Complete Solutions to Exercises **1.8**

1. Matrices **B**, **C**, **D** and **G** are elementary matrices.

Matrix **B** is an elementary matrix because we can produce matrix **B** by multiplying the first row of the identity matrix by -1.

Matrix **C** is an elementary matrix because it can be obtained by swapping the first two rows of the identity matrix.

Matrix **D** is an elementary matrix because we can carry out the single row operation of adding 3 times the first row to the second row of the identity matrix.

Matrix **G** is an elementary matrix because we can carry out the single row operation of interchanging the first and third rows of the identity matrix.

Matrices **A**, **E**, **F** and **H** are not elementary matrices.

Matrix **A** is **not** an elementary matrix because to produce the matrix **A** we need three row operations, that is multiply each row of the identity matrix by 3.

Matrix **E** is **not** an elementary matrix because to produce the matrix **E** we need two row operations on the identity matrix which are interchange the first 2 rows and multiply the last row by 2.

Matrix **F** is not an elementary matrix because to obtain matrix **F** from the identity matrix you need two operations. First multiply the first row by -1 and then subtract the last row from the first row.

Matrix **H** is not an elementary matrix because row operations dictates that we can only multiply a row by a non-zero constant. In this case the third row of the identity has been multiplied by zero.

2. *How do we find the inverse matrix in each case?*

We can write down the augmented matrix $\left(\mathbf{E} \mid \mathbf{I}\right)$ and convert this to $\left(\mathbf{I} \mid \mathbf{E}^{-1}\right)$ by elementary row operations or we can examine the given matrix and apply the inverse.

(a) Writing $\mathbf{E}_1 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$ into $\left(\mathbf{E}_1 \mid \mathbf{I}\right)$ form and labelling the rows we have

$$\begin{matrix} \mathrm{R}_1 \\ \mathrm{R}_2 \end{matrix} \left(\begin{array}{cc|cc} -1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{array}\right)$$

To get the identity on the left of the vertical line we need to multiply the first row by -1:

$$\begin{matrix} \mathrm{R}_1^* = -\mathrm{R}_1 \\ \mathrm{R}_2 \end{matrix} \left(\begin{array}{cc|cc} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 1 \end{array}\right)$$

Hence the inverse matrix is the matrix with the entries on the Right Hand Side of the vertical line. That is $\mathbf{E}_1^{-1} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$. In this case $\mathbf{E}_1 = \mathbf{E}_1^{-1}$.

The elementary matrix $\mathbf{E}_1$ corresponds to multiplying row 1 of $\mathbf{I}_2$ by -1. The inverse operation is also multiply row 1 by -1.

(b) Writing $\mathbf{E}_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ into $\left(\mathbf{E}_2 \mid \mathbf{I}\right)$ form and labelling the rows we have

$$\begin{matrix} \mathrm{R}_1 \\ \mathrm{R}_2 \end{matrix} \left(\begin{array}{cc|cc} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{array}\right)$$

Interchange the rows:

$$\begin{array}{r} R_1^* = R_2 \\ R_2^* = R_1 \end{array} \left(\begin{array}{cc|cc} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{array}\right)$$

Hence the inverse matrix is the entries on the right, $\mathbf{E}_2^{-1} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. Again $\mathbf{E}_2 = \mathbf{E}_2^{-1}$.

The elementary matrix $\mathbf{E}_2$ corresponds to swapping rows 1 and 2 of $\mathbf{I}_2$. The inverse of this is swapping rows 1 and 2 again.

(c) Writing $\mathbf{E}_3 = \begin{pmatrix} 1 & 0 \\ 0 & -2 \end{pmatrix}$ into $\left(\mathbf{E}_3 \mid \mathbf{I}\right)$ form and labelling the rows we have

$$\begin{array}{r} R_1 \\ R_2 \end{array} \left(\begin{array}{cc|cc} 1 & 0 & 1 & 0 \\ 0 & -2 & 0 & 1 \end{array}\right)$$

Multiplying the bottom row by $-\dfrac{1}{2}$ gives

$$\begin{array}{r} R_1 \\ R_2^\dagger = -\dfrac{1}{2} R_2 \end{array} \left(\begin{array}{cc|cc} 1 & 0 & 1 & 0 \\ 0 & -\dfrac{1}{2}\times(-2) & 0 & -\dfrac{1}{2}\times 1 \end{array}\right)$$

Simplifying the entries in the bottom row gives

$$\begin{array}{r} R_1 \\ R_2^\dagger \end{array} \left(\begin{array}{cc|cc} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & -1/2 \end{array}\right)$$

Hence $\mathbf{E}_3^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & -1/2 \end{pmatrix}$. In this case $\mathbf{E}_3^{-1}$ does **not** equal $\mathbf{E}_3$.

The elementary matrix $\mathbf{E}_3$ corresponds to multiply row 2 by -2. The inverse operation is to multiply row 2 by $-1/2$.

(d) Writing $\mathbf{E}_4 = \begin{pmatrix} -5 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ into $\left(\mathbf{E}_4 \mid \mathbf{I}\right)$ form and labelling the rows we have

$$\begin{array}{r} R_1 \\ R_2 \\ R_3 \end{array} \left(\begin{array}{ccc|ccc} -5 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right)$$

Multiplying the first row by $-\dfrac{1}{5}$ gives

$$\begin{array}{r} R_1^* = -\dfrac{1}{5} R_1 \\ R_2 \\ R_3 \end{array} \left(\begin{array}{ccc|ccc} -\dfrac{1}{5}(-5) & 0 & 0 & -\dfrac{1}{5}(1) & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right)$$

Simplifying the entries in the top row gives

$$\begin{array}{r} R_1^* \\ R_2 \\ R_3 \end{array} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -1/5 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right)$$

Hence $\mathbf{E}_4^{-1} = \begin{pmatrix} -1/5 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$.

$\mathbf{E}_4$ corresponds to multiply row 1 by -5. The inverse operation is multiply row 1 by $-1/5$

(e) Similarly for matrix $\mathbf{E}_5 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -\sqrt{2} & 0 \\ 0 & 0 & 1 \end{pmatrix}$ we have the row operation of multiplying row 2 of $\mathbf{I}_3$ by $-\sqrt{2}$ therefore the inverse row operation is to multiply row 2 of $\mathbf{I}_3$ by $-\dfrac{1}{\sqrt{2}}$. Hence

$\mathbf{E}_5^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1/\sqrt{2} & 0 \\ 0 & 0 & 1 \end{pmatrix}$.

(f) Similarly the inverse of $\mathbf{E}_6 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \pi \end{pmatrix}$ (multiplied row 3 of $\mathbf{I}_3$ by π) is

$$\mathbf{E}_6^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1/\pi \end{pmatrix} \qquad [\text{multiplied row 3 of } \mathbf{I}_3 \text{ by } 1/\pi]$$

3. Let $\mathbf{A} = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$.

(a) The elementary matrix $\mathbf{E} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ is obtained from the identity matrix by multiplying the middle row by -1 therefore the matrix multiplication **EA** produces the same row operation, that is it multiplies the middle row by -1:

$$\mathbf{EA} = \begin{pmatrix} a & b & c \\ -d & -e & -f \\ g & h & i \end{pmatrix}$$

(b) The elementary matrix $\mathbf{E} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$ is produced from the identity matrix by swapping the top and bottom rows therefore the matrix multiplication **EA** gives the same effect of interchanging these rows:

$$\mathbf{EA} = \begin{pmatrix} g & h & i \\ d & e & f \\ a & b & c \end{pmatrix}$$

(c) The elementary matrix $\mathbf{E}=\begin{pmatrix} k & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ is obtained from the identity matrix by multiplying the top row by k $(k \neq 0)$ therefore the matrix multiplication **EA** gives the general 3×3 matrix with the top row multiplied by k:

$$\mathbf{EA}=\begin{pmatrix} ka & kb & kc \\ d & e & f \\ g & h & i \end{pmatrix}$$

(d) Similarly for the elementary matrix $\mathbf{E}=\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1/k \end{pmatrix}$ we have

$$\mathbf{EA}=\begin{pmatrix} a & b & c \\ d & e & f \\ -g/k & -h/k & -i/k \end{pmatrix}$$

4. (a) We write $\mathbf{A}=\begin{pmatrix} 1 & 2 \\ -1 & 4 \end{pmatrix}$ into $\left(\mathbf{A} \mid \mathbf{I}\right)$ and convert this to $\left(\mathbf{I} \mid \mathbf{A}^{-1}\right)$ by elementary row operations.

$$\begin{array}{c} \mathrm{R}_1 \\ \mathrm{R}_2 \end{array}\left(\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ -1 & 4 & 0 & 1 \end{array}\right)$$

Carrying out the row operation $\mathrm{R}_2+\mathrm{R}_1$ gives

$$\begin{array}{c} \mathrm{R}_1 \\ \mathrm{R}_2^{\$}=\mathrm{R}_2+\mathrm{R}_1 \end{array}\left(\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ -1+1 & 4+2 & 0+1 & 1+0 \end{array}\right)$$

Simplifying the bottom row

$$\begin{array}{c} \mathrm{R}_1 \\ \mathrm{R}_2^{\$} \end{array}\left(\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 0 & 6 & 1 & 1 \end{array}\right)$$

We need to convert the entry 2 to 0 by executing the following row operation:

$$\begin{array}{c} \mathrm{R}_1^{*}=3\mathrm{R}_1-\mathrm{R}_2^{\$} \\ \mathrm{R}_2^{\$} \end{array}\left(\begin{array}{cc|cc} 3(1)-0 & 3(2)-6 & 3(1)-1 & 3(0)-1 \\ 0 & 6 & 1 & 1 \end{array}\right)$$

Simplifying the arithmetic in the top row gives:

$$\begin{array}{c} \mathrm{R}_1^{*} \\ \mathrm{R}_2^{\$} \end{array}\left(\begin{array}{cc|cc} 3 & 0 & 2 & -1 \\ 0 & 6 & 1 & 1 \end{array}\right)$$

Dividing the top row by 3 and the bottom row by 6 gives

$$\begin{array}{c} \mathrm{R}_1^{*}/3 \\ \mathrm{R}_2^{\$}/6 \end{array}\left(\begin{array}{cc|cc} 1 & 0 & 2/3 & -1/3 \\ 0 & 1 & 1/6 & 1/6 \end{array}\right)$$

Hence the inverse matrix are the entries on the Right Hand Side of the vertical line because we have $\left(\mathbf{I} \mid \mathbf{A}^{-1}\right)$, hence $\mathbf{A}^{-1}=\begin{pmatrix} 2/3 & -1/3 \\ 1/6 & 1/6 \end{pmatrix}=\frac{1}{6}\begin{pmatrix} 4 & -2 \\ 1 & 1 \end{pmatrix}$.

