Complete Solutions to Exercises 5.6

1\. (a) *What does* $(S\circ T)\left(\begin{bmatrix}1\\2\end{bmatrix}\right)$ *mean?*

By using definition (5-11) we have

$$(S\circ T)\left(\begin{bmatrix}1\\2\end{bmatrix}\right)=S\left(T\left(\begin{bmatrix}1\\2\end{bmatrix}\right)\right)$$

$$=S\left(\begin{bmatrix}2\\1\end{bmatrix}\right)\qquad\left[\text{Because } T\left(\begin{bmatrix}x\\y\end{bmatrix}\right)=\begin{pmatrix}y\\x\end{pmatrix}\right]$$

$$=S\left(\begin{bmatrix}2\\1\end{bmatrix}\right)=\begin{pmatrix}1\\0\end{pmatrix}\qquad\left[\text{Because } S\left(\begin{bmatrix}x\\y\end{bmatrix}\right)=\begin{pmatrix}y\\0\end{pmatrix}\right]$$

Similarly we have

$$(T\circ S)\left(\begin{bmatrix}1\\2\end{bmatrix}\right)=T\left(S\left(\begin{bmatrix}1\\2\end{bmatrix}\right)\right)$$

$$=T\left(\begin{bmatrix}2\\0\end{bmatrix}\right)\qquad\left[\text{Because } S\left(\begin{bmatrix}x\\y\end{bmatrix}\right)=\begin{pmatrix}y\\0\end{pmatrix}\right]$$

$$=T\left(\begin{bmatrix}2\\0\end{bmatrix}\right)=\begin{pmatrix}0\\2\end{pmatrix}\qquad\left[\text{Because } T\left(\begin{bmatrix}x\\y\end{bmatrix}\right)=\begin{pmatrix}y\\x\end{pmatrix}\right]$$

Note that $(S\circ T)\left(\begin{bmatrix}1\\2\end{bmatrix}\right)=\begin{pmatrix}1\\0\end{pmatrix}$ does **not equal** $(T\circ S)\left(\begin{bmatrix}1\\2\end{bmatrix}\right)=\begin{pmatrix}0\\2\end{pmatrix}$.

(b) By using definition (5-11) we have

$$(S\circ T)\left(\begin{bmatrix}1\\1\end{bmatrix}\right)=S\left(T\left(\begin{bmatrix}1\\1\end{bmatrix}\right)\right)$$

$$=S\left(\begin{bmatrix}1\\1\end{bmatrix}\right)\qquad\left[\text{Because } T\left(\begin{bmatrix}x\\y\end{bmatrix}\right)=\begin{pmatrix}y\\x\end{pmatrix}\right]$$

$$=S\left(\begin{bmatrix}1\\1\end{bmatrix}\right)=\begin{pmatrix}1\\0\end{pmatrix}\qquad\left[\text{Because } S\left(\begin{bmatrix}x\\y\end{bmatrix}\right)=\begin{pmatrix}y\\0\end{pmatrix}\right]$$

Similarly we have

$$(T\circ S)\left(\begin{bmatrix}1\\1\end{bmatrix}\right)=T\left(S\left(\begin{bmatrix}1\\1\end{bmatrix}\right)\right)$$

$$=T\left(\begin{bmatrix}1\\0\end{bmatrix}\right)\qquad\left[\text{Because } S\left(\begin{bmatrix}x\\y\end{bmatrix}\right)=\begin{pmatrix}y\\0\end{pmatrix}\right]$$

$$=T\left(\begin{bmatrix}1\\0\end{bmatrix}\right)=\begin{pmatrix}0\\1\end{pmatrix}\qquad\left[\text{Because } T\left(\begin{bmatrix}x\\y\end{bmatrix}\right)=\begin{pmatrix}y\\x\end{pmatrix}\right]$$

Again $(S\circ T)\neq(T\circ S)$

(5-11) $\qquad (S\circ T)(\mathbf{u})=S(T(\mathbf{u}))$

2. We are given that

$$T\left(ax^2+bx+c\right)=\left(ax^2+bx+c\right)' \text{ and } S\left(ax^2+bx+c\right)=ax^2+bx$$

(a) We need to find $(T\circ S)(\mathbf{p})$ where $\mathbf{p}=ax^2+bx+c$. *What does $T\circ S$ mean?*

$$\begin{aligned}(T\circ S)\left(ax^2+bx+c\right)&=T\left(S\left(ax^2+bx+c\right)\right)\\&=T\left(ax^2+bx\right)\\&=2ax+b \qquad \left[\text{Differentiating}\right]\end{aligned}$$

(b) Similarly finding the transformation the other way we have

$$\begin{aligned}(S\circ T)\left(ax^2+bx+c\right)&=S\left(T\left(ax^2+bx+c\right)\right)\\&=S(2ax+b) \qquad \left[\text{Differentiating}\right]\\&=2ax\end{aligned}$$

(c) We need to find $T\circ T$:

$$\begin{aligned}(T\circ T)\left(ax^2+bx+c\right)&=T\left(T\left(ax^2+bx+c\right)\right)\\&=T(2ax+b) \qquad \left[\text{Differentiating}\right]\\&=2a \qquad \left[\text{Differentiating}\right]\end{aligned}$$

(d) For $S\circ S$ we have

$$\begin{aligned}(S\circ S)\left(ax^2+bx+c\right)&=S\left(S\left(ax^2+bx+c\right)\right)\\&=S\left(ax^2+bx\right)\\&=ax^2+bx\end{aligned}$$

3. *How do we find the standard matrix for the given transformation T?*
Read off the coefficients of x and y. Let $\mathbf{A}$ be this matrix then

$$\mathbf{A}=\begin{pmatrix}1 & 2\\ 1 & -3\end{pmatrix} \qquad \left[\text{Because } T\left(\begin{bmatrix}x\\ y\end{bmatrix}\right)=\begin{pmatrix}x+2y\\ x-3y\end{pmatrix}\right]$$

