Complete Solutions to Exercises 7.1

*MATLAB: To find the eigenvalues of a matrix **A** the command is eig(A). To find eigenvalues and eigenvectors enter the command [V, d]=eig(A) this shows the eigenvectors down the columns of V and the eigenvalues along the diagonal entries of d.*

1. We substitute the given matrix, **A**, into $\det(\mathbf{A}-\lambda\mathbf{I})=\mathbf{0}$:

(a) We have

$$\det\begin{pmatrix}7-\lambda & 3\\ 0 & -4-\lambda\end{pmatrix}=\left[(7-\lambda)(-4-\lambda)-0\right]$$
$$=-(7-\lambda)(4+\lambda)=0$$

Thus the eigenvalues are $\lambda=-4,\ 7$.

Let $\mathbf{u}=\begin{pmatrix}x\\ y\end{pmatrix}$ be the eigenvector for $\lambda=7$. We have

$$\begin{pmatrix}7-7 & 3\\ 0 & -4-7\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix}=\begin{pmatrix}0\\ 0\end{pmatrix}$$
$$\begin{pmatrix}0 & 3\\ 0 & -11\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix}=\begin{pmatrix}0\\ 0\end{pmatrix}$$

Multiplying out the matrices gives

$$3y=0$$
$$-11y=0$$

Thus y=0 and x is any real number apart from zero. A particular value of x can be 1. So a particular eigenvector for $\lambda=7$ is $\mathbf{u}=\begin{pmatrix}1\\ 0\end{pmatrix}$.

Similarly let $\mathbf{v}$ be an eigenvector for $\lambda=-4$:

$$\begin{pmatrix}7-(-4) & 3\\ 0 & -4-(-4)\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix}=\begin{pmatrix}0\\ 0\end{pmatrix}$$
$$\begin{pmatrix}11 & 3\\ 0 & 0\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix}=\begin{pmatrix}0\\ 0\end{pmatrix}$$

Multiplying out the first row yields

$$11x+3y=0$$
$$x=-\frac{3}{11}y$$

If $y=1$then $x=-3/11$, thus $\mathbf{v}=\begin{pmatrix}-3/11\\ 1\end{pmatrix}$ or using smallest positive integers gives $\begin{pmatrix}-3\\ 11\end{pmatrix}$.

(b)

$$\det\begin{pmatrix}5-\lambda & -2\\ 4 & -1-\lambda\end{pmatrix}=(5-\lambda)(-1-\lambda)+8$$
$$=-(5-\lambda)(1+\lambda)+8=-\left[5+4\lambda-\lambda^2\right]+8=\lambda^2-4\lambda-5+8$$

Putting this quadratic to zero and solving

$$\lambda^2-4\lambda-5+8=\lambda^2-4\lambda+3=0$$
$$(\lambda-3)(\lambda-1)=0 \text{ gives } \lambda_1=1,\ \lambda_2=3$$

Let $\mathbf{u}=\begin{pmatrix}x\\y\end{pmatrix}$ be the eigenvector for $\lambda=1$:

$$\begin{pmatrix}5-1 & -2\\4 & -1-1\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix}$$
$$\begin{pmatrix}4 & -2\\4 & -2\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix}$$

Multiplying out the matrix

$$4x-2y=0$$
$$4x-2y=0$$

Solving these gives $x=1,\ y=2$. Thus $\mathbf{u}=\begin{pmatrix}1\\2\end{pmatrix}$ is an eigenvector for $\lambda=1$.

Let $\mathbf{v}=\begin{pmatrix}x\\y\end{pmatrix}$ be the eigenvector for $\lambda=3$:

$$\begin{pmatrix}5-3 & -2\\4 & -1-3\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix}$$
$$\begin{pmatrix}2 & -2\\4 & -4\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix}$$

Multiplying gives

$$2x-2y=0$$
$$4x-4y=0$$

Solving these gives $x=y=1$. An eigenvector corresponding to $\lambda=3$ is $\begin{pmatrix}1\\1\end{pmatrix}$.

