Complete solutions to Exercises **1.5**

1. Given $\mathbf{A}=\begin{pmatrix}2&-2&5\\0&-1&7\end{pmatrix}$, $\mathbf{B}=\begin{pmatrix}-1&3&7\\2&-9&6\end{pmatrix}$ and $\mathbf{C}=\begin{pmatrix}9&5&8\\-6&-1&6\end{pmatrix}$ we have:

(a) $\mathbf{A}+\mathbf{B}=\begin{pmatrix}2&-2&5\\0&-1&7\end{pmatrix}+\begin{pmatrix}-1&3&7\\2&-9&6\end{pmatrix}=\begin{pmatrix}1&1&12\\2&-10&13\end{pmatrix}$

(b) $(\mathbf{A}+\mathbf{B})+\mathbf{C}=\underbrace{\begin{pmatrix}1&1&12\\2&-10&13\end{pmatrix}}_{=\mathbf{A}+\mathbf{B}\text{ by Part (a)}}+\underbrace{\begin{pmatrix}9&5&8\\-6&-1&6\end{pmatrix}}_{=\mathbf{C}}=\begin{pmatrix}10&6&20\\-4&-11&19\end{pmatrix}$

(c) $\mathbf{A}+(\mathbf{B}+\mathbf{C})\underset{\text{By (1.5)(b)}}{=}(\mathbf{A}+\mathbf{B})+\mathbf{C}\underset{\text{By Part (b) above}}{=}\begin{pmatrix}10&6&20\\-4&-11&19\end{pmatrix}$

(d) $\mathbf{B}+(\mathbf{C}+\mathbf{A})=(\mathbf{B}+\mathbf{C})+\mathbf{A}=(\mathbf{A}+\mathbf{B})+\mathbf{C}\underset{\text{By Part (b)}}{=}\begin{pmatrix}10&6&20\\-4&-11&19\end{pmatrix}$

(e) $\mathbf{C}+\mathbf{O}\underset{\text{By Theorem (1-1)(d)}}{=}\mathbf{C}=\begin{pmatrix}9&5&8\\-6&-1&6\end{pmatrix}$

(f) $\mathbf{B}+\mathbf{B}+\mathbf{B}+\mathbf{B}+\mathbf{B}=5\mathbf{B}=5\begin{pmatrix}-1&3&7\\2&-9&6\end{pmatrix}=\begin{pmatrix}-5&15&35\\10&-45&30\end{pmatrix}$

(g) 5**B** is evaluated in part (f).

(h) $\mathbf{C}+(-\mathbf{C})=\mathbf{C}-\mathbf{C}=\mathbf{O}=\begin{pmatrix}0&0&0\\0&0&0\end{pmatrix}$.

2. (a) We need to find a matrix **B** such that

$$\mathbf{A}+\mathbf{B}=\mathbf{O}\ \text{ where }\ \mathbf{A}=\begin{pmatrix}5&-1&-2\\1&-3&2\end{pmatrix}$$

By Theorem (1-1) part (e) we have $\mathbf{A}+(-\mathbf{A})=\mathbf{O}$therefore

$$\mathbf{B}=-\mathbf{A}=-\begin{pmatrix}5&-1&-2\\1&-3&2\end{pmatrix}=\begin{pmatrix}-5&1&2\\-1&3&-2\end{pmatrix}$$

(b) Since $\mathbf{A}+(-\mathbf{C})=\mathbf{A}-\mathbf{C}$ we have

$$\begin{aligned}\mathbf{A}+(-\mathbf{C})&=\mathbf{A}-\mathbf{C}\\&=\begin{pmatrix}5&-1&-2\\1&-3&2\end{pmatrix}-\begin{pmatrix}1&2&3\\4&5&6\end{pmatrix}=\begin{pmatrix}4&-3&-5\\-3&-8&-4\end{pmatrix}\end{aligned}$$