Let $\mathbf{B}$ be the standard matrix for the given other linear transformation S:

$$\mathbf{B}=\begin{pmatrix}2 & 1\\ -1 & -1\end{pmatrix} \qquad \left[\text{Because } S\left(\begin{bmatrix}x\\ y\end{bmatrix}\right)=\begin{pmatrix}2x+y\\ -x-y\end{pmatrix}\right]$$

By Proposition (5-20) we have

$$\begin{aligned}(S\circ T)(\mathbf{u})&=\mathbf{BA}[\mathbf{u}]\\&=\begin{pmatrix}2 & 1\\ -1 & -1\end{pmatrix}\begin{pmatrix}1 & 2\\ 1 & -3\end{pmatrix}\mathbf{u}=\begin{pmatrix}3 & 1\\ -2 & 1\end{pmatrix}\mathbf{u}\end{aligned}$$

The standard matrix for $S\circ T$ is $\begin{pmatrix}3 & 1\\ -2 & 1\end{pmatrix}$.

$$\begin{aligned}(T\circ S)(\mathbf{u})&=\mathbf{AB}[\mathbf{u}]\\&=\begin{pmatrix}1 & 2\\ 1 & -3\end{pmatrix}\begin{pmatrix}2 & 1\\ -1 & -1\end{pmatrix}\mathbf{u}=\begin{pmatrix}0 & -1\\ 5 & 4\end{pmatrix}\mathbf{u}\end{aligned}$$

The standard matrix for $T \circ S$ is $\begin{pmatrix} 0 & -1 \\ 5 & 4 \end{pmatrix}$.

(a) To find $(S \circ T)\left(\begin{bmatrix} 1 \\ 5 \end{bmatrix}\right)$ we substitute $\mathbf{u} = \begin{pmatrix} 1 \\ 5 \end{pmatrix}$ into the above $(S \circ T)(\mathbf{u}) = \begin{pmatrix} 3 & 1 \\ -2 & 1 \end{pmatrix}\mathbf{u}$:

$$(S \circ T)\left(\begin{bmatrix} 1 \\ 5 \end{bmatrix}\right) = \begin{pmatrix} 3 & 1 \\ -2 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 5 \end{pmatrix} = \begin{pmatrix} 8 \\ 3 \end{pmatrix}$$

(b) Also in the above manner we have

$$(T \circ S)\left(\begin{bmatrix} 1 \\ 5 \end{bmatrix}\right) = \begin{pmatrix} 0 & -1 \\ 5 & 4 \end{pmatrix}\begin{pmatrix} 1 \\ 5 \end{pmatrix} = \begin{pmatrix} -5 \\ 25 \end{pmatrix}$$

(c) The transformation matrix for T is $\mathbf{A} = \begin{pmatrix} 1 & 2 \\ 1 & -3 \end{pmatrix}$ therefore

$$\begin{aligned}(T \circ T)(\mathbf{u}) &= \mathbf{AA}[\mathbf{u}] \\ &= \begin{pmatrix} 1 & 2 \\ 1 & -3 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 1 & -3 \end{pmatrix}\mathbf{u} = \begin{pmatrix} 3 & -4 \\ -2 & 11 \end{pmatrix}\mathbf{u}\end{aligned}$$

We can find $(T \circ T)\left(\begin{bmatrix} 1 \\ 5 \end{bmatrix}\right)$ by substituting $\mathbf{u} = \begin{pmatrix} 1 \\ 5 \end{pmatrix}$ into the above $(T \circ T)(\mathbf{u}) = \begin{pmatrix} 3 & -4 \\ -2 & 11 \end{pmatrix}\mathbf{u}$:

$$(T \circ T)\left(\begin{bmatrix} 1 \\ 5 \end{bmatrix}\right) = \begin{pmatrix} 3 & -4 \\ -2 & 11 \end{pmatrix}\begin{pmatrix} 1 \\ 5 \end{pmatrix} = \begin{pmatrix} -17 \\ 53 \end{pmatrix}$$

(d) Similarly we can find $(S \circ S)\left(\begin{bmatrix} 1 \\ 5 \end{bmatrix}\right)$:

$$\begin{aligned}(S \circ S)(\mathbf{u}) &= \mathbf{BB}[\mathbf{u}] \\ &= \begin{pmatrix} 2 & 1 \\ -1 & -1 \end{pmatrix}\begin{pmatrix} 2 & 1 \\ -1 & -1 \end{pmatrix}\mathbf{u} = \begin{pmatrix} 3 & 1 \\ -1 & 0 \end{pmatrix}\mathbf{u}\end{aligned}$$

Substituting $\mathbf{u} = \begin{pmatrix} 1 \\ 5 \end{pmatrix}$ into the above $(S \circ S)(\mathbf{u}) = \begin{pmatrix} 3 & 1 \\ -1 & 0 \end{pmatrix}\mathbf{u}$ gives

$$(S \circ S)(\mathbf{u}) = \begin{pmatrix} 3 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 5 \end{pmatrix} = \begin{pmatrix} 8 \\ -1 \end{pmatrix}$$

4. The standard matrix $\mathbf{A}$ for $T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{pmatrix} 2x - y + 10z \\ 3x - 5y + 6z \\ x + 3y - 9z \end{pmatrix}$ is

$$\mathbf{A} = \begin{pmatrix} 2 & -1 & 10 \\ 3 & -5 & 6 \\ 1 & 3 & -9 \end{pmatrix}$$

