = 1$. Since

$$\|\mathbf{A}\| = \|\mathbf{B}\| = \|\mathbf{C}\| = \|\mathbf{D}\| = 1$$

therefore the matrices are normalized so the set $S = \{\mathbf{A}, \ \mathbf{B}, \ \mathbf{C}, \ \mathbf{D}\}$ is orthonormal.

11. We need to find $\langle f(t), \ z(t) \rangle = \int_0^1 f(t)z(t)\,\mathrm{d}t$ where $f(t) = 100\sin(100t)$ and $z(t) = \cos(10t)$. Substituting these and using the given hint we have

$$\langle f(t),\ z(t)\rangle = \int_0^1 \left[100\sin(100t)\cos(10t)\right] \mathrm{d}t$$

$$= 100\int_0^1 \left[\sin(100t)\cos(10t)\right] \mathrm{d}t$$

$$= 50\int_0^1 \left[\sin(110t) + \sin(90t)\right] \mathrm{d}t \qquad \left[\begin{array}{l}\text{Because } \sin(100t)\cos(10t) \\ = \dfrac{1}{2}\left[\sin(100t+10t)+\sin(100t-10t)\right]\end{array}\right]$$

$$= 50\left[-\frac{\cos(110t)}{110} - \frac{\cos(90t)}{90}\right]_0^1 \qquad \left[\text{Because } \int \sin(kt)\,dt = -\frac{\cos(kt)}{k}\right]$$

$$= -50\left(\left[\frac{\cos(110)}{110} + \frac{\cos(90)}{90}\right] - \left(\frac{1}{110} + \frac{1}{90}\right)\right)$$

$$= -50\left(\left[\frac{-0.999}{110} + \frac{-0.448}{90}\right] - \left(\frac{1}{110} + \frac{1}{90}\right)\right)$$

$$= 1.713$$

12. Need to prove Proposition (4-7) which claims:
Every non-zero vector **w** in an inner product space *V* can be normalized by setting $\mathbf{u} = \dfrac{\mathbf{w}}{\|\mathbf{w}\|}$.

Proof.

Let **w** be a non-zero arbitrary vector in *V* and $\mathbf{u} = \dfrac{\mathbf{w}}{\|\mathbf{w}\|}$. *What do we need to prove?*

Required to show that $\|\mathbf{u}\| = 1$. We have

$$\|\mathbf{u}\| = \left\|\frac{\mathbf{w}}{\|\mathbf{w}\|}\right\| = \left\|\frac{1}{\|\mathbf{w}\|}\mathbf{w}\right\| \qquad \left[\text{Writing } \frac{x}{y} = \frac{1}{y}x\right]$$

$$= \left|\frac{1}{\|\mathbf{w}\|}\right|\|\mathbf{w}\| \qquad \left[\text{By (4-2) part (c) which is } \|k\mathbf{u}\| = |k|\|\mathbf{u}\|\right]$$

$$= \frac{1}{\|\mathbf{w}\|}\|\mathbf{w}\| = 1 \qquad \left[\frac{1}{\|\mathbf{w}\|} > 0 \text{ therefore } \left|\frac{1}{\|\mathbf{w}\|}\right| = \frac{1}{\|\mathbf{w}\|}\right]$$

Since the vector has length of 1 so every **non-zero** vector can be normalized.

■

13. *Proof.*
Since **u** and **v** are orthogonal so we have

$$\begin{aligned}\|\mathbf{u}-\mathbf{v}\|^2 &= \langle \mathbf{u}-\mathbf{v},\ \mathbf{u}-\mathbf{v}\rangle \\ &= \langle \mathbf{u},\ \mathbf{u}\rangle + \langle \mathbf{u},\ -\mathbf{v}\rangle + \langle -\mathbf{v},\ \mathbf{u}\rangle + \langle -\mathbf{v},\ -\mathbf{v}\rangle \\ &= \|\mathbf{u}\|^2 - \langle \mathbf{u},\ \mathbf{v}\rangle - \langle \mathbf{v},\ \mathbf{u}\rangle + \langle \mathbf{v},\ \mathbf{v}\rangle \\ &= \|\mathbf{u}\|^2 - \underbrace{\langle \mathbf{u},\ \mathbf{v}\rangle}_{=0} - \underbrace{\langle \mathbf{v},\ \mathbf{u}\rangle}_{=0} + \|\mathbf{v}\|^2 \qquad \left[\text{Because } \mathbf{u} \text{ and } \mathbf{v} \text{ are orthogonal}\right] \\ &= \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2\end{aligned}$$