(c) We have

$$\begin{aligned}\mathbf{A}-\mathbf{B}-\mathbf{C}&=\mathbf{A}-(-\mathbf{A})-\mathbf{C}\qquad[\text{Because }\mathbf{B}=-\mathbf{A}]\\&=\mathbf{A}+\mathbf{A}+\mathbf{C}\\&=2\mathbf{A}+\mathbf{C}=2\begin{pmatrix}5&-1&-2\\1&-3&2\end{pmatrix}-\begin{pmatrix}1&2&3\\4&5&6\end{pmatrix}\\&=\begin{pmatrix}10&-2&-4\\2&-6&4\end{pmatrix}-\begin{pmatrix}1&2&3\\4&5&6\end{pmatrix}=\begin{pmatrix}9&-4&-7\\-2&-11&-2\end{pmatrix}\end{aligned}$$

$$\mathbf{A}-(\mathbf{B}-\mathbf{C})=\mathbf{A}-(-\mathbf{A}-\mathbf{C}) \qquad [\text{Because } \mathbf{B}=-\mathbf{A}]$$
$$=\mathbf{A}+\mathbf{A}+\mathbf{C}$$
$$=2\mathbf{A}+\mathbf{C}=\underbrace{\begin{pmatrix}10 & -2 & -4\\ 2 & -6 & 4\end{pmatrix}}_{\text{By Above}}+\begin{pmatrix}1 & 2 & 3\\ 4 & 5 & 6\end{pmatrix}=\begin{pmatrix}11 & 0 & -1\\ 6 & -1 & 10\end{pmatrix}$$

3. Using $\mathbf{A}=\begin{pmatrix}1 & 5 & -2\\ 3 & 7 & -9\end{pmatrix}$, $\mathbf{B}=\begin{pmatrix}-1 & 0 & 7\\ -3 & 2 & 1\\ -4 & 1 & 6\end{pmatrix}$ and $\mathbf{C}=\begin{pmatrix}7 & 8 & -1\\ -8 & 6 & -6\\ 6 & -3 & 5\end{pmatrix}$.

(a) Evaluating **AB** we have

$$\mathbf{AB}=\begin{pmatrix}1 & 5 & -2\\ 3 & 7 & -9\end{pmatrix}\begin{pmatrix}-1 & 0 & 7\\ -3 & 2 & 1\\ -4 & 1 & 6\end{pmatrix}$$
$$=\begin{pmatrix}-8 & 8 & 0\\ 12 & 5 & -26\end{pmatrix}$$

(b) We cannot evaluate **BA**

$$\mathbf{BA}=\begin{pmatrix}-1 & 0 & 7\\ -3 & 2 & 1\\ -4 & 1 & 6\end{pmatrix}\begin{pmatrix}1 & 5 & -2\\ 3 & 7 & -9\end{pmatrix}$$

because the number of columns (3) of the Left Hand Matrix, **B**, does **not** equal the number of rows (2) of the Right Hand Matrix, **A**.

(c) Evaluating $(\mathbf{B}+\mathbf{C})$ first and then multiplying the result by **A**:

$$\mathbf{A}(\mathbf{B}+\mathbf{C})=\begin{pmatrix}1 & 5 & -2\\ 3 & 7 & -9\end{pmatrix}\left[\begin{pmatrix}-1 & 0 & 7\\ -3 & 2 & 1\\ -4 & 1 & 6\end{pmatrix}+\begin{pmatrix}7 & 8 & -1\\ -8 & 6 & -6\\ 6 & -3 & 5\end{pmatrix}\right]$$
$$=\underbrace{\begin{pmatrix}1 & 5 & -2\\ 3 & 7 & -9\end{pmatrix}}_{=\mathbf{A}}\underbrace{\begin{pmatrix}6 & 8 & 6\\ -11 & 8 & -5\\ 2 & -2 & 11\end{pmatrix}}_{=\mathbf{B}+\mathbf{C}}=\begin{pmatrix}-53 & 52 & -41\\ -77 & 98 & -116\end{pmatrix}$$

(d) By Propositiion (1 -3)(b) we have $\mathbf{AB}+\mathbf{AC}=\mathbf{A}(\mathbf{B}+\mathbf{C})$ and we have worked out the Right Hand side of this equation in part (c) above, therefore

$$\mathbf{AB}+\mathbf{AC}=\mathbf{A}(\mathbf{B}+\mathbf{C})=\begin{pmatrix}-53 & 52 & -41\\ -77 & 98 & -116\end{pmatrix}$$

(e) We have already evaluated $(\mathbf{B}+\mathbf{C})$ in part (c) above. All we need to do is to multiply this result by **A**:

$$(\mathbf{B}+\mathbf{C})\mathbf{A}=\underbrace{\begin{pmatrix}6 & 8 & 6\\ -11 & 8 & -5\\ 2 & -2 & 11\end{pmatrix}}_{=\mathbf{B}+\mathbf{C}}\underbrace{\begin{pmatrix}1 & 5 & -2\\ 3 & 7 & -9\end{pmatrix}}_{=\mathbf{A}}$$

But we **cannot** do this multiplication because the number of columns (3) of the Left Hand

Matrix, $(\mathbf{B}+\mathbf{C})$, does not equal the number of rows (2) of the Right Hand Matrix **A**.

