## Complete Solutions to Exercises 7.2

MATLAB.

To find the characteristic polynomial of a matrix **A** enter the command poly(A).

1.The eigenvalues are determined by the characteristic equation

$$\begin{aligned}\det(\mathbf{A}-\lambda\mathbf{I}) &= \begin{pmatrix} 5-\lambda & 2 \\ -2 & 1-\lambda \end{pmatrix} \\ &= (5-\lambda)(1-\lambda)+4 \\ &= 5-5\lambda-\lambda+\lambda^2+4 \\ &= \lambda^2-6\lambda+9=0\end{aligned}$$

*How do we solve this quadratic equation* $\lambda^2-6\lambda+9=0$*?*

By factorizing $\lambda^2-6\lambda+9=(\lambda-3)(\lambda-3)=(\lambda-3)^2=0$ gives $\lambda_{1,\,2}=3$. *How do we find the corresponding eigenvectors?*

Substituting $\mathbf{A}=\begin{pmatrix} 5 & 2 \\ -2 & 1 \end{pmatrix}$, $\lambda=\lambda_{1,\,2}=3$ into $(\mathbf{A}-\lambda\mathbf{I})\mathbf{u}=\mathbf{O}$ where $\mathbf{u}=\begin{pmatrix} x \\ y \end{pmatrix}$ and $\mathbf{O}=\begin{pmatrix} 0 \\ 0 \end{pmatrix}$:

$$\begin{pmatrix} 5-3 & 2 \\ -2 & 1-3 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

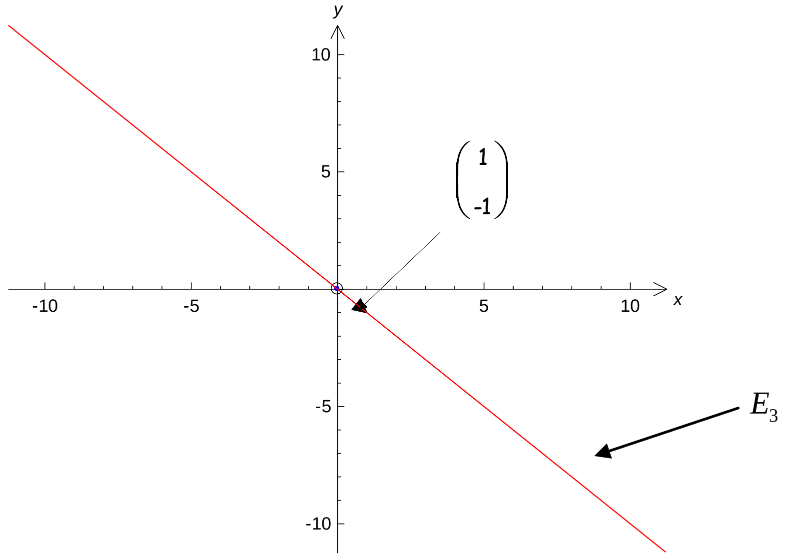
$$\begin{pmatrix} 2 & 2 \\ -2 & -2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Expanding this out gives

$$\left.\begin{aligned} 2x+2y&=0 \\ -2x-2y&=0 \end{aligned}\right\} \quad \text{which yields } y=-x$$

Let $x=s$ where $s\neq 0$ then $y=-s$ which gives the eigenvector $\mathbf{u}=\begin{pmatrix} s \\ -s \end{pmatrix}=s\begin{pmatrix} 1 \\ -1 \end{pmatrix}$.

The eigenspace $E_3$ is shown below:



A basis vector for $E_3$ is $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$.

2. The eigenvalues are determined by the characteristic equation:

$$\begin{aligned}
\det(\mathbf{A}-\lambda\mathbf{I}) &= \begin{pmatrix} -12-\lambda & 7 \\ -7 & 2-\lambda \end{pmatrix} \\
&= (-12-\lambda)(2-\lambda)+49 \\
&= (\lambda+12)(\lambda-2)+49 \\
&= \lambda^2+10\lambda-24+49 \\
&= \lambda^2+10\lambda+25 \\
&= (\lambda+5)^2 = 0
\end{aligned}$$

This gives $\lambda_{1,\,2} = -5$. *How do we find the corresponding eigenvectors?*

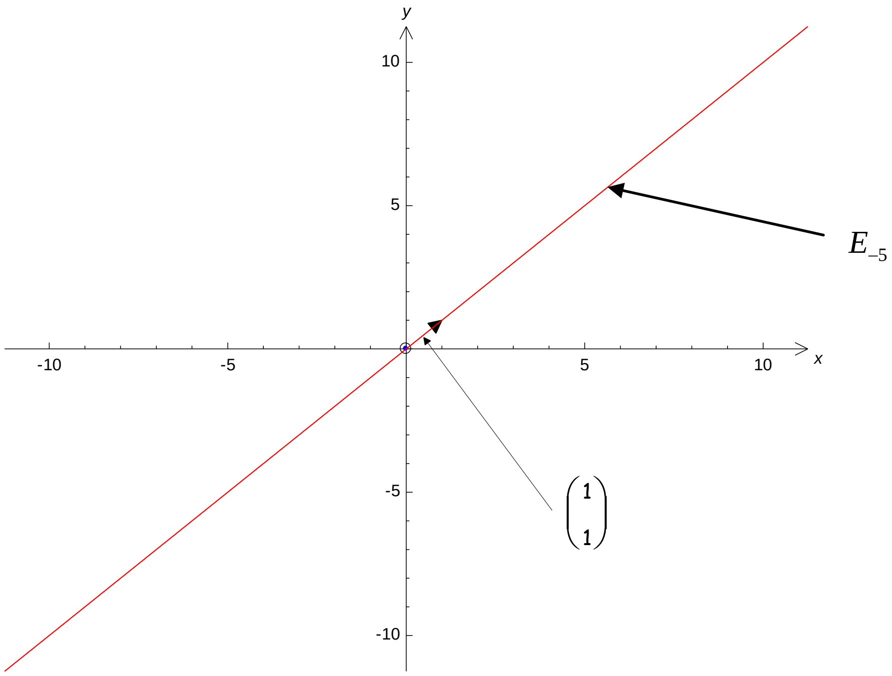
Substituting $\mathbf{A} = \begin{pmatrix} -12 & 7 \\ -7 & 2 \end{pmatrix}$, $\lambda_{1,\,2} = -5$ into $(\mathbf{A}-\lambda\mathbf{I})\mathbf{u} = \mathbf{O}$ where $\mathbf{u} = \begin{pmatrix} x \\ y \end{pmatrix}$ and $\mathbf{O} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$:

$$\begin{pmatrix} -12+5 & 7 \\ -7 & 2+5 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} -7 & 7 \\ -7 & 7 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Expanding this out and solving gives $y = x$. Let $x = s$ where $s \neq 0$ then $y = s$ which gives the eigenvector of the form $\mathbf{u} = \begin{pmatrix} s \\ s \end{pmatrix} = s\begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