(c) Substituting the given matrix into $\det(\mathbf{A}-\lambda\mathbf{I})$ yields

$$\det\begin{pmatrix}-1-\lambda & 4\\2 & 1-\lambda\end{pmatrix}\underset{\text{by (6.1)}}{=}(-1-\lambda)(1-\lambda)-(2\times 4)$$
$$=-(1+\lambda)(1-\lambda)-8$$
$$=-(1-\lambda^2)-8=-1+\lambda^2-8=\lambda^2-9$$

Solving the equation $\lambda^2-9=0$ gives

$$\lambda^2=9$$
$$\lambda=\sqrt{9}=-3,\ 3$$

Let $\mathbf{u}=\begin{pmatrix}x\\y\end{pmatrix}$ be the eigenvector for $\lambda=-3$.

$$\begin{pmatrix} -1-(-3) & 4 \\ 2 & 1-(-3) \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 2 & 4 \\ 2 & 4 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Solving gives $x=-2,\ y=1$. An eigenvector is $\begin{pmatrix} -2 \\ 1 \end{pmatrix}$ corresponding to $\lambda=-3$.

Let $\mathbf{v} = \begin{pmatrix} x \\ y \end{pmatrix}$ be an eigenvector for $\lambda = 3$:

$$\begin{pmatrix} -1-3 & 4 \\ 2 & 1-3 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} -4 & 4 \\ 2 & -2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Hence $x = y = 1$. The eigenvector $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ corresponds to $\lambda = 3$.

2. Let $\mathbf{w} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ be the eigenvector for the eigenvalue $\lambda = -5$ of the matrix in **EXAMPLE 5**.

Substituting this, $\lambda = -5$, into $(\mathbf{A} - \lambda\mathbf{I})\mathbf{w} = \mathbf{0}$ gives

$$\begin{pmatrix} 1-(-5) & 0 & 4 \\ 0 & 4-(-5) & 0 \\ 3 & 5 & -3-(-5) \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 6 & 0 & 4 \\ 0 & 9 & 0 \\ 3 & 5 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Multiplying the matrices

$$\begin{aligned} 6x \qquad + 4z &= 0 && (\dagger) \\ 9y \qquad &= 0 && (\dagger\dagger) \\ 3x + 5y + 2z &= 0 && (\dagger\dagger\dagger) \end{aligned}$$

From $(\dagger\dagger)$ we have $y = 0$. Substituting this into $(\dagger\dagger\dagger)$

$$3x + 2z = 0$$

$$3x = -2z$$

$$x = -\frac{2}{3}z$$

Let $z = a$ where $a \neq 0$, thus the general eigenvector is $a\begin{pmatrix} -2/3 \\ 0 \\ 1 \end{pmatrix}$.

Let $\mathbf{v}$ be the eigenvector for $\lambda = 4$:

$$\begin{pmatrix} 1-4 & 0 & 4 \\ 0 & 4-4 & 0 \\ 3 & 5 & -3-4 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} -3 & 0 & 4 \\ 0 & 0 & 0 \\ 3 & 5 & -7 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

The general eigenvector is

$$\mathbf{v} = s\begin{pmatrix} 20 \\ 9 \\ 15 \end{pmatrix} \text{ where } s \text{ is } \textbf{not} \text{ zero}$$

3. The eigenvalues are determined by

$$\det(\mathbf{A} - \lambda\mathbf{I}) = 0$$

Substituting the given $\mathbf{A} = \begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix}$ and subtracting λ along the leading diagonal gives

$$\begin{aligned} \det(\mathbf{A} - \lambda\mathbf{I}) &= \det\begin{pmatrix} 3-\lambda & 1 \\ 1 & 3-\lambda \end{pmatrix} \\ &= (3-\lambda)(3-\lambda) - 1 \\ &= 9 - 6\lambda + \lambda^2 - 1 \\ &= \lambda^2 - 6\lambda + 8 \end{aligned}$$

Factorizing and equating the last line to zero:

$$\lambda^2 - 6\lambda + 8 = (\lambda - 2)(\lambda - 4) = 0$$

$$\lambda_1 = 2, \quad \lambda_2 = 4$$

Next we find the eigenvectors belonging to $\lambda_1 = 2$. Using $(\mathbf{A} - \lambda\mathbf{I})\mathbf{u} = \mathbf{O}$ with $\lambda = \lambda_1 = 2$, $\mathbf{u} = \begin{pmatrix} x \\ y \end{pmatrix}$ and $\mathbf{O} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ gives

$$(\mathbf{A} - 2\mathbf{I})\mathbf{u} = \begin{pmatrix} 3-2 & 1 \\ 1 & 3-2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} \qquad \begin{bmatrix} \text{Taking Away 2 along the} \\ \text{Leading Diagonal} \end{bmatrix}$$

$$= \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Expanding out yields $x + y = 0 \quad \Rightarrow \quad x = -y$. Let $y = s$ where $s \neq 0$ then $x = -s$ and $\mathbf{u} = \begin{pmatrix} -s \\ s \end{pmatrix} = s\begin{pmatrix} -1 \\ 1 \end{pmatrix}$. Thus the eigenvector for $\lambda_1 = 2$ is $\mathbf{u} = s\begin{pmatrix} -1 \\ 1 \end{pmatrix}$.