(b) Similarly for the matrix $\mathbf{B}=\begin{pmatrix}2 & -5\\ -6 & 1\end{pmatrix}$ we have

$$\begin{array}{c} \mathrm{R}_1 \\ \mathrm{R}_2 \end{array}\left(\begin{array}{cc|cc} 2 & -5 & 1 & 0\\ -6 & 1 & 0 & 1\end{array}\right)$$

We need to transform the entries on the left of the vertical line into the identity matrix. To convert -6 to 0 we carry out the following row operation:

$$\begin{array}{c} \mathrm{R}_1 \\ \mathrm{R}_2^*=\mathrm{R}_2+3\mathrm{R}_1 \end{array}\left(\begin{array}{cc|cc} 2 & -5 & 1 & 0\\ -6+3(2) & 1+3(-5) & 0+3(1) & 1+3(0)\end{array}\right)$$

This simplifies to

$$\begin{array}{c} \mathrm{R}_1 \\ \mathrm{R}_2^* \end{array}\left(\begin{array}{cc|cc} 2 & -5 & 1 & 0\\ 0 & -14 & 3 & 1\end{array}\right)$$

Need to convert -5 to 0. *How?*

We carry out the row operation $\mathrm{R}_1-\frac{5}{14}\mathrm{R}_2^*$:

$$\begin{array}{c} \mathrm{R}_1^*=\mathrm{R}_1-\frac{5}{14}\mathrm{R}_2^* \\ \mathrm{R}_2^* \end{array}\left(\begin{array}{cc|cc} 2-\frac{5}{14}(0) & -5-\frac{5}{14}(-14) & 1-\frac{5}{14}(3) & 0-\frac{5}{14}(1)\\ 0 & -14 & 3 & 1\end{array}\right)$$

Simplifying the entries in the top row we have

$$\begin{array}{c} \mathrm{R}_1^* \\ \mathrm{R}_2^* \end{array}\left(\begin{array}{cc|cc} 2 & 0 & -1/14 & -5/14\\ 0 & -14 & 3 & 1\end{array}\right)$$

Dividing the top row by 2 and the bottom row by -14 we have

$$\begin{array}{c} \mathrm{R}_1^*/2 \\ -\mathrm{R}_2^*/14 \end{array}\left(\begin{array}{cc|cc} 1 & 0 & -1/28 & -5/28\\ 0 & 1 & -3/14 & -1/14\end{array}\right) \qquad \left[\text{We have } \left(\mathbf{I}\mid\mathbf{B}^{-1}\right)\right]$$

Remember the inverse matrix are the entries on the Right Hand Side of the vertical line,

$$\mathbf{B}^{-1}=\begin{pmatrix}-1/28 & -5/28\\ -3/14 & -1/14\end{pmatrix}=-\frac{1}{28}\begin{pmatrix}1 & 5\\ 6 & 2\end{pmatrix}.$$

(c) Writing $\mathbf{C}=\begin{pmatrix}1 & 0 & 2\\ 2 & 3 & 1\\ 3 & 6 & 0\end{pmatrix}$ into $\left(\mathbf{C}\mid\mathbf{I}\right)$ form and labelling the rows we have

$$\begin{array}{c} \mathrm{R}_1 \\ \mathrm{R}_2 \\ \mathrm{R}_3 \end{array}\left(\begin{array}{ccc|ccc} 1 & 0 & 2 & 1 & 0 & 0\\ 2 & 3 & 1 & 0 & 1 & 0\\ 3 & 6 & 0 & 0 & 0 & 1\end{array}\right)$$

Converting 2 and 3 into 0's in the first column we execute the following row operations:

$$\begin{array}{c} \mathrm{R}_1 \\ \mathrm{R}_2^\dagger=\mathrm{R}_2-2\mathrm{R}_1 \\ \mathrm{R}_3^\dagger=\mathrm{R}_3-3\mathrm{R}_1 \end{array}\left(\begin{array}{ccc|ccc} 1 & 0 & 2 & 1 & 0 & 0\\ 2-2(1) & 3-2(0) & 1-2(2) & 0-2(1) & 1-2(0) & 0-2(0)\\ 3-3(1) & 6-3(0) & 0-3(2) & 0-3(1) & 0-3(0) & 1-3(0)\end{array}\right)$$

Simplifying the entries gives

$$\begin{array}{c} R_1 \\ R_2^{\dagger} \\ R_3^{\dagger} \end{array} \left(\begin{array}{ccc|ccc} 1 & 0 & 2 & 1 & 0 & 0 \\ 0 & 3 & -3 & -2 & 1 & 0 \\ 0 & 6 & -6 & -3 & 0 & 1 \end{array}\right)$$

To convert 6 in the bottom row to 0 we carry out the row operation $R_3^{\dagger} - 2R_2^{\dagger}$:

$$\begin{array}{c} R_1 \\ R_2^{\dagger} \\ R_3^{*} = R_3^{\dagger} - 2R_2^{\dagger} \end{array} \left(\begin{array}{ccc|ccc} 1 & 0 & 2 & 1 & 0 & 0 \\ 0 & 3 & -3 & -2 & 1 & 0 \\ 0 & 6-2(3) & -6-2(-3) & -3-2(-2) & 0-2(1) & 1-2(0) \end{array}\right)$$

Simplifying the entries in the bottom row we have

$$\begin{array}{c} R_1 \\ R_2^{\dagger} \\ R_3^{*} \end{array} \left(\begin{array}{ccc|ccc} 1 & 0 & 2 & 1 & 0 & 0 \\ 0 & 3 & -3 & -2 & 1 & 0 \\ 0 & 0 & 0 & 1 & -2 & 1 \end{array}\right)$$

What can we conclude from this result?

Since we have a row of zeros on the Left Hand Side of the vertical bar therefore we conclude that the matrix **C** is non-invertible or singular, that is the matrix **C** does **not** have an inverse.

(d) We are given $\mathbf{D} = \begin{pmatrix} 1 & -1 & 1 \\ 1 & 0 & -1 \\ 0 & 0 & -1 \end{pmatrix}$. Writing $(\mathbf{D} \mid \mathbf{I})$ form and labelling the rows gives

$$\begin{array}{c} R_1 \\ R_2 \\ R_3 \end{array} \left(\begin{array}{ccc|ccc} 1 & -1 & 1 & 1 & 0 & 0 \\ 1 & 0 & -1 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 & 0 & 1 \end{array}\right)$$

Carrying out the row operations $R_2 - R_1$ and $-R_3$ gives

$$\begin{array}{c} R_1 \\ R_2^{\$} = R_2 - R_1 \\ R_3^{\$} = -R_3 \end{array} \left(\begin{array}{ccc|ccc} 1 & -1 & 1 & 1 & 0 & 0 \\ 1-1 & 0-(-1) & -1-1 & 0-1 & 1-0 & 0-0 \\ 0 & 0 & 1 & 0 & 0 & -1 \end{array}\right)$$

We have

$$\begin{array}{c} R_1 \\ R_2^{\$} \\ R_3^{\$} \end{array} \left(\begin{array}{ccc|ccc} 1 & -1 & 1 & 1 & 0 & 0 \\ 0 & 1 & -2 & -1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & -1 \end{array}\right)$$

Remember we are trying to transform the entries in the Left Hand of the vertical line into the identity matrix so therefore we carry out the following row operation:

$$\begin{array}{c} R_1^{\$} = R_1 + R_2^{\$} \\ R_2^{\$} \\ R_3^{\$} \end{array} \left(\begin{array}{ccc|ccc} 1+0 & -1+1 & 1-2 & 1-1 & 0+1 & 0+0 \\ 0 & 1 & -2 & -1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & -1 \end{array}\right)$$

Simplifying the entries in the top row gives

$$\begin{array}{c} R_1^{\$} \\ R_2^{\$} \\ R_3^{\$} \end{array} \left(\begin{array}{ccc|ccc} 1 & 0 & -1 & 0 & 1 & 0 \\ 0 & 1 & -2 & -1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & -1 \end{array}\right)$$

Just need 0's in place of the -1 and -2 in the entries to the left of the vertical line.

$$\begin{array}{c} R_1^{\$\$}=R_1^{\$}+R_3^{\$} \\ R_2^{\$\$}=R_2^{\$}+2R_3^{\$} \\ R_3^{\$} \end{array}\left(\begin{array}{ccc|ccc} 1+0 & 0+0 & -1+1 & 0+0 & 1+0 & 0-1 \\ 0+2(0) & 1+2(0) & -2+2(1) & -1+2(0) & 1+2(0) & 0+2(-1) \\ 0 & 0 & 1 & 0 & 0 & -1 \end{array}\right)$$

Simplifying the entries gives

$$\begin{array}{c} R_1^{\$\$} \\ R_2^{\$\$} \\ R_3^{\$} \end{array}\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 1 & -1 \\ 0 & 1 & 0 & -1 & 1 & -2 \\ 0 & 0 & 1 & 0 & 0 & -1 \end{array}\right) \qquad \left[\text{Because } \left(\mathbf{I} \mid \mathbf{D}^{-1}\right)\right]$$

What is the inverse matrix of $\mathbf{D}$*?*

It is the entries on the right to the vertical line, that is $\mathbf{D}^{-1}=\begin{pmatrix} 0 & 1 & -1 \\ -1 & 1 & -2 \\ 0 & 0 & -1 \end{pmatrix}$.