The eigenspace $E_{-5}$ is shown below:




A basis vector for $E_{-5}$ is $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

3. *Proof.*

Let $\mathbf{A}=\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ then the characteristic polynomial, $p(\lambda)$, is given by

$$\begin{aligned}\det(\mathbf{A}-\lambda\mathbf{I}) &= \det\begin{pmatrix} a-\lambda & b \\ c & d-\lambda \end{pmatrix} \\ &= (a-\lambda)(d-\lambda)-bc \\ &= (\lambda-a)(\lambda-d)-bc \qquad [\text{Removing 2 minus signs}] \\ &= \lambda^2-d\lambda-a\lambda+ad-bc \\ &= \lambda^2-\underbrace{(a+d)}_{=\text{ Trace of }\mathbf{A}}\lambda+\underbrace{ad-bc}_{=\text{ Determinant of }\mathbf{A}} \\ &= \lambda^2-tr(\mathbf{A})\lambda+\det(\mathbf{A})=p(\lambda)\end{aligned}$$

Thus we have our required result.

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4. We use the result established in question 3 above and equate this to zero, which is

$$\lambda^2-tr(\mathbf{A})\lambda+\det(\mathbf{A})=0$$

Substituting $tr(\mathbf{A})=2a$ and $\det(\mathbf{A})=a^2$ into this result gives

$$\begin{aligned}p(\lambda) &= \lambda^2-tr(\mathbf{A})\lambda+\det(\mathbf{A}) \\ &= \lambda^2-2a\lambda+a^2=(\lambda-a)^2=0 \text{ gives } \lambda=\lambda_{1,\ 2}=a\end{aligned}$$

Thus the given matrix **A** has an eigenvalue $\lambda=a$ with multiplicity of 2.

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5. *What type of matrix is* $\mathbf{A}=\begin{pmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 7 & 9 & 1 \end{pmatrix}$*?*

Triangular (lower) matrix. *How do we find the eigenvalues of a triangular matrix **A**?*

| |
|---|
| Proposition (7-3) which says that if a matrix **A** is a diagonal or triangular matrix then the eigenvalues of **A** are the entries along the leading diagonal. |

Using this we have the eigenvalues $\lambda_{1,\ 2,\ 3}=1$ that is the eigenvalue is 1 with multiplicity 3. *How do we find the eigenvector **u**?*

By substituting $\lambda=1$ into $(\mathbf{A}-\lambda\mathbf{I})\mathbf{u}=\mathbf{O}$:

$$(\mathbf{A}-\lambda\mathbf{I})\mathbf{u}=\begin{pmatrix} 1-1 & 0 & 0 \\ -3 & 1-1 & 0 \\ 7 & 9 & 1-1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \qquad \left[\mathbf{u}=\begin{pmatrix} x \\ y \\ z \end{pmatrix} \text{ and } \mathbf{O}=\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}\right]$$

Expanding this gives

$$\begin{pmatrix} 0 & 0 & 0 \\ -3 & 0 & 0 \\ 7 & 9 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$0x + 0y + 0z = 0 \qquad (\dagger)$$
$$-3x + 0y + 0z = 0 \qquad (\dagger\dagger)$$
$$7x + 9y + 0z = 0 \qquad (\dagger\dagger\dagger)$$

From the middle equation $-3x = 0$ we have

$$x = 0$$

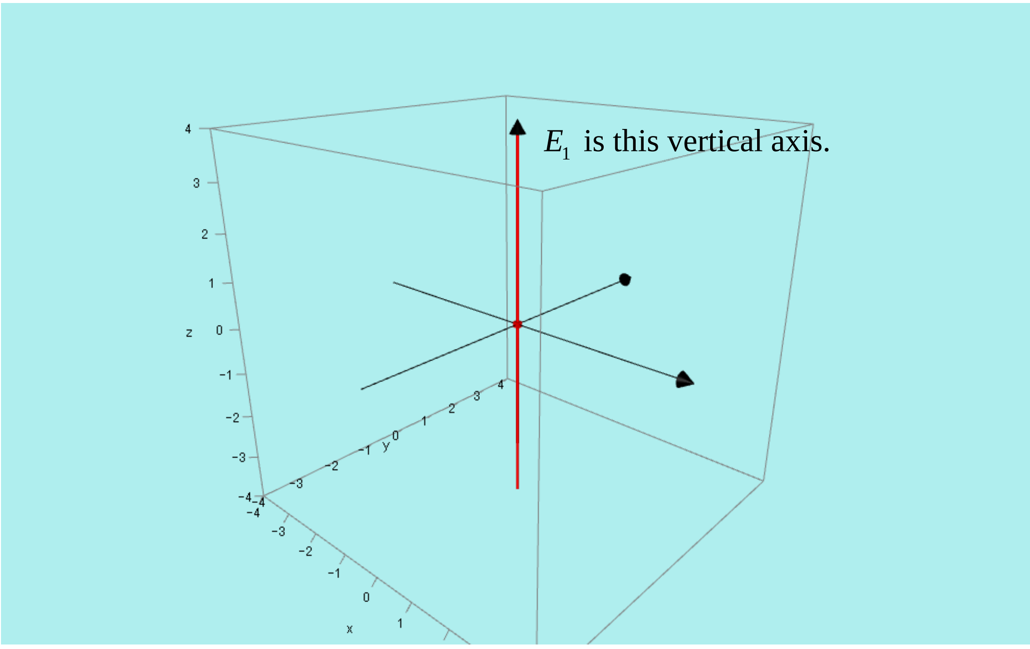
Substituting $x = 0$ into the bottom equation we have

$$7(0) + 9y + 0 = 0 \quad \text{gives} \quad y = 0$$

*What is the value of z?*

$z$ can have any non-zero value, $z = s$ where $s \neq 0$. Thus our eigenvector **u** is of the form

$$\mathbf{u} = \begin{pmatrix} 0 \\ 0 \\ s \end{pmatrix} = s\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \quad \text{where } s \neq 0$$




A basis vector for $E_1$ is $\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$.