Next we find the eigenvectors belonging to $\lambda_2 = 4$. Using $(\mathbf{A} - \lambda\mathbf{I})\mathbf{v} = \mathbf{O}$ with $\lambda = \lambda_2 = 4$, $\mathbf{v} = \begin{pmatrix} x \\ y \end{pmatrix}$ and $\mathbf{O} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ gives

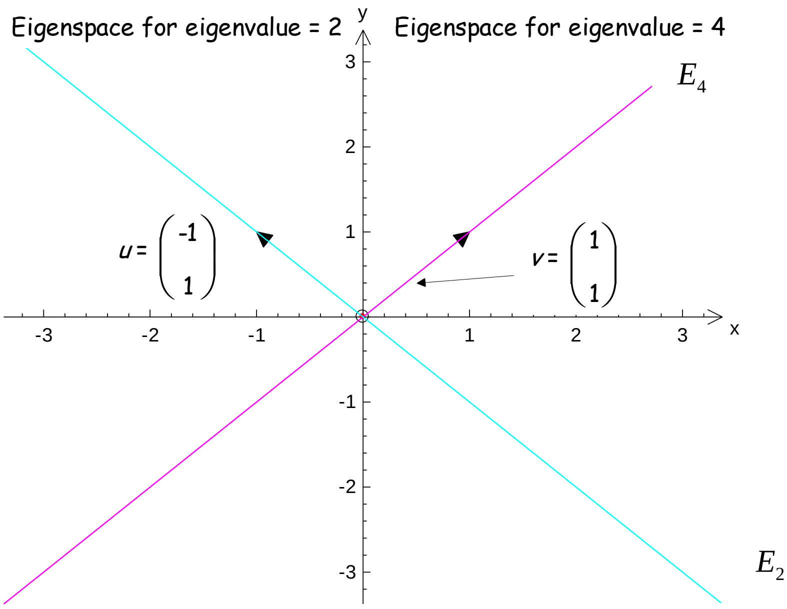
$$(\mathbf{A}-4\mathbf{I})\mathbf{v}=\begin{pmatrix}3-4 & 1\\ 1 & 3-4\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix} \qquad \begin{bmatrix}\text{Taking Away 4 along}\\ \text{the Leading Diagonal}\end{bmatrix}$$
$$=\begin{pmatrix}-1 & 1\\ 1 & -1\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix}=\begin{pmatrix}0\\ 0\end{pmatrix}$$

Expanding out yields

$$-x+y=0 \;\Rightarrow\; y=x$$
$$x-y=0 \;\Rightarrow\; x=y$$

We have $y=x$. Let $y=s$ where $s\neq 0$ then $x=s$ and $\mathbf{v}=\begin{pmatrix}s\\ s\end{pmatrix}=s\begin{pmatrix}1\\ 1\end{pmatrix}$. Thus the eigenvector for $\lambda_2=4$ is $\mathbf{v}=s\begin{pmatrix}1\\ 1\end{pmatrix}$.

The 2 eigenspaces are given by $E_2=\left\{s\begin{pmatrix}-1\\ 1\end{pmatrix}\right\}$ and $E_4=\left\{s\begin{pmatrix}1\\ 1\end{pmatrix}\right\}$. Plotting these on $\mathbb{R}^2$ gives



4. The eigenvalues are determined by

$$\det(\mathbf{A}-\lambda\mathbf{I})=0$$

Substituting the given $\mathbf{A}=\begin{pmatrix}5 & -2\\ 7 & -4\end{pmatrix}$ and subtracting λ along the leading diagonal gives

$$\det(\mathbf{A}-\lambda\mathbf{I})=\det\begin{pmatrix}5-\lambda & -2\\ 7 & -4-\lambda\end{pmatrix}$$
$$=(5-\lambda)(-4-\lambda)+14$$
$$=\underbrace{--}_{=+}(\lambda-5)(4+\lambda)+14 \qquad [\text{Taking Out Minus Signs}]$$
$$=4\lambda+\lambda^2-20-5\lambda+14 \qquad [\text{Expanding}]$$
$$=\lambda^2-\lambda-6 \qquad [\text{Simplifying}]$$