Let $\mathbf{B}$ be the standard matrix for the other given linear transformation S:

$$\mathbf{B}=\begin{pmatrix}-1 & -1 & -1\\ 1 & 1 & 2\\ 1 & 2 & 3\end{pmatrix} \qquad \left[\text{Because } S\left(\begin{bmatrix}x\\ y\\ z\end{bmatrix}\right)=\begin{pmatrix}-x-y-z\\ x+y+2z\\ x+2y+3z\end{pmatrix}\right]$$

By Proposition (5-20) we have

$$\begin{aligned}(S\circ T)(\mathbf{u}) &= \mathbf{BA}[\mathbf{u}]\\ &= \begin{pmatrix}-1 & -1 & -1\\ 1 & 1 & 2\\ 1 & 2 & 3\end{pmatrix}\begin{pmatrix}2 & -1 & 10\\ 3 & -5 & 6\\ 1 & 3 & -9\end{pmatrix}\mathbf{u}=\begin{pmatrix}-6 & 3 & -7\\ 7 & 0 & -2\\ 11 & -2 & -5\end{pmatrix}\mathbf{u}\end{aligned}$$

The standard matrix for $S\circ T$ is $\begin{pmatrix}-6 & 3 & -7\\ 7 & 0 & -2\\ 11 & -2 & -5\end{pmatrix}$.

We also need to find $(T\circ S)$ which means that the matrix multiplication **AB**:

$$\begin{aligned}(T\circ S)(\mathbf{u}) &= \mathbf{AB}[\mathbf{u}]\\ &= \begin{pmatrix}2 & -1 & 10\\ 3 & -5 & 6\\ 1 & 3 & -9\end{pmatrix}\begin{pmatrix}-1 & -1 & -1\\ 1 & 1 & 2\\ 1 & 2 & 3\end{pmatrix}\mathbf{u}=\begin{pmatrix}7 & 17 & 26\\ -2 & 4 & 5\\ -7 & -16 & -22\end{pmatrix}\mathbf{u}\end{aligned}$$

The standard matrix for $T\circ S$ is $\begin{pmatrix}7 & 17 & 26\\ -2 & 4 & 5\\ -7 & -16 & -22\end{pmatrix}$.

(a) To find $(S\circ T)\left(\begin{bmatrix}1\\ 2\\ 3\end{bmatrix}\right)$ we need to substitute $\mathbf{u}=\begin{pmatrix}1\\ 2\\ 3\end{pmatrix}$ into the above

$$(S\circ T)(\mathbf{u})=\begin{pmatrix}-6 & 3 & -7\\ 7 & 0 & -2\\ 11 & -2 & -5\end{pmatrix}\begin{pmatrix}1\\ 2\\ 3\end{pmatrix}=\begin{pmatrix}-21\\ 1\\ -8\end{pmatrix}$$

(b) To find $(T\circ S)\left(\begin{bmatrix}1\\ 2\\ 3\end{bmatrix}\right)$ we substitute $\mathbf{u}=\begin{pmatrix}1\\ 2\\ 3\end{pmatrix}$ into the above:

$$(T\circ S)(\mathbf{u})=\begin{pmatrix}7 & 17 & 26\\ -2 & 4 & 5\\ -7 & -16 & -22\end{pmatrix}\begin{pmatrix}1\\ 2\\ 3\end{pmatrix}=\begin{pmatrix}119\\ 21\\ -105\end{pmatrix}$$

(c) Similarly for $(T\circ T)\left(\begin{bmatrix}1\\ 2\\ 3\end{bmatrix}\right)$ we have

$$\left(T\circ T\right)\left(\begin{bmatrix}1\\2\\3\end{bmatrix}\right)=\mathbf{AA}\begin{pmatrix}1\\2\\3\end{pmatrix}$$
$$=\begin{pmatrix}2&-1&10\\3&-5&6\\1&3&-9\end{pmatrix}\begin{pmatrix}2&-1&10\\3&-5&6\\1&3&-9\end{pmatrix}\begin{pmatrix}1\\2\\3\end{pmatrix}$$
$$=\begin{pmatrix}11&33&-76\\-3&40&-54\\2&-43&109\end{pmatrix}\begin{pmatrix}1\\2\\3\end{pmatrix}=\begin{pmatrix}-151\\-85\\243\end{pmatrix}$$

(d) Repeating the above procedure we have

$$\left(S\circ S\right)\left(\begin{bmatrix}1\\2\\3\end{bmatrix}\right)=\mathbf{BB}\begin{pmatrix}1\\2\\3\end{pmatrix}$$
$$=\begin{pmatrix}-1&-1&-1\\1&1&2\\1&2&3\end{pmatrix}\begin{pmatrix}-1&-1&-1\\1&1&2\\1&2&3\end{pmatrix}\begin{pmatrix}1\\2\\3\end{pmatrix}$$
$$=\begin{pmatrix}-1&-2&-4\\2&4&7\\4&7&12\end{pmatrix}\begin{pmatrix}1\\2\\3\end{pmatrix}=\begin{pmatrix}-17\\31\\54\end{pmatrix}$$

5. (a) We are given that $T\left(ae^x+bxe^x+cx^2e^x\right)=\left(ae^x+bxe^x+cx^2e^x\right)'$ and

$$S\left(ae^x+bxe^x+cx^2e^x\right)=ce^x+bxe^x+ax^2e^x$$

We need to find a matrix which represents the composite transformation $S\circ T$.
The matrix **A** which represents the transformation T is the same as the one in Example 39 because we have the same transformation and vector space V. Thus

$$\mathbf{A}=\begin{pmatrix}1&1&0\\0&1&2\\0&0&1\end{pmatrix}$$

We need to find the matrix which represents the other transformation

$$S\left(ae^x+bxe^x+cx^2e^x\right)=ce^x+bxe^x+ax^2e^x$$

We first find the transformation of each vector in $\left\{e^x,\ xe^x,\ x^2e^x\right\}$ under S:

$$S\left(e^x\right)=x^2e^x=0\left(e^x\right)+0\left(xe^x\right)+1\left(x^2e^x\right)$$
$$S\left(xe^x\right)=xe^x=0\left(e^x\right)+1\left(xe^x\right)+0\left(x^2e^x\right)$$
$$S\left(x^2e^x\right)=e^x=1\left(e^x\right)+0\left(xe^x\right)+0\left(x^2e^x\right)$$