6. (a) We are given the matrix $\mathbf{A} = \begin{pmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 2 \end{pmatrix}$. We have a diagonal matrix therefore by Proposition (7-3) we have the eigenvalues $\lambda_{1,\,2} = 5$ and $\lambda_3 = 2$.

Let **u** be an eigenvector belonging to $\lambda_{1,\,2} = 5$. Substituting $\lambda = \lambda_{1,\,2} = 5$ into the $(\mathbf{A} - \lambda\mathbf{I})\mathbf{u} = \mathbf{O}$:

$$\begin{pmatrix} 5-5 & 0 & 0 \\ 0 & 5-5 & 0 \\ 0 & 0 & 2-5 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \qquad \left[\text{Remember } \mathbf{u} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \text{ and } \mathbf{O} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}\right]$$

Expanding and simplifying gives

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -3 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

From the bottom equation we have $-3z = 0$ which gives $z = 0$. We have 1 non-zero equation but 3 unknowns therefore there are $3-1=2$ free variables, which are *x* and *y*. These can take any values, so let

$$x = s \text{ and } y = t$$

Our eigenvector **u** is of the form $\mathbf{u} = \begin{pmatrix} s \\ t \\ 0 \end{pmatrix} = s\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + t\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ where **both** *s* and *t* are **not zero**.

A basis vectors for $E_5$ is $\left\{\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}\right\}$.

We also have an eigenvector **v** associated with the other eigenvalue $\lambda_3 = 2$:

$$\begin{pmatrix} 5-2 & 0 & 0 \\ 0 & 5-2 & 0 \\ 0 & 0 & 2-2 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad \Rightarrow \quad \begin{pmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Writing out the equations gives

$$3x = 0,\ 3y = 0 \text{ and } 0 = 0$$

From the first 2 equations we have $x = 0,\ y = 0$. Note that *z* can be any non-zero real number, that is $z = s$ where $s \neq 0$. Our eigenvector **v** belonging to $\lambda_3 = 2$ is of the form

$\mathbf{v} = \begin{pmatrix} 0 \\ 0 \\ s \end{pmatrix} = s\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ where $s \neq 0$. A basis vector for $E_2$ is $\left\{\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}\right\}$.

(b) The eigenvalues of the given matrix $\mathbf{B} = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 5 & 1 \\ 0 & 0 & 9 \end{pmatrix}$ are the entries on the leading diagonal because we have a (upper) triangular matrix. Thus

$$\lambda_1 = 1,\ \lambda_2 = 5 \text{ and } \lambda_3 = 9$$

Let **u** be the eigenvector belonging to $\lambda_1 = 1$. We can find **u** by substituting $\lambda = \lambda_1 = 1$ into $(\mathbf{B} - \lambda\mathbf{I})\mathbf{u} = \mathbf{O}$:

$$(\mathbf{B}-\lambda\mathbf{I})\mathbf{u}=\begin{pmatrix}1-1 & 2 & 3\\ 0 & 5-1 & 1\\ 0 & 0 & 9-1\end{pmatrix}\begin{pmatrix}x\\ y\\ z\end{pmatrix}=\begin{pmatrix}0\\ 0\\ 0\end{pmatrix}$$

$$\begin{pmatrix}0 & 2 & 3\\ 0 & 4 & 1\\ 0 & 0 & 8\end{pmatrix}\begin{pmatrix}x\\ y\\ z\end{pmatrix}=\begin{pmatrix}0\\ 0\\ 0\end{pmatrix}$$

Expanding this gives the simultaneous equations

$$0x+2y+3z=0 \qquad (\dagger)$$

$$0x+4y+\ z=0 \qquad (\dagger\dagger)$$

$$0x+0y+8z=0 \qquad (\dagger\dagger\dagger)$$

From the bottom equation $(\dagger\dagger\dagger)$ we have $z=0$. Substituting this, $z=0$, into the middle equation $(\dagger\dagger)$ gives

$$4y=0 \text{ which yields } y=0$$

From the first equation $(\dagger)$ we have $x$ can be any non-zero number, that is $x=s$ where $s\neq 0$. Our eigenvector **u** belonging to $\lambda_1=1$ is of the form

$$\mathbf{u}=\begin{pmatrix}s\\ 0\\ 0\end{pmatrix}=s\begin{pmatrix}1\\ 0\\ 0\end{pmatrix} \qquad \text{where } s\neq 0$$

Let **v** be the eigenvector belonging to $\lambda_2=5$. We can find **v** by substituting $\lambda=\lambda_2=5$ into $(\mathbf{B}-\lambda\mathbf{I})\mathbf{v}=\mathbf{O}$:

$$(\mathbf{B}-\lambda\mathbf{I})\mathbf{v}=\begin{pmatrix}1-5 & 2 & 3\\ 0 & 5-5 & 1\\ 0 & 0 & 9-5\end{pmatrix}\begin{pmatrix}x\\ y\\ z\end{pmatrix}=\begin{pmatrix}0\\ 0\\ 0\end{pmatrix}$$

$$\begin{pmatrix}-4 & 2 & 3\\ 0 & 0 & 1\\ 0 & 0 & 4\end{pmatrix}\begin{pmatrix}x\\ y\\ z\end{pmatrix}=\begin{pmatrix}0\\ 0\\ 0\end{pmatrix}$$

From the bottom row we have $z=0$. Substituting this, $z=0$, into the first row gives

$$-4x+2y=0 \text{ which gives } y=2x$$

Let $x=s$ where $s\neq 0$ then $y=2s$ and our eigenvector **v** belonging to $\lambda_2=5$ is of the form:

$$\mathbf{v}=\begin{pmatrix}s\\ 2s\\ 0\end{pmatrix}=s\begin{pmatrix}1\\ 2\\ 0\end{pmatrix} \qquad \text{where } s\neq 0$$