Factorizing and equating the last line to zero:

$$\lambda^2-\lambda-6=(\lambda+2)(\lambda-3)=0$$
$$\lambda_1=-2,\quad \lambda_2=3$$

Next we find the eigenvectors belonging to $\lambda_1=-2$. Using $(\mathbf{A}-\lambda\mathbf{I})\mathbf{u}=\mathbf{O}$ with $\lambda=\lambda_1=-2$, $\mathbf{u}=\begin{pmatrix}x\\y\end{pmatrix}$ and $\mathbf{O}=\begin{pmatrix}0\\0\end{pmatrix}$ gives

$$(\mathbf{A}+2\mathbf{I})\mathbf{u}=\begin{pmatrix}5+2 & -2\\ 7 & -4+2\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}$$
$$=\begin{pmatrix}7 & -2\\ 7 & -2\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix}$$

Expanding out yields

$$7x-2y=0 \quad\Rightarrow\quad 7x=2y$$

Let $y=7s$ where $s\neq 0$ then $x=2s$ and we have

$$\mathbf{u}=\begin{pmatrix}2s\\7s\end{pmatrix}=s\begin{pmatrix}2\\7\end{pmatrix}$$

Thus the eigenvector for $\lambda_1=-2$ is $\mathbf{u}=s\begin{pmatrix}2\\7\end{pmatrix}$ where the simplest eigenvector is $\begin{pmatrix}2\\7\end{pmatrix}$. Next we find the eigenvectors belonging to $\lambda_2=3$. Using $(\mathbf{A}-\lambda\mathbf{I})\mathbf{v}=\mathbf{O}$ with $\lambda=\lambda_2=3$, $\mathbf{v}=\begin{pmatrix}x\\y\end{pmatrix}$ and $\mathbf{O}=\begin{pmatrix}0\\0\end{pmatrix}$ gives

$$(\mathbf{A}-3\mathbf{I})\mathbf{v}=\begin{pmatrix}5-3 & -2\\ 7 & -4-3\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}$$
$$=\begin{pmatrix}2 & -2\\ 7 & -7\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix}$$

Expanding out yields

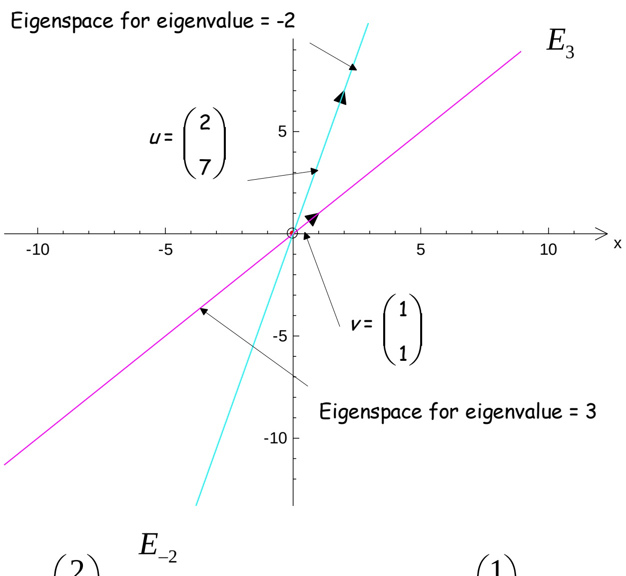
$$2x-2y=0 \quad\Rightarrow\quad y=x$$
$$7x-7y=0 \quad\Rightarrow\quad x=y$$

We have $y=x$. Let $y=s$ where $s\neq 0$ then $x=s$ and $\mathbf{v}=\begin{pmatrix}s\\s\end{pmatrix}=s\begin{pmatrix}1\\1\end{pmatrix}$. Thus the eigenvector for $\lambda_2=3$ is $\mathbf{v}=s\begin{pmatrix}1\\1\end{pmatrix}$ where the simplest eigenvector is $\begin{pmatrix}1\\1\end{pmatrix}$.