■

14. (a) Vectors **u** and **v** are orthonormal therefore we have $\langle \mathbf{u},\ \mathbf{v}\rangle = 0$, $\|\mathbf{u}\| = 1$ and $\|\mathbf{v}\| = 1$. To find $\|\mathbf{u}+\mathbf{v}\|$ we first determine $\|\mathbf{u}+\mathbf{v}\|^2$ and then take the square root. We can use Pythagoras Theorem (4-5) because **u** and **v** are orthogonal:

$$\|\mathbf{u}+\mathbf{v}\|^2 = \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2 = 1+1 = 2$$

Thus taking the square root of both sides gives $\|\mathbf{u}+\mathbf{v}\| = \sqrt{2}$.

(b) By using the result of question 13 above we have $\|\mathbf{u}-\mathbf{v}\| = \sqrt{2}$.

15. (a) We need to prove $\|\mathbf{u}_1+\mathbf{u}_2+\mathbf{u}_3+\cdots+\mathbf{u}_n\|^2 = \|\mathbf{u}_1\|^2+\|\mathbf{u}_2\|^2+\|\mathbf{u}_3\|^2+\cdots+\|\mathbf{u}_n\|^2$:

How?

By using mathematical induction.

Proof.

Clearly the result holds for $n = 2$ because by theorem (4-5) we have

$$\|\mathbf{u}_1+\mathbf{u}_2\|^2 = \|\mathbf{u}_1\|^2+\|\mathbf{u}_2\|^2$$

Assume the result is true for $n = k$, that is

$$\|\mathbf{u}_1+\mathbf{u}_2+\mathbf{u}_3+\cdots+\mathbf{u}_k\|^2 = \|\mathbf{u}_1\|^2+\|\mathbf{u}_2\|^2+\|\mathbf{u}_3\|^2+\cdots+\|\mathbf{u}_k\|^2 \qquad (*)$$

We need to prove the result for $n = k+1$, that is we are required to prove

$$\|\mathbf{u}_1+\mathbf{u}_2+\mathbf{u}_3+\cdots+\mathbf{u}_k+\mathbf{u}_{k+1}\|^2 = \|\mathbf{u}_1\|^2+\|\mathbf{u}_2\|^2+\|\mathbf{u}_3\|^2+\cdots+\|\mathbf{u}_k\|^2+\|\mathbf{u}_{k+1}\|^2$$

Examining the Left Hand Side we have

$$\begin{aligned}\|\mathbf{u}_1+\mathbf{u}_2+\mathbf{u}_3+\cdots+\mathbf{u}_k+\mathbf{u}_{k+1}\|^2 &= \|(\mathbf{u}_1+\mathbf{u}_2+\mathbf{u}_3+\cdots+\mathbf{u}_k)+\mathbf{u}_{k+1}\|^2 \\ &= \|(\mathbf{u}_1+\mathbf{u}_2+\mathbf{u}_3+\cdots+\mathbf{u}_k)\|^2+\|\mathbf{u}_{k+1}\|^2 \qquad [\text{By (4-5)}] \\ &= \underbrace{\|\mathbf{u}_1\|^2+\|\mathbf{u}_2\|^2+\|\mathbf{u}_3\|^2+\cdots+\|\mathbf{u}_k\|^2}_{\text{By }(*)}+\|\mathbf{u}_{k+1}\|^2\end{aligned}$$

This completes the proof.