Let **w** be the eigenvector belonging to $\lambda_3=9$. We can find **w** by substituting $\lambda=\lambda_3=9$ into $(\mathbf{B}-\lambda\mathbf{I})\mathbf{w}=\mathbf{O}$:

$$(\mathbf{B}-\lambda\mathbf{I})\mathbf{w}=\begin{pmatrix}1-9 & 2 & 3\\ 0 & 5-9 & 1\\ 0 & 0 & 9-9\end{pmatrix}\begin{pmatrix}x\\ y\\ z\end{pmatrix}=\begin{pmatrix}0\\ 0\\ 0\end{pmatrix}$$

$$\begin{pmatrix}-8 & 2 & 3\\ 0 & -4 & 1\\ 0 & 0 & 0\end{pmatrix}\begin{pmatrix}x\\ y\\ z\end{pmatrix}=\begin{pmatrix}0\\ 0\\ 0\end{pmatrix}$$

From the middle row we have

$$-4y+z=0 \text{ which gives } z=4y$$

Let $y=s$ then $z=4s$. Substituting these into the top row gives

$$-8x+2y+3z=0$$

$$-8x+2s+3(4s)=0 \quad\Rightarrow\quad 8x=14s \quad\Rightarrow\quad x=\frac{14}{8}s=\frac{7}{4}s$$

Our eigenvector **w** belonging to $\lambda_3=9$ is

$$\mathbf{w}=\begin{pmatrix}x\\ y\\ z\end{pmatrix}=\begin{pmatrix}7s/4\\ s\\ 4s\end{pmatrix}=\frac{s}{4}\begin{pmatrix}7\\ 4\\ 16\end{pmatrix} \qquad \text{where } s\neq 0$$

Basis vectors for eigenspaces $E_1,\ E_5$ and $E_9$ are $\begin{pmatrix}1\\ 0\\ 0\end{pmatrix},\ \begin{pmatrix}1\\ 2\\ 0\end{pmatrix}$ and $\begin{pmatrix}7\\ 4\\ 16\end{pmatrix}$ respectively.

(c) The eigenvalues of the matrix $\mathbf{C}=\begin{pmatrix}-2 & 0 & 0\\ 2 & -2 & 0\\ 4 & 10 & -2\end{pmatrix}$ are the entries on the leading diagonal because we have a (lower) triangular matrix. Thus

$$\lambda_{1,\ 2,\ 3}=-2$$

Let **u** be the eigenvector belonging to $\lambda_{1,\ 2,\ 3}=-2$. We can find **u** by substituting $\lambda=\lambda_{1,\ 2,\ 3}=-2$ into $(\mathbf{C}-\lambda\mathbf{I})\mathbf{u}=\mathbf{O}$:

$$(\mathbf{C}-\lambda\mathbf{I})\mathbf{u}=\begin{pmatrix}-2+2 & 0 & 0\\ 2 & -2+2 & 0\\ 4 & 10 & -2+2\end{pmatrix}\begin{pmatrix}x\\ y\\ z\end{pmatrix}=\begin{pmatrix}0\\ 0\\ 0\end{pmatrix}$$

$$\begin{pmatrix}0 & 0 & 0\\ 2 & 0 & 0\\ 4 & 10 & 0\end{pmatrix}\begin{pmatrix}x\\ y\\ z\end{pmatrix}=\begin{pmatrix}0\\ 0\\ 0\end{pmatrix}$$

From the middle row we have

$$2x=0 \;\Rightarrow\; x=0$$

Substituting $x=0$ into the last row

$$4(0)+10y+0=0 \quad\Rightarrow\quad y=0$$

$z$ can be any non-zero real number, let $z=s$ where $s\neq 0$. Therefore the eigenvector is of the form

$$\mathbf{u}=\begin{pmatrix}0\\0\\s\end{pmatrix}=s\begin{pmatrix}0\\0\\1\end{pmatrix} \quad \text{where } s\neq 0$$

A basis vector for $E_{-2}$ is $\begin{pmatrix}0\\0\\1\end{pmatrix}$.

7. (a) Since the given matrix is a (upper) triangular matrix therefore the eigenvalues are the diagonal entries which are 7, 7, 5 and 5. Thus we have
$\lambda_{1,\,2}=7$ and $\lambda_{3,\,4}=5$. Let **u** be the eigenvector for $\lambda=\lambda_{1,\,2}=7$:

$$(\mathbf{A}-\lambda\mathbf{I})\mathbf{u}=\begin{pmatrix}7-7&0&2&3\\0&7-7&4&6\\0&0&5-7&-3\\0&0&0&5-7\end{pmatrix}\begin{pmatrix}x\\y\\z\\w\end{pmatrix}=\begin{pmatrix}0\\0\\0\\0\end{pmatrix}$$

$$\begin{pmatrix}0&0&2&3\\0&0&4&6\\0&0&-2&-3\\0&0&0&-2\end{pmatrix}\begin{pmatrix}x\\y\\z\\w\end{pmatrix}=\begin{pmatrix}0\\0\\0\\0\end{pmatrix}$$

By expanding the bottom row we have $w=0$. Substituting this into the third row gives $z=0$.
Thus we have $z=0$ and $w=0$. Note that $x$ and $y$ can be any real numbers provided both are **not** zero. Let $x=s$ and $y=t$ therefore the eigenvector **u** belonging to $\lambda_{1,\,2}=7$ is of the form

$$\mathbf{u}=\begin{pmatrix}x\\y\\z\\w\end{pmatrix}=\begin{pmatrix}s\\t\\0\\0\end{pmatrix}=s\begin{pmatrix}1\\0\\0\\0\end{pmatrix}+t\begin{pmatrix}0\\1\\0\\0\end{pmatrix} \quad \text{where } s \text{ and } t \text{ are not both zero}$$

Let **v** be the eigenvector belonging to $\lambda=\lambda_{3,\,4}=5$:

$$(\mathbf{A}-\lambda\mathbf{I})\mathbf{v}=\begin{pmatrix}7-5&0&2&3\\0&7-5&4&6\\0&0&5-5&-3\\0&0&0&5-5\end{pmatrix}\begin{pmatrix}x\\y\\z\\w\end{pmatrix}=\begin{pmatrix}0\\0\\0\\0\end{pmatrix} \qquad \left[\mathbf{v}=\begin{pmatrix}x\\y\\z\\w\end{pmatrix}\right]$$

$$\begin{pmatrix}2&0&2&3\\0&2&4&6\\0&0&0&-3\\0&0&0&0\end{pmatrix}\begin{pmatrix}x\\y\\z\\w\end{pmatrix}=\begin{pmatrix}0\\0\\0\\0\end{pmatrix}$$

From the penultimate row we have $w=0$. By the second row we have

$$y+2z=0 \quad\Rightarrow\quad y=-2z$$

Let $z=s$ where $s\neq 0$ then $y=-2s$ and from the first row we have

$$2x+2z+3w=0 \quad \underset{\text{Because } w=0}{\Rightarrow} \quad x=-z \quad \Rightarrow \quad x=-s$$

Our eigenvector $\mathbf{v}=\begin{pmatrix}x\\y\\z\\w\end{pmatrix}$ belonging to $\lambda=\lambda_{3,\ 4}=5$ is of the form $\mathbf{v}=\begin{pmatrix}-s\\-2s\\s\\0\end{pmatrix}=s\begin{pmatrix}-1\\-2\\1\\0\end{pmatrix}$

because $x=-s,\ y=-2s,\ z=s$ and $w=0$.