Let **B** be the matrix for this transformation S. Therefore

$$\mathbf{B}=\begin{pmatrix}S\left(e^x\right) & S\left(xe^x\right) & S\left(x^2e^x\right)\end{pmatrix}=\begin{pmatrix}0&0&1\\0&1&0\\1&0&0\end{pmatrix}$$

(i) The matrix for the composite transformation $S\circ T$ is the matrix multiplication **BA**:

$$\mathbf{BA}=\begin{pmatrix}0&0&1\\0&1&0\\1&0&0\end{pmatrix}\begin{pmatrix}1&1&0\\0&1&2\\0&0&1\end{pmatrix}=\begin{pmatrix}0&0&1\\0&1&2\\1&1&0\end{pmatrix}$$

(ii) The matrix representation $T\circ S$ is the matrix multiplication **AB** which is:

$$\mathbf{AB}=\begin{pmatrix}1&1&0\\0&1&2\\0&0&1\end{pmatrix}\begin{pmatrix}0&0&1\\0&1&0\\1&0&0\end{pmatrix}=\begin{pmatrix}0&1&1\\2&1&0\\1&0&0\end{pmatrix}$$

(iii) For $T\circ T$ the matrix representing this transformation is $\mathbf{AA}=\mathbf{A}^2$:

$$\mathbf{AA}=\begin{pmatrix}1&1&0\\0&1&2\\0&0&1\end{pmatrix}\begin{pmatrix}1&1&0\\0&1&2\\0&0&1\end{pmatrix}=\begin{pmatrix}1&2&2\\0&1&4\\0&0&1\end{pmatrix}$$

(iv) For $S\circ S$ the matrix representation is **BB**:

$$\mathbf{BB}=\begin{pmatrix}0&0&1\\0&1&0\\1&0&0\end{pmatrix}\begin{pmatrix}0&0&1\\0&1&0\\1&0&0\end{pmatrix}=\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}=\mathbf{I}$$

Note that the matrix representing $S\circ S$ is the identity matrix **I**. Thus $S\circ S$ is the identity transformation. (You can check this by carrying out the operation $S\circ S$ for a general vector in V such as $ae^x+bxe^x+cx^2e^x$). Since $S\circ S=I$ so

$$S^{-1}(\mathbf{p})=S(\mathbf{p})=ce^x+bxe^x+ax^2e^x$$

(b) The matrix representing T is $\mathbf{A}=\begin{pmatrix}1&1&0\\0&1&2\\0&0&1\end{pmatrix}$ which was determined in Example 39. The inverse of this matrix is $\mathbf{A}^{-1}=\begin{pmatrix}1&-1&2\\0&1&-2\\0&0&1\end{pmatrix}$. By (5-21):

Proposition (5-21). Let $T:\ V\to W$ be a linear transform of finite dimensional spaces V and W. Then T is **invertible** $\Leftrightarrow$ the matrix **A** which represents the transformation T is invertible.

We conclude that the transformation T is invertible.

From calculus we know that integration is the inverse of differentiation so the matrix $\mathbf{A}^{-1}$ represents integration. The coordinate vector of x^2e^x is

$$x^2e^x=0\left(e^x\right)+0\left(xe^x\right)+1\left(x^2e^x\right)\text{ gives }\begin{pmatrix}0\\0\\1\end{pmatrix}$$

Hence

$$\mathbf{A}^{-1}\begin{pmatrix}0\\0\\1\end{pmatrix}=\begin{pmatrix}1&-1&2\\0&1&-2\\0&0&1\end{pmatrix}\begin{pmatrix}0\\0\\1\end{pmatrix}=\begin{pmatrix}2\\-2\\1\end{pmatrix}\quad\Rightarrow\quad 2\left(e^x\right)-2\left(xe^x\right)+1\left(x^2e^x\right)$$

Therefore $\int x^2e^x\mathrm{d}x=2e^x-2xe^x+x^2e^x+C$. Remember we need to add the constant of integration.

6. We have

$$T\left(ax^2+bx+c\right)=-ax^2-bx-c \text{ and } S\left(ax^2+bx+c\right)=-bx-c$$

(a) We need to find the matrix for $S\circ T$. First we find a matrix which represents the transformation $T\left(ax^2+bx+c\right)=-ax^2-bx-c$. *How?*

By finding the transformation of each basis vector in $B=\{1,\ x,\ x^2\}$:

$$T(1)=-1=-1(1)+0(x)+0\left(x^2\right)$$
$$T(x)=-x=0(1)-1(x)+0\left(x^2\right)$$
$$T\left(x^2\right)=-x^2=0(1)+0(x)-1\left(x^2\right)$$

What is the matrix **A** *which represents the above transformation T?*
We have

$$\mathbf{A}=\left(T(1)\quad T(x)\quad T\left(x^2\right)\right)=\begin{pmatrix}-1&0&0\\0&-1&0\\0&0&-1\end{pmatrix}=-1\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}=-\mathbf{I}$$

Similarly for $S\left(ax^2+bx+c\right)=-bx-c$ we have

$$S(1)=S\left(0x^2+0x+1\right)=-1=-1(1)+0(x)+0\left(x^2\right)$$
$$S(x)=S\left(0x^2+x+0\right)=-x=0(1)-1(x)+0\left(x^2\right)$$
$$S\left(x^2\right)=S\left(1x^2+0x+0\right)=0=0(1)+0(x)+0\left(x^2\right)$$