(f) By Propositiion (1-3) part (c) we have $\mathbf{BA}+\mathbf{CA}=(\mathbf{B}+\mathbf{C})\mathbf{A}$. The Right Hand Side of this equation, $(\mathbf{B}+\mathbf{C})\mathbf{A}$ is impossible by part (e) above, therefore $\mathbf{BA}+\mathbf{CA}$ is impossible.

(g) By Propositiion (1-3) part (d) we have $\mathbf{CO}=\mathbf{O}=\begin{pmatrix}0&0&0\\0&0&0\\0&0&0\end{pmatrix}$.

(h) Since matrices **B** and **C** are of the same size therefore we have

$$\mathbf{BO}+\mathbf{CO}=\mathbf{O}+\mathbf{O}=\mathbf{O}=\begin{pmatrix}0&0&0\\0&0&0\\0&0&0\end{pmatrix}$$

(i) By Propositiion (1-3) part (d) we have $\mathbf{OB}+\mathbf{OC}=\mathbf{BO}+\mathbf{CO}=\mathbf{O}$

(j) We cannot evaluate $(\mathbf{A}+\mathbf{B})\mathbf{C}$ because **A** and **B** are of different size, therefore we **cannot** add them.

(k) By Propositiion (1-3) part (c) we have $\mathbf{AC}+\mathbf{BC}=(\mathbf{A}+\mathbf{B})\mathbf{C}$ therefore we cannot evaluate $\mathbf{AC}+\mathbf{BC}$ because the Right Hand Side of the equation is $(\mathbf{A}+\mathbf{B})\mathbf{C}$ which is impossible by part (j) above.

4. (a) We have $\mathbf{AI}=\begin{pmatrix}1&2&3\\4&5&6\\7&8&9\end{pmatrix}\times\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}=\begin{pmatrix}1&2&3\\4&5&6\\7&8&9\end{pmatrix}$ and

(b) $\mathbf{IA}=\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}\times\begin{pmatrix}1&2&3\\4&5&6\\7&8&9\end{pmatrix}=\begin{pmatrix}1&2&3\\4&5&6\\7&8&9\end{pmatrix}$. Notice that $\mathbf{AI}=\mathbf{IA}=\mathbf{A}$.

5. *What does* $(\mathbf{AB})_{11}$ *mean?*

It is the first entry in the matrix multiplication **AB**. The notation $(\mathbf{AB})_{11}$ is the first row of matrix **A** times the first column of matrix **B** where $\mathbf{A}=\begin{pmatrix}1&2&3\\4&5&6\end{pmatrix}$ and $\mathbf{B}=\begin{pmatrix}2&-1\\2&8\\7&5\end{pmatrix}$.

$$(\mathbf{AB})_{11}=\begin{pmatrix}1&2&3\end{pmatrix}\times\begin{pmatrix}2\\2\\7\end{pmatrix}=(1\times2)+(2\times2)+(3\times7)=27$$

Similarly we have

$$(\mathbf{AB})_{12} = \begin{pmatrix} 1 & 2 & 3 \end{pmatrix} \times \begin{pmatrix} -1 \\ 8 \\ 5 \end{pmatrix} = (1\times(-1)) + (2\times 8) + (3\times 5) = 30$$

$$(\mathbf{AB})_{21} = \begin{pmatrix} 4 & 5 & 6 \end{pmatrix} \times \begin{pmatrix} 2 \\ 2 \\ 7 \end{pmatrix} = (4\times 2) + (5\times 2) + (6\times 7) = 60$$

$$(\mathbf{AB})_{22} = \begin{pmatrix} 4 & 5 & 6 \end{pmatrix} \times \begin{pmatrix} -1 \\ 8 \\ 5 \end{pmatrix} = (4\times(-1)) + (5\times 8) + (6\times 5) = 66$$

Remember the matrix multiplication $\mathbf{AB} = \begin{pmatrix} (\mathbf{AB})_{11} & (\mathbf{AB})_{12} \\ (\mathbf{AB})_{21} & (\mathbf{AB})_{22} \end{pmatrix}$ therefore $\mathbf{AB}$ is $\begin{pmatrix} 27 & 30 \\ 60 & 66 \end{pmatrix}$.