The effect of multiplying by the matrix **A** is

$$\mathbf{Au}=-2\mathbf{u} \text{ where } \mathbf{u}=s\begin{pmatrix}2\\7\end{pmatrix} \text{ and } \mathbf{Av}=3\mathbf{v} \text{ where } \mathbf{v}=s\begin{pmatrix}1\\1\end{pmatrix}$$

The 2 eigenspaces are given by $E_{-2}=\left\{s\begin{pmatrix}2\\7\end{pmatrix}\right\}$ and $E_3=\left\{s\begin{pmatrix}1\\1\end{pmatrix}\right\}$. Plotting these on $\mathbb{R}^2$ gives



A basis vector for E_{-2} is $\begin{pmatrix} 2 \\ 7 \end{pmatrix}$ and a basis vector for E_3 is $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

5. We are given $\mathbf{A} = \begin{pmatrix} -2 & 8 \\ 5 & 1 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} -4 & 16 \\ 10 & 2 \end{pmatrix}$.

(i) The eigenvalues of matrix **A** are determined by

$$\det(\mathbf{A} - \lambda\mathbf{I}) = 0$$

Substituting the given $\mathbf{A} = \begin{pmatrix} -2 & 8 \\ 5 & 1 \end{pmatrix}$ and subtracting λ along the leading diagonal gives

$$\begin{aligned}
\det(\mathbf{A} - \lambda\mathbf{I}) &= \det\begin{pmatrix} -2-\lambda & 8 \\ 5 & 1-\lambda \end{pmatrix} \\
&= (-2-\lambda)(1-\lambda) - 40 \\
&= \underbrace{--}_{=+}(\lambda+2)(\lambda-1) - 40 && \text{[Taking Out Minus Signs]} \\
&= \lambda^2 - \lambda + 2\lambda - 2 - 40 && \text{[Expanding]} \\
&= \lambda^2 + \lambda - 42 && \text{[Simplifying]}
\end{aligned}$$

Factorizing and equating the last line to zero:

$$\lambda^2 + \lambda - 42 = (\lambda+7)(\lambda-6) = 0$$

$$\lambda_1 = -7, \quad \lambda_2 = 6$$

(ii) We have

$$\det(\mathbf{B} - \lambda\mathbf{I}) = 0$$

Substituting the given $\mathbf{B} = \begin{pmatrix} -4 & 16 \\ 10 & 2 \end{pmatrix}$ and subtracting λ along the leading diagonal gives

$$\det(\mathbf{B}-\lambda\mathbf{I}) = \det\begin{pmatrix} -4-\lambda & 16 \\ 10 & 2-\lambda \end{pmatrix}$$
$$= (-4-\lambda)(2-\lambda)-160$$
$$= (\lambda+4)(\lambda-2)-160 \qquad [\text{Taking Out Minus Signs}]$$
$$= \lambda^2+2\lambda-8-160 \qquad [\text{Expanding}]$$
$$= \lambda^2+2\lambda-168 \qquad [\text{Simplifying}]$$

Factorizing and equating the last line to zero:

$$\lambda^2+2\lambda-168 = (\lambda+14)(\lambda-12) = 0$$
$$\lambda_3 = -14, \quad \lambda_4 = 12$$

(iii) Note that matrix **B** is twice matrix **A**, that is $\mathbf{B} = 2\mathbf{A}$ which transforms over to the eigenvalues because

$$\lambda_3 = -14 = 2\times(-7) = 2\lambda_1 \text{ and } \lambda_4 = 12 = 2\times 6 = 2\lambda_2$$
$$\lambda_3 = 2\lambda_1 \text{ and } \lambda_4 = 2\lambda_2$$

Prediction is given by question 4.

6. We need to prove if the matrix $\mathbf{B} = r\mathbf{A}$ where r be a real number and λ is the eigenvalue of matrix **A** then the eigenvalue of **B** is $r\lambda$.

Proof.

Let λ be an eigenvalue of **A** with eigenvector **u**. By definition of eigenvalues (7.1) we have $\mathbf{Au} = \lambda\mathbf{u}$. We are given that $\mathbf{B} = r\mathbf{A}$ so evaluating **Bu** gives

$$\mathbf{Bu} = (r\mathbf{A})\mathbf{u} = r(\mathbf{Au}) = r(\lambda\mathbf{u}) = (r\lambda)\mathbf{u}$$

Since $\mathbf{Bu} = (r\lambda)\mathbf{u}$ we conclude that $r\lambda$ is an eigenvalue of **B** with eigenvector **u**. This completes our proof.