(e) Similarly we find the inverse of $\mathbf{E}=\begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}$ by writing it out in $\left(\mathbf{E} \mid \mathbf{I}\right)$ form and converting this into $\left(\mathbf{I} \mid \mathbf{E}^{-1}\right)$ by elementary row operations.

$$\begin{array}{c} R_1 \\ R_2 \\ R_3 \end{array}\left(\begin{array}{ccc|ccc} 2 & -1 & 0 & 1 & 0 & 0 \\ -1 & 2 & -1 & 0 & 1 & 0 \\ 0 & -1 & 2 & 0 & 0 & 1 \end{array}\right)$$

We convert the first -1 in the second row into 0 by executing $2R_2+R_1$:

$$\begin{array}{c} R_1 \\ R_2^*=2R_2+R_1 \\ R_3 \end{array}\left(\begin{array}{ccc|ccc} 2 & -1 & 0 & 1 & 0 & 0 \\ 2(-1)+2 & 2(2)-1 & 2(-1)+0 & 2(0)+1 & 2(1)+0 & 2(0)+0 \\ 0 & -1 & 2 & 0 & 0 & 1 \end{array}\right)$$

Simplifying the entries gives

$$\begin{array}{c} R_1 \\ R_2^* \\ R_3 \end{array}\left(\begin{array}{ccc|ccc} 2 & -1 & 0 & 1 & 0 & 0 \\ 0 & 3 & -2 & 1 & 2 & 0 \\ 0 & -1 & 2 & 0 & 0 & 1 \end{array}\right)$$

We need to transform the -2 in the middle row into 0 by carrying out the row operation $R_2^*+R_3$:

$$\begin{array}{c} R_1 \\ R_2^{**}=R_2^*+R_3 \\ R_3 \end{array}\left(\begin{array}{ccc|ccc} 2 & -1 & 0 & 1 & 0 & 0 \\ 0+0 & 3-1 & -2+2 & 1+0 & 2+0 & 0+1 \\ 0 & -1 & 2 & 0 & 0 & 1 \end{array}\right)$$

We have

$$\begin{array}{c} R_1 \\ R_2^{**} \\ R_3 \end{array}\left(\begin{array}{ccc|ccc} 2 & -1 & 0 & 1 & 0 & 0 \\ 0 & 2 & 0 & 1 & 2 & 1 \\ 0 & -1 & 2 & 0 & 0 & 1 \end{array}\right)$$

Next we convert the -1 in the top row to 0.

$$\begin{array}{c} R_1^* = 2R_1 + R_2^{**} \\ R_2^{**} \\ R_3 \end{array} \left(\begin{array}{ccc|ccc} 2(2)+0 & 2(-1)+2 & 0 & 2(1)+1 & 2(0)+2 & 2(0)+1 \\ 0 & 2 & 0 & 1 & 2 & 1 \\ 0 & -1 & 2 & 0 & 0 & 1 \end{array}\right)$$

Simplifying the entries gives

$$\begin{array}{c} R_1^* \\ R_2^{**} \\ R_3 \end{array} \left(\begin{array}{ccc|ccc} 4 & 0 & 0 & 3 & 2 & 1 \\ 0 & 2 & 0 & 1 & 2 & 1 \\ 0 & -1 & 2 & 0 & 0 & 1 \end{array}\right)$$

Need to convert the -1 in the bottom row to 0.

$$\begin{array}{c} R_1^* \\ R_2^{**} \\ R_3^* = 2R_3 + R_2^{**} \end{array} \left(\begin{array}{ccc|ccc} 4 & 0 & 0 & 3 & 2 & 1 \\ 0 & 2 & 0 & 1 & 2 & 1 \\ 0 & 2(-1)+2 & 2(2)+0 & 2(0)+1 & 2(0)+2 & 2(1)+1 \end{array}\right)$$

We have

$$\begin{array}{c} R_1^* \\ R_2^{**} \\ R_3^* \end{array} \left(\begin{array}{ccc|ccc} 4 & 0 & 0 & 3 & 2 & 1 \\ 0 & 2 & 0 & 1 & 2 & 1 \\ 0 & 0 & 4 & 1 & 2 & 3 \end{array}\right)$$

Multiply the middle row by 2:

$$\begin{array}{c} R_1^* \\ 2R_2^{**} \\ R_3^* \end{array} \left(\begin{array}{ccc|ccc} 4 & 0 & 0 & 3 & 2 & 1 \\ 0 & 4 & 0 & 2 & 4 & 2 \\ 0 & 0 & 4 & 1 & 2 & 3 \end{array}\right)$$

If we divide this augmented matrix by 4 then we obtain the identity on the left and the inverse on the right. Thus dividing through by 4 or taking out a ¼ gives the inverse $\mathbf{E}^{-1} = \frac{1}{4}\begin{pmatrix} 3 & 2 & 1 \\ 2 & 4 & 2 \\ 1 & 2 & 3 \end{pmatrix}$.

(f) Writing $\mathbf{F} = \begin{pmatrix} 1 & 3 & 4 \\ -1 & 1 & 1 \\ 2 & 1 & -2 \end{pmatrix}$ into $(\mathbf{F} \mid \mathbf{I})$ form and converting this into $(\mathbf{I} \mid \mathbf{F}^{-1})$ by elementary row operations gives the inverse matrix $\mathbf{F}^{-1}$.

$$\begin{array}{c} R_1 \\ R_2 \\ R_3 \end{array} \left(\begin{array}{ccc|ccc} 1 & 3 & 4 & 1 & 0 & 0 \\ -1 & 1 & 1 & 0 & 1 & 0 \\ 2 & 1 & -2 & 0 & 0 & 1 \end{array}\right)$$

We execute the following row operations

$$\begin{array}{c} R_1 \\ R_2^\dagger = R_2 + R_1 \\ R_3^\dagger = R_3 - 2R_1 \end{array} \left(\begin{array}{ccc|ccc} 1 & 3 & 4 & 1 & 0 & 0 \\ -1+1 & 1+3 & 1+4 & 0+1 & 1+0 & 0+0 \\ 2-2(1) & 1-2(3) & -2-2(4) & 0-2(1) & 0-2(0) & 1-2(0) \end{array}\right)$$

Simplifying the arithmetic gives

$$\begin{array}{c} R_1 \\ R_2^\dagger \\ R_3^\dagger \end{array} \left(\begin{array}{ccc|ccc} 1 & 3 & 4 & 1 & 0 & 0 \\ 0 & 4 & 5 & 1 & 1 & 0 \\ 0 & -5 & -10 & -2 & 0 & 1 \end{array}\right)$$

What are we trying to achieve?
The identity matrix on left of the vertical line. *How do we convert the* -5 *in the bottom row to 0?*

Carry out the row operation $R_3^{\dagger}+\frac{5}{4}R_2^{\dagger}$:

$$\begin{array}{r} R_1 \\ R_2^{\dagger} \\ R_3^{\dagger\dagger}=R_3^{\dagger}+\frac{5}{4}R_2^{\dagger} \end{array}\left(\begin{array}{ccc|ccc} 1 & 3 & 4 & 1 & 0 & 0 \\ 0 & 4 & 5 & 1 & 1 & 0 \\ 0 & -5+\frac{5}{4}(4) & -10+\frac{5}{4}(5) & -2+\frac{5}{4}(1) & 0+\frac{5}{4}(1) & 1+\frac{5}{4}(0) \end{array}\right)$$

Simplifying the entries gives

$$\begin{array}{r} R_1 \\ R_2^{\dagger} \\ R_3^{\dagger\dagger} \end{array}\left(\begin{array}{ccc|ccc} 1 & 3 & 4 & 1 & 0 & 0 \\ 0 & 4 & 5 & 1 & 1 & 0 \\ 0 & 0 & -15/4 & -3/4 & 5/4 & 1 \end{array}\right)$$

Multiply the last row by $-\frac{4}{15}$ so that we get a 1 in place of $-\frac{15}{4}$:

$$\begin{array}{r} R_1 \\ R_2^{\dagger} \\ R_3^{\$}=-\frac{4}{15}R_3^{\dagger\dagger} \end{array}\left(\begin{array}{ccc|ccc} 1 & 3 & 4 & 1 & 0 & 0 \\ 0 & 4 & 5 & 1 & 1 & 0 \\ 0 & 0 & -\frac{4}{15}\left(-\frac{15}{4}\right) & -\frac{3}{4}\left(-\frac{4}{15}\right) & \frac{5}{4}\left(-\frac{4}{15}\right) & -\frac{4}{15} \end{array}\right)$$

Simplifying the entries in the last row gives

$$\begin{array}{r} R_1 \\ R_2^{\dagger} \\ R_3^{\$} \end{array}\left(\begin{array}{ccc|ccc} 1 & 3 & 4 & 1 & 0 & 0 \\ 0 & 4 & 5 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1/5 & -1/3 & -4/15 \end{array}\right)$$