(b) The eigenvalues are the leading diagonal entries because we have a diagonal matrix. Thus $\lambda_{1,\ 2,\ 3}=1$ and $\lambda_4=3$. Let **u** be the eigenvector corresponding to $\lambda_{1,\ 2,\ 3}=1$:

$$\begin{pmatrix}1-1&0&0&0\\0&1-1&0&0\\0&0&1-1&0\\0&0&0&3-1\end{pmatrix}\begin{pmatrix}x\\y\\z\\w\end{pmatrix}=\begin{pmatrix}0\\0\\0\\0\end{pmatrix}$$

$$\begin{pmatrix}0&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&2\end{pmatrix}\begin{pmatrix}x\\y\\z\\w\end{pmatrix}=\begin{pmatrix}0\\0\\0\\0\end{pmatrix}$$

We have 4 unknowns- *x*, *y*, *z* and *w* but only 1 non-zero equation therefore we have $4-1=3$ free variables. From the last row we have $w=0$. Note that other 3 are the free variables which means they can take any values provided **all 3** are not zero.

Let $x=s,\ y=t$ and $z=r$. Our eigenvector **u** is given by

$$\mathbf{u}=\begin{pmatrix}s\\t\\r\\0\end{pmatrix}=s\begin{pmatrix}1\\0\\0\\0\end{pmatrix}+t\begin{pmatrix}0\\1\\0\\0\end{pmatrix}+r\begin{pmatrix}0\\0\\1\\0\end{pmatrix} \text{ provided all } r,\ s \text{ and } t \text{ are } \textbf{not zero}$$

Let **v** be an eigenvector for $\lambda_4=3$. Substituting this into the above we have

$$\begin{pmatrix}1-3&0&0&0\\0&1-3&0&0\\0&0&1-3&0\\0&0&0&3-3\end{pmatrix}\begin{pmatrix}x\\y\\z\\w\end{pmatrix}=\begin{pmatrix}0\\0\\0\\0\end{pmatrix}$$

$$\begin{pmatrix}-2&0&0&0\\0&-2&0&0\\0&0&-2&0\\0&0&0&0\end{pmatrix}\begin{pmatrix}x\\y\\z\\w\end{pmatrix}=\begin{pmatrix}0\\0\\0\\0\end{pmatrix}$$

From the first 3 rows we have $x=0,\ y=0$ and $z=0$. The variable *w* is free therefore it can take any value, *s*, but not zero. Thus we have

$$\mathbf{v}=\begin{pmatrix}x\\y\\z\\w\end{pmatrix}=\begin{pmatrix}0\\0\\0\\s\end{pmatrix}=s\begin{pmatrix}0\\0\\0\\1\end{pmatrix} \qquad \text{where } s\neq 0$$

(c) Again we have a diagonal matrix with eigenvalues given by the entries along the leading diagonal. Thus $\lambda_{1,\ 2,\ 3,\ 4} = 3$. Let **u** be the eigenvector belonging to this eigenvalue. We find **u** be substituting this $\lambda = \lambda_{1,\ 2,\ 3,\ 4} = 3$ into $(\mathbf{C} - \lambda\mathbf{I})\mathbf{u} = \mathbf{O}$:

$$(\mathbf{C} - 3\mathbf{I})\mathbf{u} = \begin{pmatrix} 3-3 & 0 & 0 & 0 \\ 0 & 3-3 & 0 & 0 \\ 0 & 0 & 3-3 & 0 \\ 0 & 0 & 0 & 3-3 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Since **all** the rows are zero therefore the number of free variables are $4 - 0 = 4$. Let

$x = s,\ \ y = t,\ \ z = r$ and $w = q$ where **all** are not zero

Our eigenvector **u** belonging to $\lambda_{1,\ 2,\ 3,\ 4} = 3$ is of the form

$$\mathbf{u} = \begin{pmatrix} s \\ t \\ r \\ q \end{pmatrix} = s\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + t\begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} + r\begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} + q\begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

8. We need to prove Proposition (7-4) which says a matrix is invertible (non-singular) $\Leftrightarrow$ $\lambda = 0$ is **not** an eigenvalue.

*Proof.*

Suppose $\lambda = 0$ is an eigenvalue of a matrix **A**. Then substituting this $\lambda = 0$ into the characteristic equation $\det(\mathbf{A} - \lambda\mathbf{I}) = 0$ we have

$$\det(\mathbf{A}) = 0$$

By Proposition (6-13) this is true $\Leftrightarrow$ matrix **A** is **not invertibe** (singular). Thus $\lambda = 0$ **cannot** be an eigenvalue of an invertible matrix.

9. We need to prove the eigenvalue is unique.

*Proof.*

Suppose the eignvector **u** belongs to 2 eigenvalues $\lambda$ and $t$. This means we have

$\mathbf{Au} = \lambda\mathbf{u}$ and $\mathbf{Au} = t\mathbf{u}$

Subtracting these gives

$$\mathbf{Au} - \mathbf{Au} = \mathbf{O} = t\mathbf{u} - \lambda\mathbf{u} = (t - \lambda)\mathbf{u}$$

Since **u** is an eigenvector therefore it **cannot** be a zero vector. Thus $(t - \lambda)\mathbf{u} = \mathbf{O}$ yields

$$t - \lambda = 0 \quad \text{which gives } t = \lambda$$

Thus the eigenvalue is unique.

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10. We need to prove that **A** has distinct eigenvalues provided $[tr(\mathbf{A})]^2 > 4\det(\mathbf{A})$.

*Proof.*

The characteristic equation is given by $\lambda^2 - tr(\mathbf{A})\lambda + \det(\mathbf{A}) = 0$.