■

(b) We are given that $\{\mathbf{f}_1,\ \mathbf{f}_2, \mathbf{f}_3,\ \cdots,\ \mathbf{f}_n\}$ be an orthonormal set of vectors in $C[0,\ \pi]$. Since $\{\mathbf{f}_1,\ \mathbf{f}_2, \mathbf{f}_3,\ \cdots,\ \mathbf{f}_n\}$ is an orthonormal set so the vectors are orthogonal. Applying Pythagoras (4-5) we have

$$\|\mathbf{f}_1+\mathbf{f}_2+\mathbf{f}_3+\cdots+\mathbf{f}_n\|^2 = \|\mathbf{f}_1\|^2+\|\mathbf{f}_2\|^2+\|\mathbf{f}_3\|^2+\cdots+\|\mathbf{f}_n\|^2$$

Each of these vectors are normalized so they have a length of 1. Hence

$$\left\|\mathbf{f}_1+\mathbf{f}_2+\mathbf{f}_3+\cdots+\mathbf{f}_n\right\|^2=\left\|\mathbf{f}_1\right\|^2+\left\|\mathbf{f}_2\right\|^2+\left\|\mathbf{f}_3\right\|^2+\cdots+\left\|\mathbf{f}_n\right\|^2$$
$$=\underbrace{1+1+1+\cdots+1}_{n\ \text{copies}}=n$$

Taking the square root gives $\left\|\mathbf{f}_1+\mathbf{f}_2+\mathbf{f}_3+\cdots+\mathbf{f}_n\right\|=\sqrt{n}$.

16. We need to prove that the vectors $\{\mathbf{u}_1,\ \mathbf{u}_2,\ \mathbf{u}_3,\ \cdots,\ \mathbf{u}_n\}$ are orthogonal $\Leftrightarrow$ $\{k_1\mathbf{u}_1,\ k_2\mathbf{u}_2,\ k_3\mathbf{u}_3,\ \cdots,\ k_n\mathbf{u}_n\}$ are orthogonal.

Proof.

$(\Rightarrow)$ Let $\mathbf{u}_i$ and $\mathbf{u}_j$ be arbitrary vectors where $i\neq j$. Consider the inner product of $\langle k_i\mathbf{u}_i,\ k_j\mathbf{u}_j\rangle$:

$$\langle k_i\mathbf{u}_i,\ k_j\mathbf{u}_j\rangle=k_ik_j\langle\mathbf{u}_i,\ \mathbf{u}_j\rangle \qquad \left[\text{Because by (4-1) (c) } \langle k\mathbf{u},\ \mathbf{v}\rangle=k\langle\mathbf{u},\ \mathbf{v}\rangle\right]$$
$$=k_ik_j(0)=0 \qquad \left[\text{Because } \mathbf{u}_i \text{ and } \mathbf{u}_j \text{ are orthogonal}\right]$$

Hence the arbitrary vectors $k_i\mathbf{u}_i$ and $k_j\mathbf{u}_j$ are orthogonal because their inner product is zero. Since $k_i\mathbf{u}_i$ and $k_j\mathbf{u}_j$ were arbitrary vectors so

$$\{k_1\mathbf{u}_1,\ k_2\mathbf{u}_2,\ k_3\mathbf{u}_3,\ \cdots,\ k_n\mathbf{u}_n\}$$

is an orthogonal set of vectors.

$(\Leftarrow)$. We can assume $\{k_1\mathbf{u}_1,\ k_2\mathbf{u}_2,\ k_3\mathbf{u}_3,\ \cdots,\ k_n\mathbf{u}_n\}$ are a set of orthogonal vectors. Consider two arbitrary vectors in this set $k_i\mathbf{u}_i$ and $k_j\mathbf{u}_j$ where $i\neq j$. We have $\langle k_i\mathbf{u}_i,\ k_j\mathbf{u}_j\rangle=0$. Expanding this

$$\langle k_i\mathbf{u}_i,\ k_j\mathbf{u}_j\rangle=k_ik_j\langle\mathbf{u}_i,\ \mathbf{u}_j\rangle=0$$

We are given that the scalars k are non-zero therefore

$$k_ik_j\langle\mathbf{u}_i,\ \mathbf{u}_j\rangle=0 \quad\Rightarrow\quad \langle\mathbf{u}_i,\ \mathbf{u}_j\rangle=0 \quad\Rightarrow\quad \mathbf{u}_i \text{ and } \mathbf{u}_j \text{ are orthogonal}$$

This completes our proof.