6. We are given $\mathbf{A} = \begin{pmatrix} -1 & 3 & 5 \\ 4 & 1 & -7 \end{pmatrix}$, $\mathbf{B} = \begin{pmatrix} 1 & -3 & -5 \\ -4 & -1 & 7 \end{pmatrix}$, $c = -9$ and $k = 8$:

(a) We have $(ck)\mathbf{A} = (-9\times 8)\mathbf{A} = -72\begin{pmatrix} -1 & 3 & 5 \\ 4 & 1 & -7 \end{pmatrix} = \begin{pmatrix} 72 & -216 & -360 \\ -288 & -72 & 504 \end{pmatrix}$.

(b) By Theorem (1-2) property (a) we have

$$c(k\mathbf{A}) = (ck)\mathbf{A} \underset{\text{By Part (a)}}{=} \begin{pmatrix} 72 & -216 & -360 \\ -288 & -72 & 504 \end{pmatrix}$$

(c) We have

$$k(\mathbf{A}+\mathbf{B}) = 8\left[\begin{pmatrix} -1 & 3 & 5 \\ 4 & 1 & -7 \end{pmatrix} + \begin{pmatrix} 1 & -3 & -5 \\ -4 & -1 & 7 \end{pmatrix}\right]$$

$$= 8\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \mathbf{O}_{23}$$

Note that we have $\mathbf{B} = -\mathbf{A}$ so $k(\mathbf{A}+\mathbf{B}) = k(\mathbf{A}+(-\mathbf{A})) = k(\mathbf{A}-\mathbf{A}) = k(\mathbf{O}) = \mathbf{O} = \mathbf{O}_{23}$.

(d) By Theorem (1-2) property (b) we have

$$k\mathbf{A} + k\mathbf{B} = k(\mathbf{A}+\mathbf{B}) \underset{\text{By Part (c)}}{=} \mathbf{O}_{23}$$

(e) We have

$$(c+k)\mathbf{A} = (-9+8)\begin{pmatrix} -1 & 3 & 5 \\ 4 & 1 & -7 \end{pmatrix}$$

$$= -1\begin{pmatrix} -1 & 3 & 5 \\ 4 & 1 & -7 \end{pmatrix} = \begin{pmatrix} 1 & -3 & -5 \\ -4 & -1 & 7 \end{pmatrix} = \mathbf{B}$$

(f) By Theorem (1-2) property (c) we have $c\mathbf{A} + k\mathbf{A} = (c+k)\mathbf{A} \underset{\text{By Part (e)}}{=} \mathbf{B}$.

(g) We have

$$(c+k)\mathbf{B} = (-9+8)\begin{pmatrix} 1 & -3 & -5 \\ -4 & -1 & 7 \end{pmatrix}$$

$$= -1\begin{pmatrix} 1 & -3 & -5 \\ -4 & -1 & 7 \end{pmatrix} = \begin{pmatrix} -1 & 3 & 5 \\ 4 & 1 & -7 \end{pmatrix} = \mathbf{A}$$

(h) By Theorem (1-2) property (c) we have $c\mathbf{B} + k\mathbf{B} = (c+k)\mathbf{B} \underset{\text{By Part (g)}}{=} \mathbf{A}$.

7. We have

$$\mathbf{Ax} = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ 3 \end{pmatrix} = 3\begin{pmatrix} 1 \\ 1 \end{pmatrix} = 3\mathbf{x}$$

which means that $\lambda = 3$ because we have $\mathbf{Ax} = 3\mathbf{x}$.

For any natural number n a good prediction is $\mathbf{A}^n = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$.

8. For the given matrix $\mathbf{A} = \begin{pmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{pmatrix}$ take out the common factor $\frac{1}{3}$ to make the arithmetic easier. That is consider $\mathbf{A} = \frac{1}{3}\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$ then

$$\mathbf{A}^2 = \frac{1}{3}\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}\frac{1}{3}\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} = \left[\frac{1}{3}\times\frac{1}{3}\right]\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}\times\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$$

$$= \frac{1}{9}\begin{pmatrix} 3 & 3 & 3 \\ 3 & 3 & 3 \\ 3 & 3 & 3 \end{pmatrix}$$

$$\underset{\substack{\text{Taking} \\ \text{Out 3}}}{=} \frac{3}{9}\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} = \frac{1}{3}\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} = \mathbf{A}$$

