Complete Solutions to Exercises 5.2

1. (a) Let $\mathbf{v}=\begin{pmatrix}x\\y\end{pmatrix}$ and $\mathbf{A}=\begin{pmatrix}1&0\\0&1\end{pmatrix}$ be the given matrix, then we have

$$T\left(\begin{pmatrix}x\\y\end{pmatrix}\right)=\begin{pmatrix}1&0\\0&1\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix}$$

Opening out the matrix gives

$$x=0 \text{ and } y=0$$

Thus $\mathbf{v}=\begin{pmatrix}0\\0\end{pmatrix}=\mathbf{O}$ and we conclude $\ker(T)=\{\mathbf{O}\}$.

(b) Similarly we let $\mathbf{v}=\begin{pmatrix}x\\y\end{pmatrix}$ and $\mathbf{A}=\begin{pmatrix}1&1\\1&1\end{pmatrix}$ be the given matrix, then

$$T\left(\begin{pmatrix}x\\y\end{pmatrix}\right)=\begin{pmatrix}1&1\\1&1\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix}$$

Opening out the matrix gives

$$x+y=0$$
$$x+y=0$$

Solving these gives $x=-y$. Let $y=r$ where r is any real number then $x=-r$ and we have $\mathbf{v}=\begin{pmatrix}-r\\r\end{pmatrix}=r\begin{pmatrix}-1\\1\end{pmatrix}$. Thus $\ker(T)=\left\{r\begin{pmatrix}-1\\1\end{pmatrix}\ \middle|\ r\in\mathbb{R}\right\}$.

(c) The given transformation $T\left(\begin{pmatrix}x\\y\end{pmatrix}\right)=\begin{pmatrix}0\\0\end{pmatrix}$ is the zero transformation which means **all** vectors in $\mathbb{R}^2$ are transformed to the zero vector so the kernel of T is **all** the domain, which is $\mathbb{R}^2$.

(d) *What is the kernel of the given transformation* $T\left(\begin{pmatrix}x\\y\\z\end{pmatrix}\right)=\begin{pmatrix}0\\0\\x\end{pmatrix}$?

The vectors $\begin{pmatrix}0\\y\\z\end{pmatrix}$ are transformed to the zero vector because the given transformation

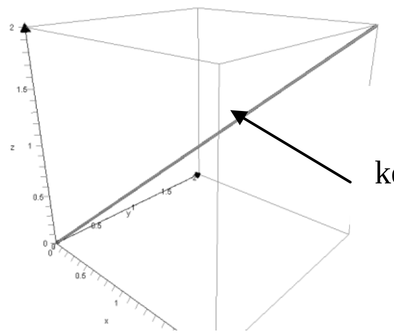
$$T\left(\begin{pmatrix}0\\y\\z\end{pmatrix}\right)=\begin{pmatrix}0\\0\\0\end{pmatrix}=\mathbf{O}$$

Thus $\ker(T)=\left\{\begin{pmatrix}0\\a\\b\end{pmatrix}\ \middle|\ a\in\mathbb{R},\ b\in\mathbb{R}\right\}$. This means the kernel of T is the yz plane.

(e) *How do we find the kernel of the given transformation* $T\left(\begin{pmatrix}x\\y\\z\end{pmatrix}\right)=\begin{pmatrix}y-z\\x-z\end{pmatrix}$?

$$T\begin{pmatrix}\begin{pmatrix}x\\y\\z\end{pmatrix}\end{pmatrix}=\begin{pmatrix}y-z\\x-z\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix}$$

We have the simultaneous equations $y-z=0$ and $x-z=0$. Thus $y=z$ and $x=z$. Let $z=r$ where r is any real number then $y=x=z=r$. Thus the kernel of T is $\left\{r\begin{pmatrix}1\\1\\1\end{pmatrix}\;\middle|\; r\in\mathbb{R}\right\}$. This can be illustrated as:



$$\ker(T)=\left\{r\begin{pmatrix}1\\1\\1\end{pmatrix}\;\middle|\; r\in\mathbb{R}\right\}$$