Need to convert the 4 in the first row and the 5 in the second row into 0's. *How?*
We execute the row operations $R_1-4R_3^{\$}$ and $R_2^{\dagger}-5R_3^{\$}$:

$$\begin{array}{r} R_1^{*}=R_1-4R_3^{\$} \\ R_2^{\dagger\dagger}=R_2^{\dagger}-5R_3^{\$} \\ R_3^{\$} \end{array}\left(\begin{array}{ccc|ccc} 1 & 3 & 4-4(1) & 1-4(1/5) & 0-4(-1/3) & 0-4(-4/15) \\ 0 & 4 & 5-5(1) & 1-5(1/5) & 1-5(-1/3) & 0-5(-4/15) \\ 0 & 0 & 1 & 1/5 & -1/3 & -4/15 \end{array}\right)$$

Simplifying the entries:

$$\begin{array}{r} R_1^{*} \\ R_2^{\dagger\dagger} \\ R_3^{\$} \end{array}\left(\begin{array}{ccc|ccc} 1 & 3 & 0 & 1/5 & 4/3 & 16/15 \\ 0 & 4 & 0 & 0 & 8/3 & 4/3 \\ 0 & 0 & 1 & 1/5 & -1/3 & -4/15 \end{array}\right)$$

Need to convert the 4 in the middle row to 1. Divide middle row by 4:

$$\begin{array}{r} R_1^{*} \\ R_2^{*}=R_2^{\dagger\dagger}/4 \\ R_3^{\$} \end{array}\left(\begin{array}{ccc|ccc} 1 & 3 & 0 & 1/5 & 4/3 & 16/15 \\ 0 & 4/4 & 0 & 0 & 8/12 & 4/12 \\ 0 & 0 & 1 & 1/5 & -1/3 & -4/15 \end{array}\right)$$

Simplifying the entries gives

$$\begin{array}{c} R_1^* \\ R_2^* \\ R_3^\$ \end{array}\left(\begin{array}{ccc|ccc} 1 & 3 & 0 & 1/5 & 4/3 & 16/15 \\ 0 & 1 & 0 & 0 & 2/3 & 1/3 \\ 0 & 0 & 1 & 1/5 & -1/3 & -4/15 \end{array}\right)$$

Subtracting 3 times the second row from the first row gives a 0 in place of the 3 which results in the identity matrix on the Left Hand Side of the vertical line:

$$\begin{array}{r} R_1^{**} = R_1^* - 3R_2^* \\ R_2^* \\ R_3^\$ \end{array}\left(\begin{array}{ccc|ccc} 1 & 3-3(1) & 0 & 1/5 & 4/3-3(2/3) & 16/15-3(1/3) \\ 0 & 1 & 0 & 0 & 2/3 & 1/3 \\ 0 & 0 & 1 & 1/5 & -1/3 & -4/15 \end{array}\right)$$

Simplifying the entries gives

$$\begin{array}{c} R_1^{**} \\ R_2^* \\ R_3^\$ \end{array}\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1/5 & -2/3 & 1/15 \\ 0 & 1 & 0 & 0 & 2/3 & 1/3 \\ 0 & 0 & 1 & 1/5 & -1/3 & -4/15 \end{array}\right)$$

Hence the inverse matrix is the matrix with the entries on the Right Hand Side of the vertical line, $\mathbf{F}^{-1} = \begin{pmatrix} 1/5 & -2/3 & 1/15 \\ 0 & 2/3 & 1/3 \\ 1/5 & -1/3 & -4/15 \end{pmatrix} = \frac{1}{15}\begin{pmatrix} 3 & -10 & 1 \\ 0 & 10 & 5 \\ 3 & -5 & -4 \end{pmatrix}$.

(g) Spot that in the matrix $\mathbf{G} = \begin{pmatrix} -2 & 5 & 3 & 1 \\ -9 & 2 & -5 & 6 \\ 2 & 4 & 8 & 16 \\ 4 & 8 & 16 & 32 \end{pmatrix}$ the bottom row is 2 times the third row.

What can we conclude about this matrix?

If we subtract 2 times the third row from the bottom row we will get a row of zeros which means that the matrix **G** will **not** have an inverse. First we write $(\mathbf{G} \mid \mathbf{I})$:

$$\begin{array}{c} R_1 \\ R_2 \\ R_3 \\ R_4 \end{array}\left(\begin{array}{cccc|cccc} -2 & 5 & 3 & 1 & 1 & 0 & 0 & 0 \\ -9 & 2 & -5 & 6 & 0 & 1 & 0 & 0 \\ 2 & 4 & 8 & 16 & 0 & 0 & 1 & 0 \\ 4 & 8 & 16 & 32 & 0 & 0 & 0 & 1 \end{array}\right)$$

Carrying out the row operation $R_4 - 2R_3$:

$$\begin{array}{r} R_1 \\ R_2 \\ R_3 \\ R_4^* = R_4 - 2R_3 \end{array}\left(\begin{array}{cccc|cccc} -2 & 5 & 3 & 1 & 1 & 0 & 0 & 0 \\ -9 & 2 & -5 & 6 & 0 & 1 & 0 & 0 \\ 2 & 4 & 8 & 16 & 0 & 0 & 1 & 0 \\ 4-2(2) & 8-2(4) & 16-2(8) & 32-2(16) & 0 & 0 & -2 & 1 \end{array}\right)$$

Simplifying the entries gives

$$\begin{array}{c} R_1 \\ R_2 \\ R_3 \\ R_4^* \end{array}\left(\begin{array}{cccc|cccc} -2 & 5 & 3 & 1 & 1 & 0 & 0 & 0 \\ -9 & 2 & -5 & 6 & 0 & 1 & 0 & 0 \\ 2 & 4 & 8 & 16 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -2 & 1 \end{array}\right)$$

Hence the matrix **G** is non-invertible or singular which means it does **not** have an inverse.

(h) We have $\mathbf{H}=\begin{pmatrix}1&2&-2&3\\-2&1&-5&-6\\-5&-10&9&-15\\-6&-12&27&-18\end{pmatrix}$ and write this into $(\mathbf{H}\mid\mathbf{I})$ form:

$$\begin{array}{c}R_1\\R_2\\R_3\\R_4\end{array}\left(\begin{array}{cccc|cccc}1&2&-2&3&1&0&0&0\\-2&1&-5&-6&0&1&0&0\\-5&-10&9&-15&0&0&1&0\\-6&-12&27&-18&0&0&0&1\end{array}\right)$$

Carrying out the elementary row operations:

$$\begin{array}{c}R_1\\R_2^*=R_2+2R_1\\R_3^*=R_3+5R_1\\R_4^*=R_4+6R_1\end{array}\left(\begin{array}{cccc|cccc}1&2&-2&3&1&0&0&0\\-2+2(1)&1+2(2)&-5+2(-2)&-6+2(3)&2&1&0&0\\-5+5(1)&-10+5(2)&9+5(-2)&-15+5(3)&5&0&1&0\\-6+6(1)&-12+6(2)&27+6(-2)&-18+6(3)&6&0&0&1\end{array}\right)$$

Simplifying the entries gives

$$\begin{array}{c}R_1\\R_2^*\\R_3^*\\R_4^*\end{array}\left(\begin{array}{cccc|cccc}1&2&-2&3&1&0&0&0\\0&5&-9&0&2&1&0&0\\0&0&-1&0&5&0&1&0\\0&0&15&0&6&0&0&1\end{array}\right)$$

If we carry out the row operation $15R_3^*+R_4^*$ then we will have a row of zeros:

$$\begin{array}{c}R_1\\R_2^*\\R_3^{**}=15R_3^*+R_4^*\\R_4^*\end{array}\left(\begin{array}{cccc|cccc}1&2&-2&3&1&0&0&0\\0&5&-9&0&2&1&0&0\\0&0&15(-1)+15&0&15(5)+6&0&15(1)+0&0\\0&0&15&0&6&0&0&1\end{array}\right)$$

Simplifying the arithmetic in the third row gives

$$\begin{array}{c}R_1\\R_2^*\\R_3^{**}\\R_4^*\end{array}\left(\begin{array}{cccc|cccc}1&2&-2&3&1&0&0&0\\0&5&-9&0&2&1&0&0\\0&0&0&0&81&0&15&0\\0&0&15&0&6&0&0&1\end{array}\right)$$

Since we have a row of zeros therefore the matrix **H** is non-invertible (singular) which means $\mathbf{H}^{-1}$ does not exist.

(When you evaluate $\mathbf{H}^{-1}$ in MATLAB you receive the message "Warning: Matrix is close to singular or badly scaled." It continues and gives the following output:

```
 Results may be inaccurate. RCOND = 1.912529e-018.

ans =

  1.0e+015 *

    9.1198   -0.0000    1.6888    0.1126
    0.0000    0.0000         0    0.0000
    0.0000         0         0    0.0000
```

-3.0399 0 -0.5629 -0.0375

However the output in MAPLE is "Error, (in rtable/Power) singular matrix." Clearly the MAPLE output is unambiguous and concurs with our evaluation that the matrix **H** is singular or non-invertible).