Similarly we have

$$\mathbf{A}^3 = \underbrace{\mathbf{A}^2}_{=\mathbf{A}\text{ by above}}\mathbf{A} = \mathbf{AA} = \mathbf{A}^2 = \mathbf{A} = \frac{1}{3}\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$$

$$\mathbf{A}^4 = \underbrace{\mathbf{A}^3}_{=\mathbf{A}\text{ by above}}\mathbf{A} = \mathbf{AA} = \mathbf{A}^2 = \mathbf{A} = \frac{1}{3}\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$$

Generally for a natural number n we have $\mathbf{A}^n = \mathbf{A} = \frac{1}{3}\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$. Hence the formula for

$$\mathbf{x}_n = \mathbf{A}^n\mathbf{x} = \mathbf{A}\mathbf{x}$$

9. (a) We need to show the following is false:

$$\mathbf{AB} = \mathbf{O} \;\Rightarrow\; \mathbf{A} = \mathbf{B}$$

Let matrices **A** and **B** both be 2 by 2. Choose $\mathbf{A} = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ then

$$\mathbf{AB} = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a+c & b+d \\ a+c & b+d \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

Equating the first entry gives

$$a + c = 0 \;\Rightarrow\; a = -c.$$

Equating the second entry yields:

$$b + d = 0 \;\Rightarrow\; b = -d$$

Hence we can select any values for $a,\ b,\ c$ and d subject to $a = -c,\ b = -d$. Let $c = -1$ then $a = 1$ and let $d = -2$ then $b = 2$. This means we have $\mathbf{B} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ -1 & -2 \end{pmatrix}$.

Remember we can select an infinite number of values for $a,\ b,\ c$ and d as long as they satisfy $a = -c,\ b = -d$.

(b) We need to show that $\mathbf{AB} - \mathbf{BA} = \mathbf{O}$ does **not imply** that $\mathbf{A} = \mathbf{B}$.
Let $\mathbf{A} = \mathbf{O}$ [zero matrix] and **B** be any non-zero matrix of the same size so that we can carry out the matrix multiplications $\mathbf{AB}$ and $\mathbf{BA}$. Then

$$\mathbf{AB} = \mathbf{O} \text{ and } \mathbf{BA} = \mathbf{O}$$

Therefore $\mathbf{AB} - \mathbf{BA} = \mathbf{O}$ but matrices are not equal, that is $\mathbf{A} \neq \mathbf{B}$.

10. The Maple commands are:

> $T := \begin{bmatrix} 0.6 & 0.7 \\ 0.4 & 0.3 \end{bmatrix}$

$$T := \begin{bmatrix} 0.6 & 0.7 \\ 0.4 & 0.3 \end{bmatrix}$$

> $p := \begin{bmatrix} 0.5 \\ 0.5 \end{bmatrix}$

$$p := \begin{bmatrix} 0.5 \\ 0.5 \end{bmatrix}$$

> $T.p$

$$\begin{bmatrix} 0.6499999999999999912 \\ 0.3499999999999999976 \end{bmatrix}$$

> $T^2.p$

$$\begin{bmatrix} 0.6350000000000000008 \\ 0.3649999999999999992 \end{bmatrix}$$

> $T^{10}p$

$$\begin{bmatrix} 0.636363636349999818 \\ 0.363636363649999961 \end{bmatrix}$$

> $T^{100}p$

$$\begin{bmatrix} 0.636363636363634688 \\ 0.363636363636362647 \end{bmatrix}$$

> $T^{100000}.p$

$$\begin{bmatrix} 0.636363636361909846 \\ 0.363636363635377158 \end{bmatrix}$$

For large k we have $\mathbf{p}_k = \mathbf{T}^k\mathbf{p} = \begin{pmatrix} 0.636 \\ 0.364 \end{pmatrix}$.

11. Need to prove $\mathbf{AO} = \mathbf{OA} = \mathbf{O}$.

Proof.