2. *What is* $\ker(T)$ *equal to when* $T(\mathbf{A})=\mathbf{A}^T$ *?*

Well it is the n by n matrix which is transformed to the zero matrix after transposition:

$$T\begin{pmatrix}\begin{pmatrix}a_{11}&\cdots&a_{1n}\\\vdots&\ddots&\vdots\\a_{n1}&\cdots&a_{nn}\end{pmatrix}\end{pmatrix}=\begin{pmatrix}a_{11}&\cdots&a_{1n}\\\vdots&\ddots&\vdots\\a_{n1}&\cdots&a_{nn}\end{pmatrix}^T$$

$$\underset{\text{Transposing}}{=}\begin{pmatrix}a_{11}&\cdots&a_{n1}\\\vdots&\ddots&\vdots\\a_{1n}&\cdots&a_{nn}\end{pmatrix}=\begin{pmatrix}0&\cdots&0\\\vdots&\ddots&\vdots\\0&\cdots&0\end{pmatrix}=\mathbf{O}$$

This means all the entries of the matrix are zero, $a_{ij}=0$, which suggests we have the zero matrix. Thus $\ker(T)=\{\mathbf{O}\}$ where $\mathbf{O}$ is the zero n by n matrix.

What is range(T) equal to?

These are the matrices transformed to the vector space M_{nn} under T:

$$T\begin{pmatrix}\begin{pmatrix}a_{11}&\cdots&a_{1n}\\\vdots&\ddots&\vdots\\a_{n1}&\cdots&a_{nn}\end{pmatrix}\end{pmatrix}=\begin{pmatrix}a_{11}&\cdots&a_{1n}\\\vdots&\ddots&\vdots\\a_{n1}&\cdots&a_{nn}\end{pmatrix}^T=\begin{pmatrix}a_{11}&\cdots&a_{n1}\\\vdots&\ddots&\vdots\\a_{1n}&\cdots&a_{nn}\end{pmatrix}$$

Thus range$(T)=M_{nn}$.

3. *What is the kernel of the given linear transformation* $T(\mathbf{p})=p'(x)$ *equal to?*

The kernel in this case is the set of polynomials which gets transformed to the zero vector under the linear transformation T. The transformation differentiates the given polynomial $p(x)$ so we have

$$T(\mathbf{p}) = p'(x)$$

Which polynomials give zero after differentiation?

The constant polynomials, that is $T(\mathbf{c}) = c' = 0$ where c is a constant.

[You can also show this as follows:

Let $p(x) = ax^2 + bx + c$ then

$$\begin{aligned} T(\mathbf{p}) = p'(x) &= \left(ax^2 + bx + c\right)' \\ &= 2ax + b = 0 \end{aligned}$$

This gives $a = 0$ and $b = 0$. Substituting these values into $p(x) = ax^2 + bx + c$ gives

$$p(x) = ax^2 + bx + c = 0 + 0 + c = c$$

This means we have the constant polynomial c].

Thus $\ker(T) = \{c \mid c \in \mathbb{R}\} = P_0$. Remember P_0 is the vector space of constant polynomials.

What is the range of the given transformation?

Since we are given $T: P_2 \to P_1$, let $\mathbf{p} = ax^2 + bx + c$ and apply the transformation to this vector:

$$T(\mathbf{p}) = \left(ax^2 + bx + c\right)' = 2ax + b$$

$2ax + b$ is a linear polynomial therefore the range of T is the set of linear polynomials, thus range$(T) = P_1$.

4. We need to prove that if $\mathbf{u} \in \ker(T)$ and $\mathbf{v} \in \ker(T)$ then for any scalars k and c the vector $(k\mathbf{u} + c\mathbf{v}) \in \ker(T)$.

Proof.

Consider the vector $k\mathbf{u} + c\mathbf{v}$. We have

$$\begin{aligned} T(k\mathbf{u} + c\mathbf{v}) &= kT(\mathbf{u}) + cT(\mathbf{v}) && [\text{Because } T \text{ is linear}] \\ &= k\mathbf{O} + c\mathbf{O} && [\text{Because } \mathbf{u} \in \ker(T),\ \mathbf{v} \in \ker(T)] \\ &= \mathbf{O} \end{aligned}$$

$T(k\mathbf{u} + c\mathbf{v}) = \mathbf{O}$ therefore $(k\mathbf{u} + c\mathbf{v}) \in \ker(T)$.

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5. We need to prove if $\mathbf{v} \in \ker(T)$ then $-\mathbf{v} \in \ker(T)$.

Proof.

Let $\mathbf{v} \in \ker(T)$ then $T(-\mathbf{v}) = -1T(\mathbf{v}) = -1 \times \mathbf{O} = \mathbf{O}$. Thus $-\mathbf{v} \in \ker(T)$.