Thus the matrix **B** which represents the transformation S is

$$\mathbf{B}=\begin{pmatrix}-1&0&0\\0&-1&0\\0&0&0\end{pmatrix}$$

The matrix representing $S\circ T$ is given by the matrix multiplication:

$$\mathbf{BA}=\mathbf{B}\left(-\mathbf{I}\right)=-\mathbf{B}=\begin{pmatrix}1&0&0\\0&1&0\\0&0&0\end{pmatrix}$$

(b) For $T\circ S$ carrying out the matrix multiplication **AB** we have:

$$\mathbf{AB}=\left(-\mathbf{I}\right)\mathbf{B}=-\mathbf{B}=\begin{pmatrix}1&0&0\\0&1&0\\0&0&0\end{pmatrix}$$

(c) For $T\circ T$ we have the matrix **AA**:

$$\mathbf{A}^2 = (-\mathbf{I})(-\mathbf{I}) = \mathbf{I}^2 = \mathbf{I} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

(d) For $S \circ S$ we have the matrix $\mathbf{B}^2$:

$$\mathbf{B}^2 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

To find $(S \circ T)(1+2x+3x^2)$ we use $(S \circ T)(\mathbf{u}) = \mathbf{BA}[\mathbf{u}]$ with $\mathbf{u} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ because the coefficients of constants, x and x^2 is 1, 2 and 3 respectively. Thus

$$\mathbf{BA}[\mathbf{u}] = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} \qquad \left[\text{Because } \mathbf{BA} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}\right]$$

The Right Hand column vector gives the coefficients of constants, x and x^2 therefore we have

$$(S \circ T)(1+2x+3x^2) = 1+2x$$

Similarly for $(T \circ S)(1+2x+3x^2)$ we have

$$\mathbf{AB}[\mathbf{u}] = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} \qquad \left[\text{Because } \mathbf{AB} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}\right]$$

Hence $(T \circ S)(1+2x+3x^2) = 1+2x$.

For $(T \circ T)(1+2x+3x^2)$ we have

$$\mathbf{AA}[\mathbf{u}] = \mathbf{Iu} = \mathbf{u} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \qquad [\text{Because } \mathbf{AA} = \mathbf{I}]$$

Hence $(T \circ T)(1+2x+3x^2) = 1+2x+3x^2$.

Similarly for $(S \circ S)(1+2x+3x^2)$ we have

$$\mathbf{B}^2[\mathbf{u}] = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} \qquad \left[\text{Because } \mathbf{B}^2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}\right]$$

We have $(S \circ S)(1+2x+3x^2) = 1+2x$.

7. Let **A** be the matrix representing the given transformation T. Then by Proposition (5-22) the inverse transformation is given by the inverse matrix $\mathbf{A}^{-1}$.
Reading off the coefficients gives the standard matrix:

$$\mathbf{A}=\begin{pmatrix}1 & -1\\ 1 & 1\end{pmatrix} \qquad \left[\text{Because } \; T\left(\begin{bmatrix}x\\ y\end{bmatrix}\right)=\begin{pmatrix}x-y\\ x+y\end{pmatrix}\right]$$

The inverse of this matrix is $\mathbf{A}^{-1}=\frac{1}{2}\begin{pmatrix}1 & 1\\ -1 & 1\end{pmatrix}$. See Example 40 in main text.

8. Similar to question 7 but we have a 3 by 3 matrix. Let **A** be the standard matrix which represents the given transformation:

$$\mathbf{A}=\begin{pmatrix}1 & 1 & 1\\ 1 & 1 & -1\\ 1 & -1 & -1\end{pmatrix} \qquad \left[\text{Because } \; T\left(\begin{bmatrix}x\\ y\\ z\end{bmatrix}\right)=\begin{pmatrix}x+y+z\\ x+y-z\\ x-y-z\end{pmatrix}\right]$$

By MATLAB we can find the inverse of this matrix by the following commands:

```
>> A=[1 1 1; 1 1 -1; 1 -1 -1]
A =
    1     1     1
    1     1    -1
    1    -1    -1
>> inv(A)

ans =

   0.5000         0    0.5000
        0    0.5000   -0.5000
   0.5000   -0.5000         0
```

Thus

$$\begin{array}{cc} & \begin{matrix}x & y & z\end{matrix}\\ \mathbf{A}^{-1}=\frac{1}{2} & \begin{pmatrix}1 & 0 & 1\\ 0 & 1 & -1\\ 1 & -1 & 0\end{pmatrix}\end{array}$$

$$T^{-1}\left(\begin{bmatrix}x\\ y\\ z\end{bmatrix}\right)=\frac{1}{2}\begin{pmatrix}x+z\\ y-z\\ x-y\end{pmatrix}$$

9. We are given $T(\mathbf{p})=(\mathbf{p}x)'$ where $\mathbf{p}=ax^3+bx^2+cx+d$ and $B=\{x^3,\ x^2,\ x,\ 1\}$ be a basis. *What do we need to find first?*

The matrix which represents the given transformation T. *How can we locate this matrix?*

By finding the transformation of each basis vector in $B=\{x^3,\ x^2,\ x,\ 1\}$:

$$T\left(x^3\right)=\left(x^3x\right)'=\left(x^4\right)'=4x^3$$

$$T\left(x^2\right)=\left(x^2x\right)'=\left(x^3\right)'=3x^2$$

$$T(x)=(xx)'=\left(x^2\right)'=2x$$

$$T(1)=(1x)'=(x)'=1$$

We need to write these in terms of the basis vectors $B=\{x^3,\ x^2,\ x,\ 1\}$:

$$T(x^3)=4x^3=4x^3+0(x^2)+0(x)+0(1)$$
$$T(x^2)=3x^2=0(x^3)+3(x^2)+0(x)+0(1)$$
$$T(x)=2x=0(x^3)+0(x^2)+2(x)+0(1)$$
$$T(1)=1=0(x^3)+0(x^2)+0(x)+1(1)$$