(i) Matrix **J** is very similar to the above matrix **H** with the last two entries in the bottom row changed to 12 and -19 respectively. By carrying out the same row operations we obtain the same entries in the first three rows as part (h):

$$\begin{array}{r} R_1 \\ R_2^* = R_2 + 2R_1 \\ R_3^* = R_3 + 5R_1 \\ R_4^* = R_4 + 6R_1 \end{array} \left(\begin{array}{cccc|cccc} 1 & 2 & -2 & 3 & 1 & 0 & 0 & 0 \\ -2+2(1) & 1+2(2) & -5+2(-2) & -6+2(3) & 2 & 1 & 0 & 0 \\ -5+5(1) & -10+5(2) & 9+5(-2) & -15+5(3) & 5 & 0 & 1 & 0 \\ -6+6(1) & -12+6(2) & 12+6(-2) & -19+6(3) & 6 & 0 & 0 & 1 \end{array}\right)$$

Simplifying the entries gives

$$\begin{array}{r} R_1 \\ R_2^* \\ R_3^* \\ R_4^* \end{array} \left(\begin{array}{cccc|cccc} 1 & 2 & -2 & 3 & 1 & 0 & 0 & 0 \\ 0 & 5 & -9 & 0 & 2 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 5 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 & 6 & 0 & 0 & 1 \end{array}\right)$$

Converting the -1 into 1s:

$$\begin{array}{r} R_1 \\ R_2^* \\ R_3^{**} = -R_3^* \\ R_4^{**} = -R_4^* \end{array} \left(\begin{array}{cccc|cccc} 1 & 2 & -2 & 3 & 1 & 0 & 0 & 0 \\ 0 & 5 & -9 & 0 & 2 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & -5 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & -6 & 0 & 0 & -1 \end{array}\right)$$

Converting the -2 and -9 into 0's:

$$\begin{array}{r} R_1^* = R_1 + 2R_3^{**} \\ R_2^* = R_2^* + 9R_3^{**} \\ R_3^{**} \\ R_4^{**} \end{array} \left(\begin{array}{cccc|cccc} 1 & 2 & -2+2(1) & 3 & 1+2(-5) & 0 & 0+2(-1) & 0 \\ 0 & 5 & -9+9(1) & 0 & 2+9(-5) & 1 & 0+9(-1) & 0 \\ 0 & 0 & 1 & 0 & -5 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & -6 & 0 & 0 & -1 \end{array}\right)$$

Simplifying

$$\begin{array}{r} R_1^* \\ R_2^* \\ R_3^{**} \\ R_4^{**} \end{array} \left(\begin{array}{cccc|cccc} 1 & 2 & 0 & 3 & -9 & 0 & -2 & 0 \\ 0 & 5 & 0 & 0 & -43 & 1 & -9 & 0 \\ 0 & 0 & 1 & 0 & -5 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & -6 & 0 & 0 & -1 \end{array}\right)$$

Converting the 3 in the top row into 0:

$$\begin{array}{r} R_1^{**} = R_1^* - 3R_4^{**} \\ R_2^* \\ R_3^{**} \\ R_4^{**} \end{array} \left(\begin{array}{cccc|cccc} 1 & 2 & 0 & 3-3(1) & -9-3(-6) & 0 & -2 & 0-3(-1) \\ 0 & 5 & 0 & 0 & -43 & 1 & -9 & 0 \\ 0 & 0 & 1 & 0 & -5 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & -6 & 0 & 0 & -1 \end{array}\right)$$

Simplifying

$$\begin{array}{c} R_1^{**} \\ R_2^{*} \\ R_3^{**} \\ R_4^{**} \end{array} \left(\begin{array}{cccc|cccc} 1 & 2 & 0 & 0 & 9 & 0 & -2 & 3 \\ 0 & 5 & 0 & 0 & -43 & 1 & -9 & 0 \\ 0 & 0 & 1 & 0 & -5 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & -6 & 0 & 0 & -1 \end{array}\right)$$

Converting the 2 in the top row into 0:

$$\begin{array}{c} R_1^{£} = R_1^{**} - \frac{2}{5}R_2^{*} \\ R_2^{*} \\ R_3^{**} \\ R_4^{**} \end{array} \left(\begin{array}{cccc|cccc} 1 & 2-\frac{2}{5}(5) & 0 & 0 & 9-\frac{2}{5}(-43) & 0-\frac{2}{5}(1) & -2-\frac{2}{5}(-9) & 3 \\ 0 & 5 & 0 & 0 & -43 & 1 & -9 & 0 \\ 0 & 0 & 1 & 0 & -5 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & -6 & 0 & 0 & -1 \end{array}\right)$$

We have

$$\begin{array}{c} R_1^{£} \\ R_2^{*} \\ R_3^{**} \\ R_4^{**} \end{array} \left(\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 131/5 & -2/5 & 8/5 & 3 \\ 0 & 5 & 0 & 0 & -43 & 1 & -9 & 0 \\ 0 & 0 & 1 & 0 & -5 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & -6 & 0 & 0 & -1 \end{array}\right)$$

Dividing the second row by 5 gives the identity matrix on the Left Hand Side of the vertical line:

$$\begin{array}{c} R_1^{£} \\ R_2^{*}/5 \\ R_3^{**} \\ R_4^{**} \end{array} \left(\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 131/5 & -2/5 & 8/5 & 3 \\ 0 & 1 & 0 & 0 & -43/5 & 1/5 & -9/5 & 0 \\ 0 & 0 & 1 & 0 & -5 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & -6 & 0 & 0 & -1 \end{array}\right)$$

Hence the inverse matrix is

$$\mathbf{J}^{-1} = \begin{pmatrix} 131/5 & -2/5 & 8/5 & 3 \\ -43/5 & 1/5 & -9/5 & 0 \\ -5 & 0 & -1 & 0 \\ -6 & 0 & 0 & -1 \end{pmatrix} = \frac{1}{5}\begin{pmatrix} 131 & -2 & 8 & 15 \\ -43 & 1 & -9 & 0 \\ -25 & 0 & -5 & 0 \\ -30 & 0 & 0 & -5 \end{pmatrix}$$

5. All equations are linear systems of the form $\mathbf{Ax} = \mathbf{b}$ and we evaluate $\mathbf{x}$ by $\mathbf{x} = \mathbf{A}^{-1}\mathbf{b}$. In each case the matrix $\mathbf{A}$ has exactly the entries of the matrix given in question 4 respectively apart from the last part (h) which is the matrix $\mathbf{J}$ in question 4 part (i).

(a) We need to find $\mathbf{x} = \mathbf{A}^{-1}\mathbf{b}$ where $\mathbf{A}^{-1} = \frac{1}{6}\begin{pmatrix} 4 & -2 \\ 1 & 1 \end{pmatrix}$, $\mathbf{x} = \begin{pmatrix} x \\ y \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} 3 \\ 5 \end{pmatrix}$.

Remember the inverse matrix $\mathbf{A}^{-1}$ is the one we found in solution to question 4(a).

$$\mathbf{x} = \begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{6}\begin{pmatrix} 4 & -2 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 3 \\ 5 \end{pmatrix} \qquad \left[\text{Because } \mathbf{x} = \mathbf{A}^{-1}\mathbf{b}\right]$$

$$= \frac{1}{6}\begin{pmatrix} 2 \\ 8 \end{pmatrix} = \begin{pmatrix} 2/6 \\ 8/6 \end{pmatrix} = \begin{pmatrix} 1/3 \\ 4/3 \end{pmatrix}$$

Hence $x = 1/3$ and $y = 4/3$.

(b) We need to find $\mathbf{x} = \mathbf{B}^{-1}\mathbf{b}$ where $\mathbf{B}^{-1} = \frac{1}{28}\begin{pmatrix} -1 & -5 \\ -6 & -2 \end{pmatrix}$, $\mathbf{x} = \begin{pmatrix} x \\ y \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} 3 \\ -1 \end{pmatrix}$.

Remember the inverse matrix $\mathbf{B}^{-1}$ is the one we found in solution to question 4(b).

$$\mathbf{x}=\begin{pmatrix}x\\y\end{pmatrix}=\frac{1}{28}\begin{pmatrix}-1&-5\\-6&-2\end{pmatrix}\begin{pmatrix}3\\-1\end{pmatrix}\qquad\left[\text{Remember }\mathbf{x}=\mathbf{B}^{-1}\mathbf{b}\right]$$

$$=\frac{1}{28}\begin{pmatrix}2\\-16\end{pmatrix}=\begin{pmatrix}2/28\\-16/28\end{pmatrix}=\begin{pmatrix}1/14\\-4/7\end{pmatrix}$$

Hence $x=1/14$ and $y=-4/7$.

(c) We need to find $\mathbf{x}=\mathbf{C}^{-1}\mathbf{b}$ but $\mathbf{C}^{-1}$ does **not** exist therefore the given linear system does **not have** a solution.

(d) We need to find $\mathbf{x}=\mathbf{D}^{-1}\mathbf{b}$ where $\mathbf{D}^{-1}=\begin{pmatrix}0&1&-1\\-1&1&-2\\0&0&-1\end{pmatrix}$, $\mathbf{x}=\begin{pmatrix}x\\y\\z\end{pmatrix}$ and $\mathbf{b}=\begin{pmatrix}10\\3\\5\end{pmatrix}$.

Remember the inverse matrix $\mathbf{D}^{-1}$ is the one we found in solution to question 4(d).

$$\mathbf{x}=\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}0&1&-1\\-1&1&-2\\0&0&-1\end{pmatrix}\begin{pmatrix}10\\3\\5\end{pmatrix}\qquad\left[\text{Because }\mathbf{x}=\mathbf{D}^{-1}\mathbf{b}\right]$$

$$=\begin{pmatrix}(0\times10)+(1\times3)+(-1\times5)\\(-1\times10)+(1\times3)+(-2\times5)\\(0\times10)+(0\times3)+(-1\times5)\end{pmatrix}$$

$$=\begin{pmatrix}-2\\-17\\-5\end{pmatrix}$$

Hence $x=-2$, $y=-17$ and $z=-5$.

(e) We need to find $\mathbf{x}=\mathbf{E}^{-1}\mathbf{b}$ where $\mathbf{E}^{-1}=\frac{1}{4}\begin{pmatrix}3&2&1\\2&4&2\\1&2&3\end{pmatrix}$, $\mathbf{x}=\begin{pmatrix}x\\y\\z\end{pmatrix}$ and $\mathbf{b}=\begin{pmatrix}5\\7\\3\end{pmatrix}$.