*What sort of equation is this?*

Quadratic equation $ax^2 + bx + c = 0$ which has distinct roots if

$$b^2 - 4ac > 0 \quad \text{or} \quad b^2 > 4ac$$

For our equation $\lambda^2 - tr(\mathbf{A})\lambda + \det(\mathbf{A}) = 0$ we have

$$a = 1,\ b = -tr(\mathbf{A}) \quad \text{and} \quad c = \det(\mathbf{A})$$

Substituting these into $b^2 > 4ac$ yields

$$\left[-tr(\mathbf{A})\right]^2 > 4\det(\mathbf{A}) \quad \Leftrightarrow \quad \left[tr(\mathbf{A})\right]^2 > 4\det(\mathbf{A})$$

gives distinct roots, which means we have distinct eigenvalues.

■

We have equal eigenvalues (roots) if $b^2 = 4ac$ which is $\left[tr(\mathbf{A})\right]^2 = 4\det(\mathbf{A})$.

We have complex eigenvalues (roots) if $b^2 < 4ac$ which is $\left[tr(\mathbf{A})\right]^2 < 4\det(\mathbf{A})$.

11\. (a) (i) We have a triangular matrix $\mathbf{A} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 3 & 2 & 0 & 0 \\ 5 & 2 & 3 & 0 \\ 9 & 8 & 1 & 4 \end{pmatrix}$. The eigenvalues are the entries on the leading diagonal, $\lambda_1 = 1,\ \lambda_2 = 2,\ \lambda_3 = 3$ and $\lambda_4 = 4$.

(ii) Eigenvalues of $\mathbf{A}^5$ are given by $\lambda^5$ which is

$$(\lambda_1)^5 = 1^5 = 1,\ (\lambda_2)^5 = 2^5 = 32,\ (\lambda_3)^5 = 3^5 = 243 \text{ and } (\lambda_4)^5 = 4^5 = 1024$$

(iii) Eigenvalues of $\mathbf{A}^{-1}$ are $\lambda^{-1}$:

$$(\lambda_1)^{-1} = 1,\ (\lambda_2)^{-1} = \frac{1}{2},\ (\lambda_3)^{-1} = \frac{1}{3} \text{ and } (\lambda_4)^{-1} = \frac{1}{4}$$

(iv) The determinant $\det(\mathbf{A})$ is the multiplication of all the eigenvalues

$$\det(\mathbf{A}) = 1 \times 2 \times 3 \times 4 = 24$$

(v) The trace is the addition of all the eigenvalues

$$tr(\mathbf{A}) = 1 + 2 + 3 + 4 = 10$$

(b) (i) We have a triangular matrix $\mathbf{A} = \begin{pmatrix} -1 & 3 & 4 & 7 \\ 0 & 6 & -3 & 5 \\ 0 & 0 & -8 & 9 \\ 0 & 0 & 0 & 3 \end{pmatrix}$. The eigenvalues are the entries on the leading diagonal, $\lambda_1 = -1,\ \lambda_2 = 6,\ \lambda_3 = -8$ and $\lambda_4 = 3$.

(ii) Eigenvalues of $\mathbf{A}^5$ are given by $\lambda^5$ which is

$$(\lambda_1)^5 = (-1)^5 = -1,\ (\lambda_2)^5 = 6^5 = 7776,\ (\lambda_3)^5 = (-8)^5 = -32768 \text{ and } (\lambda_4)^5 = 3^5 = 243$$

(iii) Eigenvalues of $\mathbf{A}^{-1}$ are $\lambda^{-1}$:

$$(\lambda_1)^{-1} = -1,\ (\lambda_2)^{-1} = \frac{1}{6},\ (\lambda_3)^{-1} = -\frac{1}{8} \text{ and } (\lambda_4)^{-1} = \frac{1}{3}$$

(iv) The determinant $\det(\mathbf{A})$ is the multiplication of all the eigenvalues

$$\det(\mathbf{A}) = -1\times 6\times(-8)\times 3 = 144$$

(v) The trace is the addition of all the eigenvalues

$$tr(\mathbf{A}) = -1+6-8+3 = 0$$

(c) (i) We have a triangular matrix $\mathbf{A} = \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & -4 & 0 & 0 \\ 0 & 0 & -7 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$. The eigenvalues are the entries on the leading diagonal, $\lambda_1 = 2,\ \lambda_2 = -4,\ \lambda_3 = -7$ and $\lambda_4 = 0$.

(ii) Eigenvalues of $\mathbf{A}^5$ are given by $\lambda^5$ which is

$$(\lambda_1)^5 = 2^5 = 32,\ (\lambda_2)^5 = (-4)^5 = -1024,\ (\lambda_3)^5 = (-7)^5 = -16807 \text{ and } (\lambda_4)^5 = 0^5 = 0$$

(iii) $\mathbf{A}^{-1}$ does **not** exist for the matrix **A** because $\lambda_4 = 0$ is an eigenvalue which means that the matrix **A** is **not** invertible.

(iv) The determinant $\det(\mathbf{A})$ is the multiplication of all the eigenvalues

$$\det(\mathbf{A}) = 2\times(-4)\times(-7)\times 0 = 0$$

(v) The trace is the addition of all the eigenvalues

$$tr(\mathbf{A}) = 2-4-7+0 = -9$$

12. The characteristic polynomial for the given matrix is

$$p(\lambda) = \det\begin{pmatrix} 2-\lambda & 1 \\ -2 & 1-\lambda \end{pmatrix}$$
$$= (2-\lambda)(1-\lambda)+2$$

Expanding this and simplifying gives

$$p(\lambda) = 2-3\lambda+\lambda^2+2 = \lambda^2-3\lambda+4$$

The characteristic polynomial is $p(\lambda) = \lambda^2-3\lambda+4$. Thus

$$p(\mathbf{A}) = \mathbf{A}^2-3\mathbf{A}+4\mathbf{I} = \mathbf{O}$$

Rearranging this gives

$$\mathbf{A}^2-3\mathbf{A} = -4\mathbf{I}$$
$$\mathbf{A}(\mathbf{A}-3\mathbf{I}) = -4\mathbf{I} \quad\Rightarrow\quad \mathbf{A}\underbrace{\left[-\frac{1}{4}(\mathbf{A}-3\mathbf{I})\right]}_{=\mathbf{A}^{-1}} = \mathbf{I}$$

The inverse matrix

$$\mathbf{A}^{-1} = \left[-\frac{1}{4}(\mathbf{A}-3\mathbf{I})\right] = -\frac{1}{4}\left[\begin{pmatrix} 2 & 1 \\ -2 & 1 \end{pmatrix} - \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix}\right]$$
$$= -\frac{1}{4}\begin{pmatrix} -1 & 1 \\ -2 & -2 \end{pmatrix}$$
$$= \frac{1}{4}\begin{pmatrix} 1 & -1 \\ 2 & 2 \end{pmatrix} \qquad \begin{bmatrix}\text{Taking in the} \\ \text{minus sign}\end{bmatrix}$$

13. The characteristic polynomial for $\mathbf{A}=\begin{pmatrix}6 & 5\\ 3 & 4\end{pmatrix}$ is $p(\lambda)=\lambda^2-10\lambda+9$.