What is the matrix **A** *which represents the given transformation T equal to?*

$$\mathbf{A}=\begin{pmatrix}T(x^3) & T(x^2) & T(x) & T(1)\end{pmatrix}=\begin{pmatrix}4&0&0&0\\0&3&0&0\\0&0&2&0\\0&0&0&1\end{pmatrix}$$

To find the inverse transformation we need to determine $\mathbf{A}^{-1}$. By using row operations:

$$\mathbf{A}^{-1}=\begin{pmatrix}1/4&0&0&0\\0&1/3&0&0\\0&0&1/2&0\\0&0&0&1\end{pmatrix}$$

With $\mathbf{p}=ax^3+bx^2+cx+d$ we can evaluate the coefficients of x^3, x^2, x and constants of the inverse transformation T^{-1} as follows:

$$\begin{pmatrix}1/4&0&0&0\\0&1/3&0&0\\0&0&1/2&0\\0&0&0&1\end{pmatrix}\begin{pmatrix}a\\b\\c\\d\end{pmatrix}=\begin{pmatrix}a/4\\b/3\\c/2\\d\end{pmatrix}$$

The entries in the Right Hand vector are the coefficients of x^3, x^2, x and constants.
This means that the inverse transformation is given by

$$T^{-1}(\mathbf{p})=T^{-1}\left(ax^3+bx^2+cx+d\right)=\frac{a}{4}x^3+\frac{b}{3}x^2+\frac{c}{2}x+d$$

10. We are given $T\left(\begin{bmatrix}x\\y\\z\end{bmatrix}\right)=\begin{pmatrix}x+y+z\\2x+2y+z\\y+z\end{pmatrix}$ and we need to find the inverse transformation T^{-1}. *How?*

We first find the standard matrix for T and then take the inverse of this matrix.
Reading off the coefficients of x, y and z in the above we have

$$\mathbf{A}=\begin{pmatrix}1&1&1\\2&2&1\\0&1&1\end{pmatrix}\qquad\left[\text{Because } T\left(\begin{bmatrix}x\\y\\z\end{bmatrix}\right)=\begin{pmatrix}x+y+z\\2x+2y+z\\y+z\end{pmatrix}\right]$$

By MATLAB the inverse of this matrix is

$$\mathbf{A}^{-1}=\begin{pmatrix}1&0&-1\\2&1&1\\2&-1&0\end{pmatrix}$$

Thus

$$T^{-1}\left(\begin{bmatrix}x\\y\\z\end{bmatrix}\right)=\begin{pmatrix}x-z\\2x+y+z\\2x-y\end{pmatrix}\qquad\left[\text{Because }\begin{array}{c}\begin{matrix}x & y & z\end{matrix}\\\begin{pmatrix}1&0&-1\\2&1&1\\2&-1&0\end{pmatrix}\end{array}\right]$$

11. (a) *Is the transformation* $T\left(\begin{bmatrix}x\\y\\z\end{bmatrix}\right)=\begin{pmatrix}x-z\\x+y+z\end{pmatrix}$ *invertible?*

No because the standard matrix **A** for the given linear transformation T is not a square matrix

$$\mathbf{A}=\begin{pmatrix}1&0&-1\\1&1&1\end{pmatrix}\qquad\left[\text{Because }T\left(\begin{bmatrix}x\\y\\z\end{bmatrix}\right)=\begin{pmatrix}x-z\\x+y+z\end{pmatrix}\right]$$

The inverse of a non-square matrix does **not** exist therefore the given linear transformation is **not** invertible.

(b) *Is the transformation* $T\left(\begin{bmatrix}x\\y\\z\end{bmatrix}\right)=\begin{pmatrix}0\\x+y+z\\x-y-z\end{pmatrix}$ *invertible?*

No because the standard matrix **A** for T is

$$\mathbf{A}=\begin{pmatrix}0&0&0\\1&1&1\\1&-1&-1\end{pmatrix}\qquad\left[\text{Because }T\left(\begin{bmatrix}x\\y\\z\end{bmatrix}\right)=\begin{pmatrix}0\\x+y+z\\x-y-z\end{pmatrix}\right]$$

and this matrix has a row of zeros which means it does **not** have an inverse. Hence the given transformation is **not** invertible.

(c) Since the given transformation $T(\mathbf{p})=\mathbf{p}$ is the identity transformation therefore the matrix which represents T is the identity and we know from our earlier work that $\mathbf{I}^{-1}=\mathbf{I}$. Thus the given transformation T is invertible and

$$T^{-1}(\mathbf{p})=\mathbf{p}\text{ where }\mathbf{p}=ax^3+bx^2+cx+d$$

12. (a) We need to find what $Q\circ Q$ does on the vector **x**:

$$(Q\circ Q)(\mathbf{x})=\begin{pmatrix}1&0\\0&-1\end{pmatrix}\begin{pmatrix}1&0\\0&-1\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}$$
$$=\begin{pmatrix}1&0\\0&1\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}x\\y\end{pmatrix}=\mathbf{x}$$

(b) Required to prove that $\begin{pmatrix}1&0\\0&-1\end{pmatrix}^{2n+1}=\begin{pmatrix}1&0\\0&-1\end{pmatrix}$ where n is a natural number. *How?*

By using mathematical induction:

<u>Step 1</u>: Check the result for a base case such as $n=1$.

$$\begin{pmatrix}1 & 0\\0 & -1\end{pmatrix}^{2+1}=\begin{pmatrix}1 & 0\\0 & -1\end{pmatrix}^{3}=\begin{pmatrix}1 & 0\\0 & -1\end{pmatrix}^{2}\begin{pmatrix}1 & 0\\0 & -1\end{pmatrix}=\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}\begin{pmatrix}1 & 0\\0 & -1\end{pmatrix}=\mathbf{Rf}_x$$