Remember the inverse matrix $\mathbf{E}^{-1}$ is the one we found in solution to question 4(e).

$$\mathbf{x}=\begin{pmatrix}x\\y\\z\end{pmatrix}=\frac{1}{4}\begin{pmatrix}3&2&1\\2&4&2\\1&2&3\end{pmatrix}\begin{pmatrix}5\\7\\3\end{pmatrix}$$

$$=\frac{1}{4}\begin{pmatrix}32\\44\\28\end{pmatrix}=\begin{pmatrix}8\\11\\7\end{pmatrix}$$

Hence $x=8$, $y=11$ and $z=7$.

(f) We need to find $\mathbf{x}=\mathbf{F}^{-1}\mathbf{b}$ where $\mathbf{F}^{-1}=\frac{1}{15}\begin{pmatrix}3&-10&1\\0&10&5\\3&-5&-4\end{pmatrix}$, $\mathbf{x}=\begin{pmatrix}x\\y\\z\end{pmatrix}$ and

$\mathbf{b}=\begin{pmatrix}-2\\3\\6\end{pmatrix}$. Remember the inverse matrix $\mathbf{F}^{-1}$ is the one we found in solution to question 4(f).

$$\mathbf{x}=\begin{pmatrix}x\\y\\z\end{pmatrix}=\frac{1}{15}\begin{pmatrix}3&-10&1\\0&10&5\\3&-5&-4\end{pmatrix}\begin{pmatrix}-2\\3\\6\end{pmatrix}$$

$$=\frac{1}{15}\begin{pmatrix}-30\\60\\-45\end{pmatrix}=\begin{pmatrix}-2\\4\\-3\end{pmatrix}$$

Hence $x=-2,\ y=4$ and $z=-3$.

(g) We need to find $\mathbf{x}=\mathbf{J}^{-1}\mathbf{b}$ where $\mathbf{J}^{-1}=\frac{1}{5}\begin{pmatrix}131&-2&8&15\\-43&1&-9&0\\-25&0&-5&0\\-30&0&0&-5\end{pmatrix}$, $\mathbf{x}=\begin{pmatrix}x\\y\\z\\w\end{pmatrix}$ and

$\mathbf{b}=\begin{pmatrix}8\\1\\-1\\5\end{pmatrix}$. Remember the inverse matrix $\mathbf{J}^{-1}$ is the one we found in solution to question 4(i).

$$\mathbf{x}=\begin{pmatrix}x\\y\\z\\w\end{pmatrix}=\frac{1}{5}\begin{pmatrix}131&-2&8&15\\-43&1&-9&0\\-25&0&-5&0\\-30&0&0&-5\end{pmatrix}\begin{pmatrix}8\\1\\-1\\5\end{pmatrix}$$

$$=\frac{1}{5}\begin{pmatrix}(131\times8)-(2\times1)+(8\times(-1))+(15\times5)\\(-43\times8)+(1\times1)+(-9\times(-1))+(0\times5)\\(-25\times8)+(0\times1)+(-5\times(-1))+(0\times5)\\(-30\times8)+(0\times1)+(0\times(-1))+(-5\times5)\end{pmatrix}$$

$$=\frac{1}{5}\begin{pmatrix}1113\\-334\\-195\\-265\end{pmatrix}=\begin{pmatrix}222.6\\-66.8\\-39\\-53\end{pmatrix}$$

Hence $x=222.6,\ y=-66.8,\ z=-39$ and $w=-53$.

6. Use any appropriate software such as MATLAB or Maple to carry out the calculations.
Writing out the table we have

A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q	R	S	T	U	V	W	X	Y	Z	
1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	
4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30

(a) ATTACK is represented by the numbers: 4, 23, 23, 4, 6, 14. We can write these as column vectors with 3 rows because we are given a 3 by 3 matrix. Hence

$$\begin{pmatrix} 4 \\ 23 \\ 23 \end{pmatrix} \text{ and } \begin{pmatrix} 4 \\ 6 \\ 14 \end{pmatrix}$$

Let these vectors be the column vectors of matrix **B**, that is $\mathbf{B} = \begin{pmatrix} 4 & 4 \\ 23 & 6 \\ 23 & 14 \end{pmatrix}$. The matrix multiplication is given by $\mathbf{AB} = \begin{pmatrix} 3 & 4 & 5 \\ 1 & 3 & 1 \\ 1 & 1 & 2 \end{pmatrix}\begin{pmatrix} 4 & 4 \\ 23 & 6 \\ 23 & 14 \end{pmatrix} = \begin{pmatrix} 219 & 106 \\ 96 & 36 \\ 73 & 38 \end{pmatrix}$. Multiplying this by the inverse matrix $\mathbf{A}^{-1} = \begin{pmatrix} 5 & -3 & -11 \\ -1 & 1 & 2 \\ -2 & 1 & 5 \end{pmatrix}$ gives

$$\mathbf{A}^{-1}(\mathbf{AB}) = \begin{pmatrix} 5 & -3 & -11 \\ -1 & 1 & 2 \\ -2 & 1 & 5 \end{pmatrix}\begin{pmatrix} 219 & 106 \\ 96 & 36 \\ 73 & 38 \end{pmatrix} = \begin{pmatrix} 4 & 4 \\ 23 & 6 \\ 23 & 14 \end{pmatrix} = \mathbf{B}$$

(b) Similarly by using the above table for the message 'IS YOUR PARTNER HOME' we have

I	S		Y	O	U	R		P	A	R	T	N	E	R		H	O	M	E
12	22	30	28	18	24	21	30	19	4	21	23	17	8	21	30	11	18	16	8

12, 22, 30, 28, 18, 24, 21, 30, 19, 4, 21, 23, 17, 8, 21, 30, 11, 18, 16, 8

We write these as column vectors with 3 rows. In the list we have 20 numbers but each vector only has 3 entries so the last two numbers 16 and 8 are left over. We write these numbers with a space at the end so the last vector will have the entries 16, 8 and 30 where 30 represents space:

$$\begin{pmatrix} 12 \\ 22 \\ 30 \end{pmatrix}, \begin{pmatrix} 28 \\ 18 \\ 24 \end{pmatrix}, \begin{pmatrix} 21 \\ 30 \\ 19 \end{pmatrix}, \begin{pmatrix} 4 \\ 21 \\ 23 \end{pmatrix}, \begin{pmatrix} 17 \\ 8 \\ 21 \end{pmatrix}, \begin{pmatrix} 30 \\ 11 \\ 18 \end{pmatrix}, \begin{pmatrix} 16 \\ 8 \\ 30 \end{pmatrix}$$

The matrix **B** consists of these vectors has its column vectors, that is

$$\mathbf{B} = \begin{pmatrix} 12 & 28 & 21 & 4 & 17 & 30 & 16 \\ 22 & 18 & 30 & 21 & 8 & 11 & 8 \\ 30 & 24 & 19 & 23 & 21 & 18 & 30 \end{pmatrix}$$

Evaluating the matrix multiplication **AB** we have

$$\mathbf{AB} = \begin{pmatrix} 3 & 4 & 5 \\ 1 & 3 & 1 \\ 1 & 1 & 2 \end{pmatrix}\begin{pmatrix} 12 & 28 & 21 & 4 & 17 & 30 & 16 \\ 22 & 18 & 30 & 21 & 8 & 11 & 8 \\ 30 & 24 & 19 & 23 & 21 & 18 & 30 \end{pmatrix}$$

$$= \begin{pmatrix} 274 & 276 & 278 & 211 & 188 & 224 & 230 \\ 108 & 106 & 130 & 90 & 62 & 81 & 70 \\ 94 & 94 & 89 & 71 & 67 & 77 & 84 \end{pmatrix}$$

The numbers in this matrix **AB** are the ones which are transmitted. If we multiply this matrix **AB** by the inverse matrix $\mathbf{A}^{-1}$ we should get matrix **B** which will give us our message:

$$\mathbf{A}^{-1}(\mathbf{AB}) = \begin{pmatrix} 5 & -3 & -11 \\ -1 & 1 & 2 \\ -2 & 1 & 5 \end{pmatrix}\begin{pmatrix} 274 & 276 & 278 & 211 & 188 & 224 & 230 \\ 108 & 106 & 130 & 90 & 62 & 81 & 70 \\ 94 & 94 & 89 & 71 & 67 & 77 & 84 \end{pmatrix}$$

$$= \begin{pmatrix} 12 & 28 & 21 & 4 & 17 & 30 & 16 \\ 22 & 18 & 30 & 21 & 8 & 11 & 8 \\ 30 & 24 & 19 & 23 & 21 & 18 & 30 \end{pmatrix}$$

This is our matrix **B**.