Let $\mathbf{A}$ be a general $m \times n$ matrix given by $\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}$. Then we have

jth Column

$$\mathbf{AO} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ \boxed{a_{i1}} & \boxed{a_{i2}} & \boxed{\cdots a_{in}} & \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \begin{pmatrix} 0 & 0 \cdots & \boxed{0} & \cdots 0 \\ 0 & 0 \cdots & \boxed{0} & \cdots 0 \\ \vdots & \vdots & \boxed{\vdots} & \vdots \\ 0 & 0 \cdots & \boxed{0} & \cdots 0 \end{pmatrix}$$

ith Row

Multiplying out the boxed (ith) row and (jth) column gives:

$$(\mathbf{AO})_{ij} = (a_{i1} \times 0) + (a_{i2} \times 0) + \cdots + (a_{in} \times 0) = 0$$

Since we have $(\mathbf{AO})_{ij} = 0$ for arbitrary row and column therefore all the entries of $\mathbf{AO}$ are zero which means it is the zero matrix, $\mathbf{O}$. Similarly multiplying the other way round we have $(\mathbf{OA})_{ij} = 0$ which means that $\mathbf{OA} = \mathbf{O}$. This completes our proof.

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12. Let $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{C}$ be matrices of the same size $m \times n$ (m rows by n columns) given by

$$\mathbf{A} = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \vdots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{pmatrix}, \quad \mathbf{B} = \begin{pmatrix} b_{11} & \cdots & b_{1n} \\ \vdots & \vdots & \vdots \\ b_{m1} & \cdots & b_{mn} \end{pmatrix} \text{ and } \mathbf{C} = \begin{pmatrix} c_{11} & \cdots & c_{1n} \\ \vdots & \vdots & \vdots \\ c_{m1} & \cdots & c_{mn} \end{pmatrix}$$

Need to show that $(\mathbf{A} + \mathbf{B}) + \mathbf{C} = \mathbf{A} + (\mathbf{B} + \mathbf{C})$.

Proof of (b):

We have

$$\left(\mathbf{A}+\mathbf{B}\right)+\mathbf{C}=\underbrace{\left[\begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \vdots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{pmatrix}+\begin{pmatrix} b_{11} & \cdots & b_{1n} \\ \vdots & \vdots & \vdots \\ b_{m1} & \cdots & b_{mn} \end{pmatrix}\right]}_{=\mathbf{A}+\mathbf{B}}+\begin{pmatrix} c_{11} & \cdots & c_{1n} \\ \vdots & \vdots & \vdots \\ c_{m1} & \cdots & c_{mn} \end{pmatrix}$$

$$=\begin{pmatrix} a_{11}+b_{11} & \cdots & a_{1n}+b_{1n} \\ \vdots & \vdots & \vdots \\ a_{m1}+b_{m1} & \cdots & a_{mn}+b_{mn} \end{pmatrix}+\begin{pmatrix} c_{11} & \cdots & c_{1n} \\ \vdots & \vdots & \vdots \\ c_{m1} & \cdots & c_{mn} \end{pmatrix}$$

$$=\begin{pmatrix} a_{11}+b_{11}+c_{11} & \cdots & a_{1n}+b_{1n}+c_{1n} \\ \vdots & \vdots & \vdots \\ a_{m1}+b_{m1}+c_{m1} & \cdots & a_{mn}+b_{mn}+c_{mn} \end{pmatrix}$$

Similarly we can show that

$$\mathbf{A}+\left(\mathbf{B}+\mathbf{C}\right)=\begin{pmatrix} a_{11}+b_{11}+c_{11} & \cdots & a_{1n}+b_{1n}+c_{1n} \\ \vdots & \vdots & \vdots \\ a_{m1}+b_{m1}+c_{m1} & \cdots & a_{mn}+b_{mn}+c_{mn} \end{pmatrix}$$

This proves our required result $\left(\mathbf{A}+\mathbf{B}\right)+\mathbf{C}=\mathbf{A}+\left(\mathbf{B}+\mathbf{C}\right)$.

Proof of (d)

Required to prove $\mathbf{A}+\mathbf{O}=\mathbf{A}$ where **O** is the zero matrix. Adding the matrices **A** and **O**:

$$\mathbf{A}+\mathbf{O}=\begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \vdots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{pmatrix}+\begin{pmatrix} 0 & \cdots & 0 \\ \vdots & \vdots & \vdots \\ 0 & \cdots & 0 \end{pmatrix}$$

$$=\begin{pmatrix} a_{11}+0 & \cdots & a_{1n}+0 \\ \vdots & \vdots & \vdots \\ a_{m1}+0 & \cdots & a_{mn}+0 \end{pmatrix}=\begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \vdots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{pmatrix}=\mathbf{A}$$

Hence $\mathbf{A}+\mathbf{O}=\mathbf{A}$.

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Proof of (e).