Applying the Cayley Hamilton theorem we have

$$p(\mathbf{A})=\mathbf{A}^2-10\mathbf{A}+9\mathbf{I}=\mathbf{O}$$

Rearranging

$$\begin{aligned}\mathbf{A}^2=10\mathbf{A}-9\mathbf{I}&=10\begin{pmatrix}6 & 5\\ 3 & 4\end{pmatrix}-9\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}\\ &=\begin{pmatrix}60 & 50\\ 30 & 40\end{pmatrix}-\begin{pmatrix}9 & 0\\ 0 & 9\end{pmatrix}=\begin{pmatrix}51 & 50\\ 30 & 31\end{pmatrix}\end{aligned}$$

*How can we find* $\mathbf{A}^3$*?*

$$\mathbf{A}^3=\mathbf{A}\mathbf{A}^2=\mathbf{A}(10\mathbf{A}-9\mathbf{I})=10\mathbf{A}^2-9\mathbf{A}$$

Substituting the above results into this, $\mathbf{A}^3=10\mathbf{A}^2-9\mathbf{A}$, gives

$$\begin{aligned}\mathbf{A}^3&=10\mathbf{A}^2-9\mathbf{A}\\ &=10\begin{pmatrix}51 & 50\\ 30 & 31\end{pmatrix}-9\begin{pmatrix}6 & 5\\ 3 & 4\end{pmatrix}\\ &=\begin{pmatrix}510 & 500\\ 300 & 310\end{pmatrix}-\begin{pmatrix}54 & 45\\ 27 & 36\end{pmatrix}=\begin{pmatrix}510-54 & 500-45\\ 300-27 & 310-36\end{pmatrix}=\begin{pmatrix}456 & 455\\ 273 & 274\end{pmatrix}\end{aligned}$$

14. We are given $p(\lambda)=\lambda^3-4\lambda^2-\lambda+4$. By the Cayley Hamilton theorem we have

$$p(\mathbf{A})=\mathbf{A}^3-4\mathbf{A}^2-\mathbf{A}+4\mathbf{I}=\mathbf{O}$$

Rearranging this gives

$$\begin{aligned}\mathbf{A}^3-4\mathbf{A}^2-\mathbf{A}&=-4\mathbf{I}\\ \mathbf{A}\left(\mathbf{A}^2-4\mathbf{A}-\mathbf{I}\right)&=-4\mathbf{I}\\ \mathbf{A}\underbrace{\left[-\frac{1}{4}\left(\mathbf{A}^2-4\mathbf{A}-\mathbf{I}\right)\right]}_{=\mathbf{A}^{-1}}&=\mathbf{I} \qquad \text{[Dividing both sides by } -4\text{]}\end{aligned}$$

Thus $\mathbf{A}^{-1}=-\frac{1}{4}\left(\mathbf{A}^2-4\mathbf{A}-\mathbf{I}\right)$. *How do we find* $\mathbf{A}^4$*?*

By making $\mathbf{A}^3$ the subject of $\mathbf{A}^3-4\mathbf{A}^2-\mathbf{A}=-4\mathbf{I}$:

$$\begin{aligned}\mathbf{A}^3&=4\mathbf{A}^2+\mathbf{A}-4\mathbf{I}\\ \mathbf{A}^4=\mathbf{A}\mathbf{A}^3&=\mathbf{A}\left(4\mathbf{A}^2+\mathbf{A}-4\mathbf{I}\right)\\ &=4\mathbf{A}^3+\mathbf{A}^2-4\mathbf{A} \qquad &&\text{[Expanding]}\\ &=4\left(4\mathbf{A}^2+\mathbf{A}-4\mathbf{I}\right)+\mathbf{A}^2-4\mathbf{A} \qquad &&\text{[Replacing } \mathbf{A}^3\text{]}\\ &=16\mathbf{A}^2+4\mathbf{A}-16\mathbf{I}+\mathbf{A}^2-4\mathbf{A}\\ &=17\mathbf{A}^2-16\mathbf{I}\end{aligned}$$

Thus $\mathbf{A}^4=17\mathbf{A}^2-16\mathbf{I}$.

15. We need to prove the following: If **A** is a *n* by *n* matrix with distinct eigenvalues $\lambda_1,\ \lambda_2,\ \lambda_3,\ \cdots$ and $\lambda_m$ and then the corresponding eigenvectors $\mathbf{u}_1,\ \mathbf{u}_2,\ \mathbf{u}_3,\ \cdots$ and $\mathbf{u}_m$ are **linearly independent**.

*How do we prove this proposition?*

We use mathematical induction. We prove the result for $m=1$, assume it is true for $m=k$ and then prove it for $m=k+1$.

*Proof.*

Clearly $\mathbf{u}_1$ is linearly independent.√

Assume the result is true for $m=k$, that is the eigenvectors $\mathbf{u}_1,\ \mathbf{u}_2,\ \mathbf{u}_3,\ \cdots$ and $\mathbf{u}_k$ are linearly independent.

Required to prove the result for $m=k+1$:

Suppose the eigenvectors $\mathbf{u}_1,\ \mathbf{u}_2,\ \mathbf{u}_3,\ \cdots,\ \mathbf{u}_k$ and $\mathbf{u}_{k+1}$ are linearly **dependent**.