7. We have the equation $\mathbf{p} = \mathbf{Ap} + \mathbf{d}$. *How can we find the demand vector* ***d****?*
Transposing this we have

$$\mathbf{p} - \mathbf{Ap} = \mathbf{d}$$

$$\mathbf{Ip} - \mathbf{Ap} = (\mathbf{I} - \mathbf{A})\mathbf{p} = \mathbf{d}$$

We need the inverse of $\mathbf{I} - \mathbf{A}$ because

$$\mathbf{p} = (\mathbf{I} - \mathbf{A})^{-1}\mathbf{d} \qquad (*)$$

Substituting the input-output matrix $\mathbf{A} = \begin{pmatrix} 0.25 & 0.15 & 0.1 \\ 0.4 & 0.15 & 0.2 \\ 0.15 & 0.2 & 0.2 \end{pmatrix}$ and finding the inverse:

$$(\mathbf{I} - \mathbf{A})^{-1} = \left(\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} - \begin{pmatrix} 0.25 & 0.15 & 0.1 \\ 0.4 & 0.15 & 0.2 \\ 0.15 & 0.2 & 0.2 \end{pmatrix}\right)^{-1}$$

$$= \begin{pmatrix} 0.75 & -0.15 & -0.1 \\ -0.4 & 0.85 & -0.2 \\ -0.15 & -0.2 & 0.8 \end{pmatrix}^{-1} = \begin{pmatrix} 1.57 & 0.34 & 0.28 \\ 0.86 & 1.44 & 0.47 \\ 0.51 & 0.42 & 1.42 \end{pmatrix}$$

Substituting the given demand vector $\mathbf{d} = \begin{pmatrix} 100 & 100 & 100 \end{pmatrix}^T = \begin{pmatrix} 100 \\ 100 \\ 100 \end{pmatrix}$ and the above evaluation into (*) yields:

$$\mathbf{p} = (\mathbf{I} - \mathbf{A})^{-1}\mathbf{d} =$$

$$= \begin{pmatrix} 1.57 & 0.34 & 0.28 \\ 0.86 & 1.44 & 0.47 \\ 0.51 & 0.42 & 1.42 \end{pmatrix}\begin{pmatrix} 100 \\ 100 \\ 100 \end{pmatrix} = \begin{pmatrix} 220.04 \\ 277.58 \\ 235.40 \end{pmatrix}$$

The production for oil, energy and services should be 220.04, 277.58 and 235.40.

8. *Proof.*
By Proposition (1-10):

Let **A** be an invertible matrix then $\mathbf{A}^T$ is invertible.

We have $\mathbf{A}^T$ is invertible. By Proposition (1-16):

The linear system $\mathbf{Ax} = \mathbf{b}$ has a unique solution $\Leftrightarrow$ **A** is invertible.

Hence $\mathbf{A}^T\mathbf{x} = \mathbf{b}$ has a unique solution.

■

9. Let the matrix **A** equal $-\mathbf{B}$ then $\mathbf{Ax}=-\mathbf{Bx}=-\mathbf{c}$ because we are given $\mathbf{Bx}=\mathbf{c}$

$$(\mathbf{A}+\mathbf{B})\mathbf{x}=(-\mathbf{B}+\mathbf{B})\mathbf{x}=\mathbf{Ox}=-\mathbf{c}+\mathbf{c}=\mathbf{O}$$

O matrix is **not** invertible so $(\mathbf{A}+\mathbf{B})\mathbf{x}=\mathbf{Ox}=\mathbf{O}$ has an infinite number of solutions.

10. *Proof.* Let $\mathbf{E}_1$ be an elementary matrix. This means that by a single row operation we obtain the matrix $\mathbf{E}_1$ from the identity matrix. We know there is an elementary matrix $\mathbf{E}_2$ which reverses the row operation of $\mathbf{E}_1$ and this implies that

$$\mathbf{E}_2\mathbf{E}_1=\mathbf{I} \text{ and } \mathbf{E}_1\mathbf{E}_2=\mathbf{I} \qquad (*)$$

Hence $\mathbf{E}_1$ is invertible. Clearly from (*) we also conclude that the inverse of $\mathbf{E}_1$ is $\mathbf{E}_2$ which is also invertible.

■

11. Need to prove that if a matrix **B** is row equivalent to matrix **A** then there exists an invertible matrix **P** such that $\mathbf{B}=\mathbf{PA}$.

Proof.

We assume matrix **B** is row equivalent to matrix **A** and deduce $\mathbf{B}=\mathbf{PA}$. We are given that **B** is row equivalent to **A** therefore by definition (1-4):

(1-4) **B** is row equivalent to a matrix **A** if and only if $\mathbf{B}=\mathbf{E}_n\mathbf{E}_{n-1}\cdots\mathbf{E}_2\mathbf{E}_1\mathbf{A}$.

There are elementary matrices $\mathbf{E}_1,\ \mathbf{E}_2,\ \mathbf{E}_3\ \cdots$ and $\mathbf{E}_n$ such that

$$\mathbf{B}=\mathbf{E}_1\mathbf{E}_2\mathbf{E}_3\ \cdots\ \mathbf{E}_n\mathbf{A}$$

Let $\mathbf{P}=\mathbf{E}_1\mathbf{E}_2\mathbf{E}_3\ \cdots\ \mathbf{E}_n$ and by Proposition (1-14):

(1-14) An elementary matrix is invertible and it’s inverse is also an elementary matrix.

We know that each elementary matrix is invertible, therefore

$$\mathbf{P}^{-1}=(\mathbf{E}_1\mathbf{E}_2\mathbf{E}_3\ \cdots\ \mathbf{E}_n)^{-1}=\mathbf{E}_n^{-1}\mathbf{E}_{n-1}^{-1}\ \cdots\mathbf{E}_2^{-1}\,\mathbf{E}_1^{-1} \qquad \left\lfloor \text{Because } (\mathbf{XYZ})^{-1}=\mathbf{Z}^{-1}\mathbf{Y}^{-1}\mathbf{X}^{-1} \right\rfloor$$

Hence matrix **P** is invertible.

■

12. (a) Need to prove that if matrix **B** is row equivalent to matrix **A** then matrix **A** is row equivalent to matrix **B**.

Proof.

We are given that matrix **B** is row equivalent to matrix **A** therefore by Definition (1-4) there exists elementary matrices $\mathbf{E}_1,\ \mathbf{E}_2,\ \mathbf{E}_3,\ \cdots$ and $\mathbf{E}_n$ such that

$$\mathbf{B}=\mathbf{E}_n\mathbf{E}_{n-1}\cdots\mathbf{E}_2\mathbf{E}_1\mathbf{A}$$

Multiplying both sides by $(\mathbf{E}_n\mathbf{E}_{n-1}\cdots\mathbf{E}_2\mathbf{E}_1)^{-1}$ gives

$$(\mathbf{E}_n\mathbf{E}_{n-1}\cdots\mathbf{E}_2\mathbf{E}_1)^{-1}\mathbf{B}=\underbrace{(\mathbf{E}_n\mathbf{E}_{n-1}\cdots\mathbf{E}_2\mathbf{E}_1)^{-1}(\mathbf{E}_n\mathbf{E}_{n-1}\cdots\mathbf{E}_2\mathbf{E}_1)}_{=\mathbf{I}}\mathbf{A}$$

$$\mathbf{E}_1^{-1}\mathbf{E}_2^{-1}\cdots\mathbf{E}_{n-1}^{-1}\mathbf{E}_n^{-1}\mathbf{B}=\mathbf{IA}=\mathbf{A}$$

We have $\mathbf{A}=\mathbf{E}_1^{-1}\mathbf{E}_2^{-1}\cdots\mathbf{E}_{n-1}^{-1}\mathbf{E}_n^{-1}\mathbf{B}$ and by Proposition (1-14) we know $\mathbf{E}_j^{-1}$ is an elementary matrix. Hence **A** is row equivalent to the matrix **B**.

■

(b) Need to prove if matrix **A** is row equivalent to matrix **B** and matrix **B** is row equivalent to matrix **C** then **A** is row equivalent to **C**.

Proof.

We are given that matrix **A** is row equivalent to matrix **B** and matrix **B** is row equivalent to matrix **C** therefore there are elementary matrices $\mathbf{E}_1,\ \mathbf{E}_2,\ \mathbf{E}_3,\ \cdots$ and $\mathbf{E}_n$ and $\mathbf{F}_1,\ \mathbf{F}_2,\ \mathbf{F}_3,\ \cdots$ and $\mathbf{F}_k$ such that

$$\mathbf{B}=\mathbf{E}_n\mathbf{E}_{n-1}\cdots\mathbf{E}_2\mathbf{E}_1\mathbf{A} \quad \text{and} \quad \mathbf{C}=\mathbf{F}_k\mathbf{F}_{k-1}\cdots\mathbf{F}_2\mathbf{F}_1\mathbf{B}$$

From part (a) we have $\left(\mathbf{F}_k\mathbf{F}_{k-1}\cdots\mathbf{F}_2\mathbf{F}_1\right)^{-1}\mathbf{C}=\mathbf{B}$. Substituting this $\left(\mathbf{F}_k\mathbf{F}_{k-1}\cdots\mathbf{F}_2\mathbf{F}_1\right)^{-1}\mathbf{C}=\mathbf{B}$ into the above, $\mathbf{B}=\mathbf{E}_n\mathbf{E}_{n-1}\cdots\mathbf{E}_2\mathbf{E}_1\mathbf{A}$, gives

$$\left(\mathbf{F}_k\mathbf{F}_{k-1}\cdots\mathbf{F}_2\mathbf{F}_1\right)^{-1}\mathbf{C}=\mathbf{E}_n\mathbf{E}_{n-1}\cdots\mathbf{E}_2\mathbf{E}_1\mathbf{A}$$

Pre-multiply both sides by $\left(\mathbf{F}_k\mathbf{F}_{k-1}\cdots\mathbf{F}_2\mathbf{F}_1\right)$ gives

$$\underbrace{\left(\mathbf{F}_k\mathbf{F}_{k-1}\cdots\mathbf{F}_2\mathbf{F}_1\right)\left(\mathbf{F}_k\mathbf{F}_{k-1}\cdots\mathbf{F}_2\mathbf{F}_1\right)^{-1}}_{=\mathbf{I}}\mathbf{C}=\left(\mathbf{F}_k\mathbf{F}_{k-1}\cdots\mathbf{F}_2\mathbf{F}_1\right)\mathbf{E}_n\mathbf{E}_{n-1}\cdots\mathbf{E}_2\mathbf{E}_1\mathbf{A}$$

$$\mathbf{IC}=\left(\mathbf{F}_k\mathbf{F}_{k-1}\cdots\mathbf{F}_2\mathbf{F}_1\mathbf{E}_n\mathbf{E}_{n-1}\cdots\mathbf{E}_2\mathbf{E}_1\right)\mathbf{A}$$

$$\mathbf{C}=\left(\mathbf{F}_k\mathbf{F}_{k-1}\cdots\mathbf{F}_2\mathbf{F}_1\mathbf{E}_n\mathbf{E}_{n-1}\cdots\mathbf{E}_2\mathbf{E}_1\right)\mathbf{A}$$

The **E**'s and **F**'s are elementary matrices therefore matrix **A** is row equivalent to matrix **C** which is our required result.