Need to show that

$$\mathbf{A}+\left(-\mathbf{A}\right)=\mathbf{A}-\mathbf{A}=\mathbf{O}$$

We have

$$\mathbf{A}+\left(-\mathbf{A}\right)=\mathbf{A}+\left(\left(-1\right)\mathbf{A}\right)$$

$$=\mathbf{A}+\left(-1\right)\mathbf{A}=\mathbf{A}-\mathbf{A}$$

$$=\begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \vdots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{pmatrix}-\begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \vdots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{pmatrix}$$

$$=\begin{pmatrix} a_{11}-a_{11} & \cdots & a_{1n}-a_{1n} \\ \vdots & \vdots & \vdots \\ a_{m1}-a_{m1} & \cdots & a_{mn}-a_{mn} \end{pmatrix}=\begin{pmatrix} 0 & \cdots & 0 \\ \vdots & \vdots & \vdots \\ 0 & \cdots & 0 \end{pmatrix}=\mathbf{O}$$

Hence our required result.

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13. Let $\mathbf{A}=\begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \vdots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{pmatrix}$ and $\mathbf{B}=\begin{pmatrix} b_{11} & \cdots & b_{1n} \\ \vdots & \vdots & \vdots \\ b_{m1} & \cdots & b_{mn} \end{pmatrix}$ both be $m\times n$ (m rows by n columns) matrices and $c,\ k$ be scalars. Need to prove:

(a) $(ck)\mathbf{A}=c(k\mathbf{A})$. We have

$$(ck)\mathbf{A}=(ck)\begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \vdots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{pmatrix}$$

$$=\begin{pmatrix} (ck)a_{11} & \cdots & (ck)a_{1n} \\ \vdots & \vdots & \vdots \\ (ck)a_{m1} & \cdots & (ck)a_{mn} \end{pmatrix} \qquad \left[\text{Multiplying each entry by } ck\right]$$

$$=\begin{pmatrix} c(ka_{11}) & \cdots & c(ka_{1n}) \\ \vdots & \vdots & \vdots \\ c(ka_{m1}) & \cdots & c(ka_{mn}) \end{pmatrix} \qquad \left[\begin{array}{l}\text{Multiplying real numbers}\\ \text{is associative that is}\\ (ck)x=c(kx)\end{array}\right]$$

$$=\underbrace{c}_{\substack{\text{Factorizing}\\ \text{Out } c}}\begin{pmatrix} ka_{11} & \cdots & ka_{1n} \\ \vdots & \vdots & \vdots \\ ka_{m1} & \cdots & ka_{mn} \end{pmatrix}=c\left[\underbrace{k}_{\substack{\text{Factorizing}\\ \text{Out } k}}\underbrace{\begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \vdots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{pmatrix}}_{=\mathbf{A}}\right]=c(k\mathbf{A})$$

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(b) $k(\mathbf{A}+\mathbf{B})=k\mathbf{A}+k\mathbf{B}$. We have

$$k(\mathbf{A}+\mathbf{B})=k\left(\begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \vdots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{pmatrix}+\begin{pmatrix} b_{11} & \cdots & b_{1n} \\ \vdots & \vdots & \vdots \\ b_{m1} & \cdots & b_{mn} \end{pmatrix}\right)$$

$$=k\begin{pmatrix} a_{11}+b_{11} & \cdots & a_{1n}+b_{1n} \\ \vdots & \vdots & \vdots \\ a_{m1}+b_{m1} & \cdots & a_{mn}+b_{mn} \end{pmatrix} \qquad \left[\text{Adding the corresponding entries.}\right]$$

$$=\begin{pmatrix} ka_{11}+kb_{11} & \cdots & ka_{1n}+kb_{1n} \\ \vdots & \vdots & \vdots \\ ka_{m1}+kb_{m1} & \cdots & ka_{mn}+kb_{mn} \end{pmatrix} \qquad \left[\text{Multiplying each entry by } k.\right]$$

$$=\begin{pmatrix} ka_{11} & \cdots & ka_{1n} \\ \vdots & \vdots & \vdots \\ ka_{m1} & \cdots & ka_{mn} \end{pmatrix}+\begin{pmatrix} kb_{11} & \cdots & kb_{1n} \\ \vdots & \vdots & \vdots \\ kb_{m1} & \cdots & kb_{mn} \end{pmatrix} \qquad \left[\text{Separating into } a\text{'s and } b\text{'s}\right]$$

$$=k\underbrace{\begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \vdots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{pmatrix}}_{=\mathbf{A}}+k\underbrace{\begin{pmatrix} b_{11} & \cdots & b_{1n} \\ \vdots & \vdots & \vdots \\ b_{m1} & \cdots & b_{mn} \end{pmatrix}}_{=\mathbf{B}}=k\mathbf{A}+k\mathbf{B}$$

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14. $(\mathbf{AB})\mathbf{C} = \mathbf{A}(\mathbf{BC})$ (Associative law for multiplication). This is a difficult proof to follow because it has many symbols where the student can be confused. Go through it carefully.