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7. We need to prove that the $n\times n$ matrix, $\mathbf{O} = \begin{pmatrix} 0 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \end{pmatrix}$, only has the zero eigenvalues.

Proof. Consider the zero matrix, $\mathbf{O} = \begin{pmatrix} 0 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \end{pmatrix} = \mathbf{A}$, as the matrix **A** in the characteristic equation $\det(\mathbf{A}-\lambda\mathbf{I}) = 0$:

$$\det(\mathbf{A}-\lambda\mathbf{I}) = \det\begin{pmatrix} 0-\lambda & 0 & \cdots & 0 \\ 0 & 0-\lambda & 0 & \cdots \\ \vdots & 0 & \ddots & 0 \\ 0 & \cdots & 0 & 0-\lambda \end{pmatrix} \qquad \begin{bmatrix} \text{Taking Away } \lambda \text{ along} \\ \text{the Leading Diagonal} \end{bmatrix}$$

$$= \det\begin{pmatrix} -\lambda & 0 & \cdots & 0 \\ 0 & -\lambda & 0 & \cdots \\ \vdots & 0 & \ddots & 0 \\ 0 & \cdots & 0 & -\lambda \end{pmatrix} \underset{\text{By Proposition (6-8)}}{=} (-\lambda)^n = 0$$

The Proposition (6-8) is from chapter 6 and claims:

(6-8). The determinant of a triangular or diagonal matrix is a product of the entries along the leading diagonal.

Note that we have a *n* by *n* diagonal matrix therefore the determinant of the matrix is

$$\underbrace{(-\lambda)\times(-\lambda)\times(-\lambda)\times\cdots\times(-\lambda)}_{n\text{ copies}}=(-\lambda)^n$$

We have $(-\lambda)^n=0$ which means $\lambda=0$. Hence the only eigenvalues of the zero square matrix is 0.

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8. We have the characteristic equation given by

$$\begin{aligned}
\det(\mathbf{A}-\lambda\mathbf{I}) &= \begin{pmatrix}1-\lambda & 2 & 1\\ 2 & 1-\lambda & 1\\ 1 & 1 & 2-\lambda\end{pmatrix}\\
&=(1-\lambda)\det\begin{pmatrix}1-\lambda & 1\\ 1 & 2-\lambda\end{pmatrix}-2\det\begin{pmatrix}2 & 1\\ 1 & 2-\lambda\end{pmatrix}+\det\begin{pmatrix}2 & 1-\lambda\\ 1 & 1\end{pmatrix}\\
&=(1-\lambda)\left[(1-\lambda)(2-\lambda)-1\right]-2\left[2(2-\lambda)-1\right]+\left[2-(1-\lambda)\right]\\
&=(1-\lambda)\left[(\lambda-1)(\lambda-2)-1\right]-2\left[4-2\lambda-1\right]+\left[1+\lambda\right]\\
&=(1-\lambda)\left[\lambda^2-3\lambda+2-1\right]-2\left[3-2\lambda\right]+\left[1+\lambda\right] && \left[\text{Simplifying Brackets}\right]\\
&=(1-\lambda)\left[\lambda^2-3\lambda+1\right]-6+4\lambda+1+\lambda && \left[\text{Expanding Last 2 Brackets}\right]\\
&=(1-\lambda)\left[\lambda^2-3\lambda+1\right]-5+5\lambda && \left[\text{Simplifying}\right]\\
&=(1-\lambda)\left[\lambda^2-3\lambda+1\right]-5(1-\lambda) && \left[\text{Because } -5+5\lambda=-5(1-\lambda)\right]\\
&=(1-\lambda)\left[\lambda^2-3\lambda+1-5\right] && \left[\text{Taking Out Factor } (1-\lambda)\right]\\
&=(1-\lambda)\left[\lambda^2-3\lambda-4\right]\\
&=(1-\lambda)\left[(\lambda-4)(\lambda+1)\right] && \left[\text{Factorizing Square Brackets}\right]
\end{aligned}$$

Equating all this to zero gives

$$(1-\lambda)(\lambda-4)(\lambda+1)=0$$

$$\lambda_1=1,\ \lambda_2=4\ \text{ and }\ \lambda_3=-1$$

What else do we need to find?