■

13. Need to prove that the linear system $\mathbf{Ax}=\mathbf{b}$ has a unique solution $\Leftrightarrow$ **A** is invertible.

How do we prove this result?

By using the Propositions (1-11) and (1-13) of section G and Theorem (1-15) of section H.

> Proposition (1-11). If a linear system is described by the augmented matrix $\left(\mathbf{A}\ \middle|\ \mathbf{b}\right)$ and it is row equivalent to $\left(\mathbf{R}\ \middle|\ \mathbf{b}'\right)$ then both linear systems have the same solution set.

> Proposition (1-13). Let a consistent non-homogeneous linear system $\mathbf{Ax}=\mathbf{b}$ where $\mathbf{b}\neq\mathbf{O}$ be row equivalent to the augmented matrix $\left(\mathbf{R}\ \middle|\ \mathbf{b}'\right)$ where **R** is in reduced row echelon form and there be *n* unknowns and *r* non-zero equations in **R**. We have
> If $r<n$ (the number of non-zero equations in **R** is less than the number of unknowns) then the linear system $\mathbf{Ax}=\mathbf{b}$ has an infinite number of solutions.

> Theorem (1-15). Let **A** be a *n* by *n* matrix, then the following 4 statements are equivalent:
> (a) The matrix **A** is invertible (non-singular).
> (b) The linear system $\mathbf{Ax}=\mathbf{O}$ only has the trivial solution $\mathbf{x}=\mathbf{O}$.
> (c) The reduced row echelon form of the matrix **A** is the identity matrix **I**.
> (d) **A** is a product of elementary matrices.

We also need to use:

> Question 7 of Exercise 1.7. RREF of a square matrix is either **I** or it contains a row of zeros.

Proof.

Case I: $\mathbf{b}=\mathbf{0}$ then by Theorem (1-15) we have our required result.

Case II: $\mathbf{b} \neq \mathbf{0}$ (not the zero vector). The given statement contains $\Leftrightarrow$ which means we need to prove it both ways:

$(\Rightarrow)$. We are given that $\mathbf{Ax} = \mathbf{b}$ has a **unique** solution. Let the reduced row echelon form of the matrix **A** equal **R**, in compact notation this is written as $rref\,(\mathbf{A}) = \mathbf{R}$. By the above Proposition (1-11) $\mathbf{Rx} = \mathbf{b}'$ has the same solution as $\mathbf{Ax} = \mathbf{b}$. This means that **R** has **no** zero rows. *Why?* Because if **R** has zero rows then $\mathbf{Ax} = \mathbf{b}$ has an infinite number of solutions or **no** solution. This is a contradiction because we are given that $\mathbf{Ax} = \mathbf{b}$ has a **unique** solution. Hence **R** has **no** zero rows so by question 7 of Exercises 1.7 the matrix **R** is the identity matrix. By Theorem (1-15) we conclude that the matrix **A** is invertible because $rref\,(\mathbf{A}) = \mathbf{I}$.

$(\Leftarrow)$. We can assume that the matrix **A** is invertible. Consider the given linear system:

$$\mathbf{Ax} = \mathbf{b}$$

Suppose **y** is also a solution to this system, that is $\mathbf{Ay} = \mathbf{b}$. Subtracting these two, $\mathbf{Ax} = \mathbf{b}$ and $\mathbf{Ay} = \mathbf{b}$, we have

$$\mathbf{Ax} - \mathbf{Ay} = \mathbf{b} - \mathbf{b} = \mathbf{0}$$
$$\mathbf{A}(\mathbf{x} - \mathbf{y}) = \mathbf{0}$$

A is invertible therefore by Proposition (1-15) part (b)

$$\mathbf{A}(\mathbf{x} - \mathbf{y}) = \mathbf{O} \;\Rightarrow\; \mathbf{x} - \mathbf{y} = \mathbf{O} \;\Rightarrow\; \mathbf{x} = \mathbf{y}$$

We have $\mathbf{x} = \mathbf{y}$ therefore a unique solution.

■

14. Need to prove that the reduced row echelon form **R** has at least one row of zeros $\Leftrightarrow$ **A** is non-invertible (singular).

Proof.

This is a direct consequence of Theorem (1-15) because from this we have:

The matrix **A** is invertible (non-singular) $\Leftrightarrow$ The reduced row echelon form of the matrix **A** is the identity matrix **I**.

By mathematical logic we have statements $P \Leftrightarrow Q$ is equivalent to $\text{not}\,(P) \Leftrightarrow \text{not}(Q)$.

If P is the statement 'matrix **A** is invertible (non-singular)' then $\text{not}\,(P)$ is

A is non-invertible (singular)

Q is the statement 'reduced row echelon form of the matrix **A** is the identity matrix **I**' then $\text{not}(Q)$ is

reduced row echelon form of the matrix **A** is **not** the identity matrix

Putting these together we have $\text{not}\,(P) \Leftrightarrow \text{not}(Q)$ is:

A is non-invertible (singular) $\Leftrightarrow$ reduced row echelon form **R** of the matrix **A** is **not** the identity matrix.

By Question 7 of Exercises 1.7 which says that

RREF of a square matrix is either **I** or it contains a row of zeros.

We conclude that **R** has a row of zeros.

■

15. There are only three types of elementary matrices because there only exists three different row operations:

(a) Multiply a row by a non-zero constant.

(b) Add a multiply of one row to another.
(c) Interchange rows.

16. Need to prove that the transpose of an elementary matrix is an elementary matrix.
Proof.
From question 15 above we know there are only three types of elementary matrices.
Consider the three row operations on the identity matrix.

1. Let $\mathbf{E}_1$ be the elementary matrix of multiplying the ith row of the identity matrix **I** by a non-zero constant, *k*, that is:

$$\mathbf{E}_1 = \begin{array}{c} \begin{pmatrix} 1 & 0 & \cdots & & 0 \\ 0 & \ddots & & & \\ \vdots & 0 & k & 0 & \cdots \\ & \vdots & 0 & & \\ 0 & & \vdots & & 1 \end{pmatrix} \end{array} \begin{array}{l} \\ \\ \text{ith Row} \\ \\ \\ \end{array}$$

Transposing a matrix changes the ij (ith row and jth column) entry to the ji (jth row and ith column) entry. All the ij entries where $i \neq j$ are zero. Hence $\mathbf{E}_1^T = \mathbf{E}_1$, that is the transpose of the elementary matrix $\mathbf{E}_1$ is equal to $\mathbf{E}_1$ which is an elementary matrix.

2. Let $\mathbf{E}_2$ be the elementary matrix of swapping over the ith and jth rows of the identity matrix, that is

$$\begin{array}{ll} & \text{Cols} \quad \text{ith} \qquad \text{jth} \\ \mathbf{E}_2 = & \begin{pmatrix} 1 & 0 & \cdots & & 0 \\ 0 & \vdots & 0 & \boxed{1} & \vdots \\ \vdots & 0 & 1 & 0 & \cdots \\ & \boxed{1} & 0 & & \\ 0 & & \vdots & & 1 \end{pmatrix} \end{array} \begin{array}{l} \\ \\ \text{ith Row} \\ \\ \text{jth Row} \\ \\ \end{array}$$

This means that the 1 in the ii (ith row and ith column) entry goes to ji (jth row and ith column) entry and the 1 in the jj (jth row and jth column) goes to ij (ith row and jth column) entry. In both cases the entry is 1, that is ji entry equals the ij entry. All the other ij entries where $i \neq j$ are zero. This means that $\mathbf{E}_2^T = \mathbf{E}_2$. Hence after transposing we still have an elementary matrix.

3. Let $\mathbf{E}_3$ be the elementary matrix of adding *k* times the jth row to the ith row of the identity matrix, that is

$$\begin{array}{ll} & \qquad\qquad \text{jth Col} \\ \mathbf{E}_3 = & \begin{pmatrix} 1 & 0 & \cdots & & 0 \\ 0 & \ddots & 0 & \boxed{k} & \vdots \\ \vdots & 0 & 1 & 0 & \cdots \\ & & 0 & \ddots & \\ 0 & & \vdots & & 1 \end{pmatrix} \end{array} \begin{array}{l} \\ \\ \text{ith Row} \\ \\ \text{jth Row} \\ \\ \end{array}$$

This means $\mathbf{E}_3$ has the same entries as the identity apart from the ij entry which is *k*. Transposing $\mathbf{E}_3$ changes ij entry to the ji entry, that is

$$\mathbf{E}_3^{\,T} = \begin{array}{cl} \begin{array}{ccccc} & \text{ith Col} & & & \end{array} & \\ \begin{pmatrix} 1 & 0 & \cdots & & 0 \\ 0 & \ddots & 0 & & \vdots \\ \vdots & 0 & 1 & 0 & \cdots \\ & \boxed{k} & 0 & \ddots & \\ 0 & & \vdots & & 1 \end{pmatrix} & \begin{array}{l} \\ \text{ith Row} \\ \\ \text{jth Row} \\ \\ \end{array} \end{array}$$

We can obtain the matrix $\mathbf{E}_3^{\,T}$ by the **single** row operation of adding k times the ith row to the jth row of the identity matrix. This means that $\mathbf{E}_3^{\,T}$ is an elementary matrix.

In each of the three row operations we end up with an elementary matrix therefore we conclude that the transpose of an elementary matrix is an elementary matrix.

■