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6. We need to prove that if $T(\mathbf{v}) = \mathbf{O}$ for all $\mathbf{v}$ in the domain V then $\ker(T) = V$.

Proof.

Suppose there exists a vector $\mathbf{u} \in V$ such that $\mathbf{u}$ is not in $\ker(T)$. Since we are given that $T(\mathbf{v}) = \mathbf{O}$ for all $\mathbf{v}$ therefore $T(\mathbf{u}) = \mathbf{O}$ because $\mathbf{u} \in V$. This means that $\mathbf{u} \in \ker(T)$. This contradicts that $\mathbf{u} \notin \ker(T)$. Thus there is **no** such $\mathbf{u} \in V$ where $\mathbf{u} \notin \ker(T)$ which means that $\ker(T) = V$.

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7. We need to prove for the identity linear transformation $T(\mathbf{v}) = \mathbf{v}$ that

(a) $\ker(T) = \{\mathbf{O}\}$

Proof.

Suppose there exists $\mathbf{u} \neq \mathbf{O}$ in $\ker(T)$. Since **u** is in $\ker(T)$ therefore

$$T(\mathbf{u}) = \mathbf{O}$$

We are given the identity linear transformation therefore $T(\mathbf{u}) = \mathbf{u} \neq \mathbf{O}$. We have a contradiction because $T(\mathbf{u}) = \mathbf{O}$ and $T(\mathbf{u}) \neq \mathbf{O}$ which means there is **no** such $\mathbf{u} \neq \mathbf{O}$ in $\ker(T)$. Thus $\ker(T) = \{\mathbf{O}\}$.

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(b) $\text{range}(T) = V$

Proof.

Suppose $\text{range}(T) \neq V$ then there exists a vector **u** in *V* such that $T(\mathbf{u}) \in W$ but $T(\mathbf{u}) \notin V$. The given transformation is the identity transformation therefore

$$T(\mathbf{u}) = \mathbf{u} \in V$$

We have a contradiction because $T(\mathbf{u}) \notin V$ and $T(\mathbf{u}) \in V$ which means that our supposition is wrong and so $\text{range}(T) = V$.

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8. We need to prove if $T(\mathbf{u}) = \mathbf{x}$ and $T(\mathbf{v}) = \mathbf{x}$ then $(\mathbf{u} - \mathbf{v}) \in \ker(T)$.

Proof.

Consider the vector $\mathbf{u} - \mathbf{v}$. Since *T* is a linear transformation therefore

$$\begin{aligned} T(\mathbf{u} - \mathbf{v}) &= T(\mathbf{u}) - T(\mathbf{v}) \\ &= \mathbf{x} - \mathbf{x} = \mathbf{O} \end{aligned}$$

We have $T(\mathbf{u} - \mathbf{v}) = \mathbf{O}$ which means that $(\mathbf{u} - \mathbf{v}) \in \ker(T)$.

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9. Required to prove that if $S_1 = \{\mathbf{v}_1,\ \mathbf{v}_2,\ \cdots,\ \mathbf{v}_n\}$ spans the domain *V* then $S_2 = \{T(\mathbf{v}_1),\ T(\mathbf{v}_2),\ \cdots,\ T(\mathbf{v}_n)\}$ spans $\text{range}(T)$.

Proof.

Let $T(\mathbf{u})$ be an arbitrary vector in $\text{range}(T)$. We have $T(\mathbf{u}) \in \text{range}(T)$ therefore **u** is in the domain *V*.

We are given that $S_1 = \{\mathbf{v}_1,\ \mathbf{v}_2,\ \cdots,\ \mathbf{v}_n\}$ spans *V* therefore we can write the vector **u** as a linear combination of $\mathbf{v}_1,\ \mathbf{v}_2,\ \cdots,$ and $\mathbf{v}_n$. We have

$$\begin{aligned} \mathbf{u} &= k_1\mathbf{v}_1 + k_2\mathbf{v}_2 + \cdots + k_n\mathbf{v}_n && [k\text{'s are scalars}] \\ T(\mathbf{u}) &= T(k_1\mathbf{v}_1 + k_2\mathbf{v}_2 + \cdots + k_n\mathbf{v}_n) \\ &= k_1T(\mathbf{v}_1) + k_2T(\mathbf{v}_2) + \cdots + k_nT(\mathbf{v}_n) && [\text{Because } T \text{ is linear}] \end{aligned}$$

This means that $S_2 = \{T(\mathbf{v}_1),\ T(\mathbf{v}_2),\ \cdots,\ T(\mathbf{v}_n)\}$ spans $T(\mathbf{u})$ which is in $\text{range}(T)$. Since $T(\mathbf{u})$ was an arbitrary vector in $\text{range}(T)$ therefore

$$S_2 = \{T(\mathbf{v}_1),\ T(\mathbf{v}_2),\ \cdots,\ T(\mathbf{v}_n)\}$$

spans $\text{range}(T)$.