<u>Step 2</u>: Assume the result is true for $n=k$:

$$\begin{pmatrix}1 & 0\\0 & -1\end{pmatrix}^{2k+1}=\begin{pmatrix}1 & 0\\0 & -1\end{pmatrix} \qquad (\dagger)$$

<u>Step 3</u>: Required to show that $\begin{pmatrix}1 & 0\\0 & -1\end{pmatrix}^{2(k+1)+1}=\begin{pmatrix}1 & 0\\0 & -1\end{pmatrix}$.

$$\begin{pmatrix}1 & 0\\0 & -1\end{pmatrix}^{2(k+1)+1}=\begin{pmatrix}1 & 0\\0 & -1\end{pmatrix}^{2k+1}\begin{pmatrix}1 & 0\\0 & -1\end{pmatrix}^{2}$$

$$\underset{\text{By }(\dagger)}{=}\begin{pmatrix}1 & 0\\0 & -1\end{pmatrix}\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}=\begin{pmatrix}1 & 0\\0 & -1\end{pmatrix}=\mathbf{Rf}_x$$

Hence $(\mathbf{Rf}_x)^{\text{odd number}}=\mathbf{Rf}_x$ which means that applying the linear operator Q an *odd* number of times reflects the vector in the x – axis.

(c) We need to find the composite transformation $T\circ T$:

$$(T\circ T)(\mathbf{x})=\begin{pmatrix}\cos(\theta) & -\sin(\theta)\\ \sin(\theta) & \cos(\theta)\end{pmatrix}\begin{pmatrix}\cos(\theta) & -\sin(\theta)\\ \sin(\theta) & \cos(\theta)\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}$$

Evaluating the matrix multiplication

$$\begin{pmatrix}\cos(\theta) & -\sin(\theta)\\ \sin(\theta) & \cos(\theta)\end{pmatrix}\begin{pmatrix}\cos(\theta) & -\sin(\theta)\\ \sin(\theta) & \cos(\theta)\end{pmatrix}$$

$$=\begin{pmatrix}\cos^2(\theta)-\sin^2(\theta) & -\cos(\theta)\sin(\theta)-\sin(\theta)\cos(\theta)\\ \sin(\theta)\cos(\theta)+\cos(\theta)\sin(\theta) & -\sin^2(\theta)+\cos^2(\theta)\end{pmatrix}$$

$$\underset{\text{By Trig identities}}{=}\begin{pmatrix}\cos(2\theta) & -\sin(2\theta)\\ \sin(2\theta) & \cos(2\theta)\end{pmatrix}$$

Hence the composite transform $T\circ T$ rotates a vector by an angle of 2θ.

13. (a) We have the following values for *x*:

$$\mathbf{x}_1=T(\mathbf{x}_0)=\mathbf{A}\mathbf{x}_0=\begin{pmatrix}1/3 & 2/3\\2/3 & 1/3\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix}=\begin{pmatrix}0.3333\\0.6667\end{pmatrix}$$

$$\mathbf{x}_2=T\circ T(\mathbf{x}_0)=\mathbf{A}(\mathbf{A}\mathbf{x}_0)=\begin{pmatrix}1/3 & 2/3\\2/3 & 1/3\end{pmatrix}\begin{pmatrix}0.3333\\0.6667\end{pmatrix}=\begin{pmatrix}0.5556\\0.4444\end{pmatrix}$$

$$\mathbf{x}_3=\mathbf{A}(\mathbf{A}^2\mathbf{x}_0)=\begin{pmatrix}1/3 & 2/3\\2/3 & 1/3\end{pmatrix}\begin{pmatrix}0.5556\\0.4444\end{pmatrix}=\begin{pmatrix}0.4815\\0.5185\end{pmatrix}$$

$$\mathbf{x}_4=\mathbf{A}(\mathbf{A}^3\mathbf{x}_0)=\begin{pmatrix}1/3 & 2/3\\2/3 & 1/3\end{pmatrix}\begin{pmatrix}0.4815\\0.5185\end{pmatrix}=\begin{pmatrix}0.5062\\0.4938\end{pmatrix}$$

$$\mathbf{x}_5=\mathbf{A}(\mathbf{A}^4\mathbf{x}_0)=\begin{pmatrix}1/3 & 2/3\\2/3 & 1/3\end{pmatrix}\begin{pmatrix}0.5062\\0.4938\end{pmatrix}=\begin{pmatrix}0.4979\\0.5021\end{pmatrix}$$

(b) MATLAB should give the output $\begin{pmatrix}0.5\\0.5\end{pmatrix}$.

14. We need to prove $T \circ I_V = T$ where I_V is the identity transformation on *V*.
Proof.
(a) Let **v** be an arbitrary vector in the vector space *V*. Then

$$\begin{aligned}(T \circ I_V)(\mathbf{v}) &= T(I_V(\mathbf{v})) \\ &= T(\mathbf{v}) \qquad \left[\text{Because } I_V(\mathbf{v}) = \mathbf{v}\right]\end{aligned}$$

Thus $T \circ I_V = T$.

■

(b) We need to prove $I_W \circ T = T$ where I_W is the identity transformation on *W*.
Proof.
Let $\mathbf{v}$ be an arbitrary vector in the vector space *V*. Then

$$\begin{aligned}(I_W \circ T)(\mathbf{v}) &= I_W(T(\mathbf{v})) \\ &= T(\mathbf{v}) \qquad \left[\text{Because } I_W(T(\mathbf{v})) = T(\mathbf{v})\right]\end{aligned}$$

Hence $I_W \circ T = T$.