■

If one of the scalars is zero, $k_j=0$ say. We have $k_j\mathbf{u}_j=\mathbf{O}$ which means that

$k_j\mathbf{u}_j$ is orthogonal to all the vectors in $\{k_1\mathbf{u}_1,\ k_2\mathbf{u}_2,\ k_3\mathbf{u}_3,\ \cdots,\ k_n\mathbf{u}_n\}$

However the inner product of this vector $k_j\mathbf{u}_j=\mathbf{O}$ and $k_i\mathbf{u}_i$ is zero, that is

$$\langle k_i\mathbf{u}_i,\ k_j\mathbf{u}_j\rangle=k_ik_j\langle\mathbf{u}_i,\ \mathbf{u}_j\rangle=0 \text{ because } k_j=0$$

But $\langle\mathbf{u}_i,\ \mathbf{u}_j\rangle\neq 0$ which means the vectors $\mathbf{u}_i$ and $\mathbf{u}_j$ are not orthogonal. Hence if one of the scalars is zero then the orthogonal set

$$\{k_1\mathbf{u}_1,\ k_2\mathbf{u}_2,\ k_3\mathbf{u}_3,\ \cdots,\ k_n\mathbf{u}_n\} \not\Rightarrow \{\mathbf{u}_1,\ \mathbf{u}_2,\ \mathbf{u}_3,\ \cdots,\ \mathbf{u}_n\} \text{ are orthogonal}$$

17. We are required to prove $\|\mathbf{u}+\mathbf{v}\|\le\|\mathbf{u}\|+\|\mathbf{v}\|$.

Proof.

$$
\begin{aligned}
\|\mathbf{u}+\mathbf{v}\|^2 &= \langle \mathbf{u}+\mathbf{v},\ \mathbf{u}+\mathbf{v}\rangle \\
&= \langle \mathbf{u},\ \mathbf{u}+\mathbf{v}\rangle + \langle \mathbf{v},\ \mathbf{u}+\mathbf{v}\rangle \\
&= \langle \mathbf{u},\ \mathbf{u}\rangle + \langle \mathbf{u},\ \mathbf{v}\rangle + \langle \mathbf{v},\ \mathbf{u}\rangle + \langle \mathbf{v},\ \mathbf{v}\rangle \\
&= \|\mathbf{u}\|^2 + 2\langle \mathbf{u},\ \mathbf{v}\rangle + \|\mathbf{v}\|^2 \\
&\le \|\mathbf{u}\|^2 + 2\left|\langle \mathbf{u},\ \mathbf{v}\rangle\right| + \|\mathbf{v}\|^2 && \left[\text{Because } x \le |x|\right] \\
&\le \|\mathbf{u}\|^2 + 2\|\mathbf{u}\|\|\mathbf{v}\| + \|\mathbf{v}\|^2 && \left[\text{Using CSI (4-3) } \left|\langle \mathbf{u},\ \mathbf{v}\rangle\right| \le \|\mathbf{u}\|\|\mathbf{v}\|\ \right] \\
&= \left(\|\mathbf{u}\| + \|\mathbf{v}\|\right)^2
\end{aligned}
$$

We have

$$
\begin{aligned}
\|\mathbf{u}+\mathbf{v}\|^2 &\le \left(\|\mathbf{u}\| + \|\mathbf{v}\|\right)^2 \\
\|\mathbf{u}+\mathbf{v}\| &\le \|\mathbf{u}\| + \|\mathbf{v}\| && \left[\text{Taking Square Roots}\right]
\end{aligned}
$$

This was our required inequality.

■