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10. We need to prove that if $B = \{\mathbf{v}_1,\ \mathbf{v}_2,\ \cdots,\ \mathbf{v}_n\}$ is a basis for the domain V then $S = \{T(\mathbf{v}_1),\ T(\mathbf{v}_2),\ \cdots,\ T(\mathbf{v}_n)\}$ is a basis for $\text{range}(T)$.

Proof.

Since T is a linear transformation therefore by Proposition (5-1) part (a) we have $T(\mathbf{O}) = \mathbf{O}$.

We can write zero vector $\mathbf{O}$ in V as a linear combination of the basis vectors $B = \{\mathbf{v}_1,\ \mathbf{v}_2,\ \cdots,\ \mathbf{v}_n\}$:

$$k_1\mathbf{v}_1 + k_2\mathbf{v}_2 + \cdots + k_n\mathbf{v}_n = \mathbf{O}$$

where **all** the scalars, k's, are zero. *Why?*

Because $B = \{\mathbf{v}_1,\ \mathbf{v}_2,\ \cdots,\ \mathbf{v}_n\}$ is a basis for V therefore these vectors are linearly independent which means **all** the scalars are zero. Consider $T(\mathbf{O})$:

$$\begin{aligned} T(\mathbf{O}) &= T(k_1\mathbf{v}_1 + k_2\mathbf{v}_2 + \cdots + k_n\mathbf{v}_n) \\ &= k_1T(\mathbf{v}_1) + k_2T(\mathbf{v}_2) + \cdots + k_nT(\mathbf{v}_n) \qquad [\text{Because } T \text{ is linear}] \\ &= \mathbf{O} \end{aligned}$$

Since **all** scalars k's are zero therefore the set of vectors

$$S = \{T(\mathbf{v}_1),\ T(\mathbf{v}_2),\ \cdots,\ T(\mathbf{v}_n)\}$$

are linearly independent in $\text{range}(T)$. *What else do we need to show for the set S to form a basis for range(T)?*

They span $\text{range}(T)$. We have already established this in question 9. Thus the set S forms a basis for $\text{range}(T)$.

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11. We have to prove that if S is a subspace of V then $T(S)$ is a subspace of $\text{range}(T)$.

How do we prove this result?

By using Proposition (3-5) which says:

A non-empty subset S with vectors $\mathbf{u}$ and $\mathbf{v}$ is a subspace $\Leftrightarrow$ $k\mathbf{u} + c\mathbf{v}$ is also in S.

Proof.

Let set S be a subspace of V. We are given that T is linear which means we have $T(\mathbf{O}) = \mathbf{O}$ thus $\mathbf{O} \in T(S)$.

Since $T(S)$ is non-empty (because $\mathbf{O} \in T(S)$) let $\mathbf{u}$ and $\mathbf{v}$ be vectors in $T(S)$ and consider the vector

$$k\mathbf{u} + c\mathbf{v}$$

What do we need to show for $T(S)$ to be a subspace of range(T)?

Required to prove that $k\mathbf{u} + c\mathbf{v}$ is in $T(S)$.

Since $\mathbf{u}$ and $\mathbf{v}$ are in $T(S)$ therefore there exists vectors $\mathbf{v}_1 \in S$ and $\mathbf{v}_2 \in S$ such that

$$T(\mathbf{v}_1) = \mathbf{u} \text{ and } T(\mathbf{v}_2) = \mathbf{v}$$

We have

$$\begin{aligned} k\mathbf{u} + c\mathbf{v} &= kT(\mathbf{v}_1) + cT(\mathbf{v}_2) \\ &= T(k\mathbf{v}_1 + c\mathbf{v}_2) \end{aligned}$$

We are given that *S* is a subspace therefore by Proposition (3-5) we can say $k\mathbf{v}_1 + c\mathbf{v}_2$ is also in *S*. This means that $k\mathbf{u} + c\mathbf{v}$ is in $T(S)$. Hence we conclude that $T(S)$ is a subspace of $\text{range}(T)$.

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