Eigenvectors. Let $\mathbf{u}=\begin{pmatrix}x\\ y\\ z\end{pmatrix}$ be an eigenvector belonging to $\lambda=\lambda_1=1$:

$$\begin{aligned}
(\mathbf{A}-\lambda\mathbf{I})\mathbf{u} &= \begin{pmatrix}1-1 & 2 & 1\\ 2 & 1-1 & 1\\ 1 & 1 & 2-1\end{pmatrix}\mathbf{u}\\
&=\begin{pmatrix}0 & 2 & 1\\ 2 & 0 & 1\\ 1 & 1 & 1\end{pmatrix}\begin{pmatrix}x\\ y\\ z\end{pmatrix}=\begin{pmatrix}0\\ 0\\ 0\end{pmatrix}
\end{aligned}$$

Multiplying out the matrix gives:

$$0+2y+z=0 \qquad (*)$$
$$2x+0+z=0 \qquad (**)$$
$$x+y+z=0 \qquad (***)$$

From the middle equation we have $z=-2x$. Let $x=s$ where $s\neq 0$ then $z=-2s$. *How can we find y?*
Substituting $z=-2s$ into the top equation (*):

$$0+2y-2s=0 \text{ gives } y=s$$

Thus our eigenvector is $\mathbf{u}=\begin{pmatrix} x \\ y \\ z \end{pmatrix}=\begin{pmatrix} s \\ s \\ -2s \end{pmatrix}=s\begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix}$ belonging to $\lambda_1=1$.

Similarly we can find the other 2 eigenvectors. Substitute $\lambda=\lambda_2=4$ and **v** be the corresponding eigenvector:

$$(\mathbf{A}-\lambda\mathbf{I})\mathbf{v}=\begin{pmatrix} 1-4 & 2 & 1 \\ 2 & 1-4 & 1 \\ 1 & 1 & 2-4 \end{pmatrix}\mathbf{v}$$
$$=\begin{pmatrix} -3 & 2 & 1 \\ 2 & -3 & 1 \\ 1 & 1 & -2 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Multiplying out the matrix gives:

$$-3x+2y+z=0 \qquad (*)$$
$$2x-3y+z=0 \qquad (**)$$
$$x \ \ +y-2z=0 \qquad (***)$$

Subtracting the top two equations, (*) and (**), gives

$$\begin{array}{rl} & -3x+2y+z=0 \qquad (*) \\ - & (2x-3y+z=0) \qquad (**) \\ \hline & -5x+5y+0=0 \end{array}$$

From the last line we have $x=y$. Let $y=s$ where $s\neq 0$ then $x=s$ and from the last equation (***) we have

$$s+s-2z=0 \quad \text{gives} \quad z=s$$

Our eigenvector $\mathbf{v}=\begin{pmatrix} s \\ s \\ s \end{pmatrix}=s\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ where $s\neq 0$ and is the eigenvector belonging to $\lambda_2=4$.

Call the last eigenvector **w** which belongs to $\lambda=\lambda_3=-1$. We have

$$(\mathbf{A}-\lambda\mathbf{I})\mathbf{w}=\begin{pmatrix}1-(-1) & 2 & 1\\ 2 & 1-(-1) & 1\\ 1 & 1 & 2-(-1)\end{pmatrix}\mathbf{w}$$

$$=\begin{pmatrix}2 & 2 & 1\\ 2 & 2 & 1\\ 1 & 1 & 3\end{pmatrix}\begin{pmatrix}x\\ y\\ z\end{pmatrix}=\begin{pmatrix}0\\ 0\\ 0\end{pmatrix}$$

Multiplying out the matrix gives:

$$\begin{aligned} 2x+2y+z&=0 && (*)\\ 2x+2y+z&=0 && (**)\\ x+y+3z&=0 && (***)\end{aligned}$$

Multiply the last equation (***) by 2 and subtract the top equation (*):

$$\begin{array}{rr} & 2x+2y+6z=0\\ - & (2x+2y+z=0)\\ \hline & 0+0+5z=0\end{array}$$

From the last line we have $z=0$. Substituting this into the bottom equation (***) gives