Then we can write $\mathbf{u}_{k+1}$ as a linear combination of the other vectors:

$$c_1\mathbf{u}_1+c_2\mathbf{u}_2+c_3\mathbf{u}_3+\cdots+c_k\mathbf{u}_k=\mathbf{u}_{k+1} \qquad (*)$$

where the *c*'s are scalars. Multiplying this by matrix **A** gives

$$\mathbf{A}\left(c_1\mathbf{u}_1+c_2\mathbf{u}_2+c_3\mathbf{u}_3+\cdots+c_k\mathbf{u}_k\right)=\mathbf{A}\mathbf{u}_{k+1}$$

$$c_1\mathbf{A}\mathbf{u}_1+c_2\mathbf{A}\mathbf{u}_2+c_3\mathbf{A}\mathbf{u}_3+\cdots+c_k\mathbf{A}\mathbf{u}_k=\mathbf{A}\mathbf{u}_{k+1} \qquad \left[\text{Because } \mathbf{A}c\mathbf{u}=c\mathbf{A}\mathbf{u}\right]$$

$$c_1\lambda_1\mathbf{u}_1+c_2\lambda_2\mathbf{u}_2+c_3\lambda_3\mathbf{u}_3+\cdots+c_k\lambda_k\mathbf{u}_k=\lambda_{k+1}\mathbf{u}_{k+1} \qquad \left[\text{Because } \mathbf{A}\mathbf{u}=\lambda\mathbf{u}\right]$$

The last line follows from the definition of eigenvalue and eigenvector. Referencing the last line with $(\dagger)$:

$$c_1\lambda_1\mathbf{u}_1+c_2\lambda_2\mathbf{u}_2+c_3\lambda_3\mathbf{u}_3+\cdots+c_k\lambda_k\mathbf{u}_k=\lambda_{k+1}\mathbf{u}_{k+1} \qquad (\dagger)$$

Multiplying the above labelled (*) by $\lambda_{k+1}$ gives

$$\lambda_{k+1}c_1\mathbf{u}_1+\lambda_{k+1}c_2\mathbf{u}_2+\lambda_{k+1}c_3\mathbf{u}_3+\cdots+\lambda_{k+1}c_k\mathbf{u}_k=\lambda_{k+1}\mathbf{u}_{k+1} \qquad (**)$$

Subtracting $(**)$ and $(\dagger)$ gives

$$\underbrace{c_1\lambda_{k+1}\mathbf{u}_1+c_2\lambda_{k+1}\mathbf{u}_2+\cdots+c_k\lambda_{k+1}\mathbf{u}_k}_{=(**)}-\underbrace{\left(c_1\lambda_1\mathbf{u}_1+c_2\lambda_2\mathbf{u}_2+\cdots+c_k\lambda_k\mathbf{u}_k\right)}_{=(\dagger)}=\mathbf{O}$$

$$c_1\left(\lambda_{k+1}-\lambda_1\right)\mathbf{u}_1+c_2\left(\lambda_{k+1}-\lambda_2\right)\mathbf{u}_2+\cdots+c_k\left(\lambda_{k+1}-\lambda_k\right)\mathbf{u}_k=\mathbf{O} \qquad \left[\text{Collecting Like terms}\right]$$

Since $\mathbf{u}_1,\ \mathbf{u}_2,\ \mathbf{u}_3,\ \cdots$ and $\mathbf{u}_k$ are linearly independent therefore

$$c_1\left(\lambda_{k+1}-\lambda_1\right)=c_2\left(\lambda_{k+1}-\lambda_2\right)=\cdots=c_k\left(\lambda_{k+1}-\lambda_k\right)=0$$

None of the bracket terms are zero. *Why not?*

Because **all** the eigenvalues are **distinct**

$$\lambda_{k+1}-\lambda_j\neq 0 \text{ for } j=1,\ 2,\ 3,\ \cdots,\ k$$

Therefore all the *c*'s must be zero, that is

$$c_1=0,\ c_2=0,\ c_3=0,\ \cdots,\ c_k=0$$

Substituting these into (*) gives

$$0\mathbf{u}_1+0\mathbf{u}_2+0\mathbf{u}_3+\cdots+0\mathbf{u}_k=\mathbf{u}_{k+1}$$

$$\mathbf{O}=\mathbf{u}_{k+1}$$

This is impossible. *Why?*

Because $\mathbf{u}_{k+1}$ is an eigenvector which means it **cannot** be zero. Since we have a contradiction therefore the supposition $\mathbf{u}_1,\ \mathbf{u}_2,\ \mathbf{u}_3,\ \cdots,\ \mathbf{u}_k$ and $\mathbf{u}_{k+1}$ are linearly **dependent** is false.
Hence by mathematical induction we have $\mathbf{u}_1,\ \mathbf{u}_2,\ \mathbf{u}_3,\ \cdots,$ and $\mathbf{u}_m$ are **linearly independent**.

■

16. Let **A** be a square matrix with eigenvalues, $\lambda$, then the characteristic polynomial is given by $p(\lambda)=\det(\mathbf{A}-\lambda\mathbf{I})$. Consider the matrix $(\mathbf{A}-\lambda\mathbf{I})^T$:

$$(\mathbf{A}-\lambda\mathbf{I})^T=\mathbf{A}^T-(\lambda\mathbf{I})^T \qquad \left[\text{By (1-4) (c) } (\mathbf{A}+\mathbf{B})^T=\mathbf{A}^T+\mathbf{B}^T\right]$$
$$=\mathbf{A}^T-\lambda\mathbf{I} \qquad \left[\text{Because } \lambda\mathbf{I} \text{ is a Diagonal Matrix}\right]$$

Consider the characteristic polynomial $q(\lambda)$ for $\mathbf{A}^T$:

$$\begin{aligned}
q(\lambda)&=\det\left(\mathbf{A}^T-\lambda\mathbf{I}\right)\\
&=\det\left(\mathbf{A}^T-(\lambda\mathbf{I})^T\right)\\
&=\det\left([\mathbf{A}-\lambda\mathbf{I}]^T\right)\\
&=\det(\mathbf{A}-\lambda\mathbf{I}) \qquad \left[\text{Because by (6-5) we have } \det\left(\mathbf{B}^T\right)=\det(\mathbf{B})\right]\\
&=p(\lambda)
\end{aligned}$$

We have the **same** characteristic polynomial $q(\lambda)=p(\lambda)$. Therefore the eigenvalues $\lambda$ (roots) of $q(\lambda)=0$ are exactly the eigenvalues of $p(\lambda)=0$. Thus the matrix **A** and $\mathbf{A}^T$ have the same eigenvalues.

■