■

15. We are required to prove that $(S \circ T) \neq (T \circ S)$.
Proof.
Let $T: U \to V$ and $S: V \to W$ be linear transformations and matrices **A** and **B** be the matrix representation of *T* and *S* respectively.
By Proposition (5-20) the matrix multiplication **BA** represents the composite transformation $S \circ T$. Again by the same proposition the matrix representation of the composite transform $T \circ S$ is $\mathbf{AB}$.
Since from the theory matrices we have

$$\mathbf{BA} \neq \mathbf{AB} \quad \text{[Not Equal]}$$

therefore $(S \circ T) \neq (T \circ S)$.

■

16. We need to prove that $k(S \circ T) = (kS) \circ T = S \circ (kT)$ where *k* is any scalar and $T: U \to V$, $S: V \to W$.
Proof.
Let matrices **A** and **B** be the matrix representation of *T* and *S* respectively. Then by proposition (5-20) we have the matrix multiplication **BA** represents the composite transform $S \circ T$. From the theory of matrices we have

$$k(\mathbf{BA}) = (k\mathbf{B})\mathbf{A} = \mathbf{B}(k\mathbf{A})$$

We have $k(\mathbf{BA})$ represents the transformation $k(S \circ T)$, $(k\mathbf{B})\mathbf{A}$ represents the transformation $(kS) \circ T$ and $\mathbf{B}(k\mathbf{A})$ represents the transformation $S \circ (kT)$.
Thus we have our required result $k(S \circ T) = (kS) \circ T = S \circ (kT)$.

■

17. Required to prove T^{-1} is linear.
Proof.
Let $T(\mathbf{u}) = \mathbf{w}_1$ and $T(\mathbf{v}) = \mathbf{w}_2$. We know that *T* is linear so

$$\mathbf{w}_1 + \mathbf{w}_2 = T(\mathbf{u}) + T(\mathbf{v}) = T(\mathbf{u} + \mathbf{v})$$

Using the definition of inverse linear transform:

> Definition (5-9). Let $T: V \to W$ be a *bijective* linear transform. The inverse transformation $T^{-1}: W \to V$ is defined as:
> $$\mathbf{v} = T^{-1}(\mathbf{w}) \quad \Leftrightarrow \quad T(\mathbf{v}) = \mathbf{w}$$

We have $T^{-1}(\mathbf{w}_1 + \mathbf{w}_2) = \mathbf{u} + \mathbf{v}$, $T^{-1}(\mathbf{w}_1) = \mathbf{u}$ and $T^{-1}(\mathbf{w}_2) = \mathbf{v}$. Hence

$$T^{-1}(\mathbf{w}_1 + \mathbf{w}_2) = \mathbf{u} + \mathbf{v} = T^{-1}(\mathbf{w}_1) + T^{-1}(\mathbf{w}_2)$$

Similarly we can show that $T^{-1}(k\mathbf{w}) = kT^{-1}(\mathbf{w})$. Both conditions of linearity are proven so T^{-1} is linear.

■

18. We need to prove $\left(T^{-1}\right)^{-1} = T$.

Proof.
Let **A** be the matrix which represents the given invertible linear operator *T*. Since *T* is invertible therefore the matrix **A** is invertible. From the theory of matrices in earlier chapters we have $\left(\mathbf{A}^{-1}\right)^{-1} = \mathbf{A}$. By Proposition (5-22) we know the matrix $\mathbf{A}^{-1}$ represents the transformation T^{-1}. Since $\left(\mathbf{A}^{-1}\right)^{-1} = \mathbf{A}$ therefore we have our required result $\left(T^{-1}\right)^{-1} = T$.

■

19. We our required to prove that T^{-1} is unique.

Proof.
Let $T: V \to W$ be an invertible linear transformation and **A** be the matrix which represents this transformation. Since the transformation is invertible therefore $\mathbf{A}^{-1}$ exists and it represents the inverse transformation T^{-1}. From our work on matrices we know that $\mathbf{A}^{-1}$ is unique which means that T^{-1} is unique.

■

20. We need to prove that

> *T* is invertible $\Leftrightarrow$ **A** is invertible

Proof.
$(\Rightarrow)$. Assume *T* is in invertible. Let **A** be the matrix representation of this transform *T*. Since *T* is invertible so applying (5-12):

> Definition (5-12). Let $T: V \to W$ be a linear transform. Then *T* is invertible $\Leftrightarrow$ there exists a transformation T^{-1} such that
>
> (a) $\left(T^{-1} \circ T\right) = I_V$ where I_V is the identity transformation on the start vector space V
>
> (b) $\left(T \circ T^{-1}\right) = I_W$ where I_W is the identity transformation on the arrival vector space W

We have that there is a transform T^{-1} such that

$$T^{-1} \circ T = I_V$$

Let **B** be the matrix representing T^{-1}. Since I_V is the identity transform so by question 12 of the last Exercises 5.5 which says:

> If T is an identity linear operator then the matrix for *T* is the identity matrix $\mathbf{I}_n$.

We have the matrix for I_V is the identity $\mathbf{I}_n$. Hence

$$\mathbf{BA} = \mathbf{I}_n$$

Similarly we have $\mathbf{AB} = \mathbf{I}_n$. Hence matrix **A** is invertible.

$(\Leftarrow)$. Assume matrix **A** is invertible and represents the transform *T*. Then there exists a matrix **B** such that

$$\mathbf{AB} = \mathbf{I} \text{ and } \mathbf{BA} = \mathbf{I}$$

Let matrix **B** represent the transform *S* then

$$\mathbf{BA} \text{ represents the composite transform } S \circ T$$
$$\mathbf{AB} \text{ represents the composite transform } T \circ S$$

We have $S \circ T = T \circ S = I$ where *I* is the identity transform. By the above Definition (5-12) we conclude that the transform *T* is invertible.

■