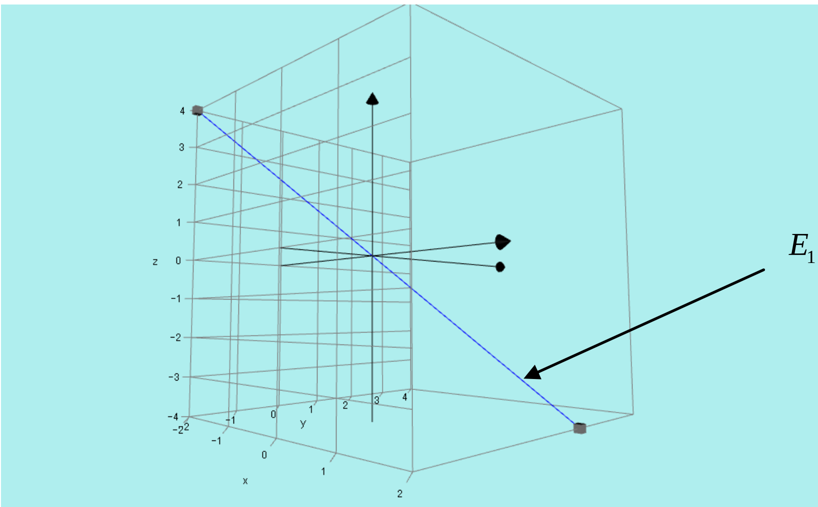
$$x+y=0 \quad\Rightarrow\quad x=-y$$

Let $y=s$ then $x=-s$ and we have

$$\mathbf{w}=\begin{pmatrix}-s\\ s\\ 0\end{pmatrix}=s\begin{pmatrix}-1\\ 1\\ 0\end{pmatrix} \text{ where } s\neq 0$$

This $\mathbf{w}$ is the eigenvector belonging to $\lambda_3=-1$.

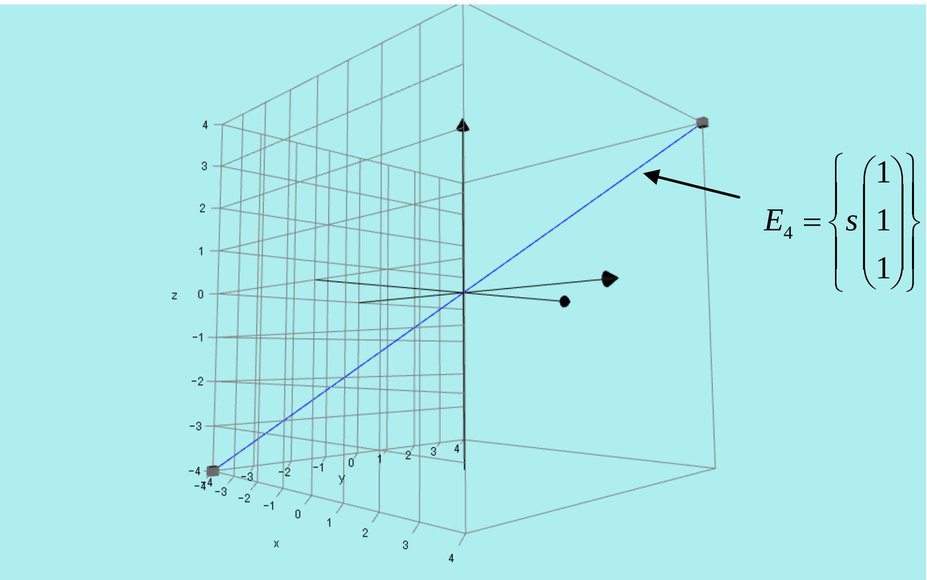
Our eigenspace for $\lambda=1$ is $E_1=\left\{s\begin{pmatrix}1\\ 1\\ -2\end{pmatrix}\right\}$ and in $\mathbb{R}^3$ is the line shown below:



$$E_1=\left\{s\begin{pmatrix}1\\ 1\\ -2\end{pmatrix}\right\}$$

A basis vector for $E_1=\left\{s\begin{pmatrix}1\\ 1\\ -2\end{pmatrix}\right\}$ is $\begin{pmatrix}1\\ 1\\ -2\end{pmatrix}$.

Our eigenspace for $\lambda = 4$ is $E_4 = \left\{ s\begin{pmatrix}1\\1\\1\end{pmatrix} \right\}$ plotted in $\mathbb{R}^3$ is the line shown below:



A basis for $E_4 = \left\{ s\begin{pmatrix}1\\1\\1\end{pmatrix} \right\}$ is $\begin{pmatrix}1\\1\\1\end{pmatrix}$. For $\lambda = -1$ we have the eigenspace

$E_{-1} = \left\{ s\begin{pmatrix}-1\\1\\0\end{pmatrix} \right\}$ plotted in $\mathbb{R}^3$ below and a basis vector for E_{-1} is $\begin{pmatrix}-1\\1\\0\end{pmatrix}$