Proof.

Let $\mathbf{A}$ be a $m \times r$ matrix, $\mathbf{B}$ be a $r \times n$ matrix and $\mathbf{C}$ be a $n \times p$ matrix. Then $\mathbf{AB}$ is a $m \times n$ matrix and $\mathbf{BC}$ is a $r \times p$ matrix. We first examine the Left Hand Side of $(\mathbf{AB})\mathbf{C} = \mathbf{A}(\mathbf{BC})$ and look at the ij entry of $(\mathbf{AB})\mathbf{C}$ and then show that it is equal to ij entry of the Right Hand Side, $\mathbf{A}(\mathbf{BC})$.

We can write $(\mathbf{AB})\mathbf{C}$ as

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$$(\mathbf{AB})\mathbf{C} = \begin{pmatrix} (\mathbf{AB})_{11} & (\mathbf{AB})_{12} & \cdots & (\mathbf{AB})_{1n} \\ \vdots & \vdots & \vdots & \vdots \\ (\mathbf{AB})_{i1} & (\mathbf{AB})_{i2} & \cdots & (\mathbf{AB})_{in} \\ \vdots & \vdots & \vdots & \vdots \\ (\mathbf{AB})_{m1} & (\mathbf{AB})_{mj} & \cdots & (\mathbf{AB})_{mn} \end{pmatrix} \begin{pmatrix} c_{11} & \cdots & c_{1j} & \cdots & c_{1p} \\ c_{21} & \cdots & c_{2j} & \cdots & c_{2p} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ c_{n1} & \cdots & c_{nj} & \cdots & c_{np} \end{pmatrix}$$

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The ij entry of $(\mathbf{AB})\mathbf{C}$ is given by

$$\begin{aligned}
\lfloor (\mathbf{AB})\mathbf{C} \rfloor_{ij} &= (\mathbf{AB})_{i1} c_{1j} + (\mathbf{AB})_{i2} c_{2j} + \cdots + (\mathbf{AB})_{in} c_{nj} \\
&= \sum_{s=1}^{n} (\mathbf{AB})_{is} c_{sj} && [\text{Using the sigma notation}] \\
&= \sum_{s=1}^{n} \left(\sum_{k=1}^{r} a_{ik} b_{ks} \right) c_{sj} && [\text{Applying } (1.13)] \\
&= \sum_{s=1}^{n} \sum_{k=1}^{r} a_{ik} \left(b_{ks} c_{sj} \right) && [\text{Using the algebra of Real Numbers}] \\
&= \sum_{k=1}^{r} a_{ik} \sum_{s=1}^{n} b_{ks} c_{sj} && [\text{Using the algebra of Real Numbers}] \\
&= \sum_{k=1}^{r} a_{ik} \left(\sum_{s=1}^{n} b_{ks} c_{sj} \right) \\
&= \sum_{k=1}^{r} \mathbf{A}_{ik} (\mathbf{BC})_{kj} && [\text{Applying } (1.13)] \\
&= \left[\mathbf{A}(\mathbf{BC}) \right]_{ij}
\end{aligned}$$

Since $\left[(\mathbf{AB})\mathbf{C} \right]_{ij} = \left[\mathbf{A}(\mathbf{BC}) \right]_{ij}$ therefore the ij entries of $(\mathbf{AB})\mathbf{C}$ and $\mathbf{A}(\mathbf{BC})$ match and we conclude that $(\mathbf{AB})\mathbf{C} = \mathbf{A}(\mathbf{BC})$ which is our required result.

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(1.13) $$(\mathbf{AB})_{ij} = \left(a_{i1} b_{1j}\right) + \left(a_{i2} b_{2j}\right) + \left(a_{i3} b_{3j}\right) + \cdots + \left(a_{ir} b_{rj}\right) = \sum_{k=1}^{r} a_{ik} b_{kj}$$